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APPLIED THERMODYNAMICS AND HEAT TRANSFER

By

I. I. Novikov and K. D. Voskresenskiy

FOREIGN TECHNOLOGY DIVISION

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BY: I. I. Novikov and K. D. Voskresenskiy

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FROM THE EDITOR

The practical possibility of utilizing atomic energy for industrial purposes, and in particular for the generation of electricity, is not subject to any doubt at the present time. This is convincingly evidenced by the experience in the operation of the first atomic electric station in the world, started in the Soviet Union in June 1954, and also the successful operation of the large atomic electric stations constructed after 1954 in England, the Soviet Union, and in the United States of America.

Two natural elements can be used for the extraction of nuclear energy, uranium and thorium. By 1955, the surveyed natural resources of uranium amounted to approximately 25 million tons, and those of thorium almost one million tons. In terms of standard fuel, the potential energy of the natural resources of uranium and thorium is approximately 25 times larger than the natural resources of organic fuel (coal, oil, natural gas). It follows therefore that the use of nuclear energy uncovers broad prospects for the expansion and development of the world's power base.

Particular interest is attached to the possibility of using atomic energy for transport purposes. In 1959, the Soviet atomic ship "Lenin" went into construction, the first large ocean vessel with nuclear equipment. Various types of atomic ships are constructed also in other countries.

The extensive program of peaceful utilization of atomic energy poses a vital problem to the higher institutions of learning, namely

the training of specialists so as to supply personnel for the atomic electric stations earmarked for construction and for many new branches of technology, which come into being with the development of applied nuclear physics and radiation chemistry.

During the last decade much theoretical and experimental material was accumulated on various problems in nuclear power, and much experience was gained in working with atomic reactors and other nuclear installations under commercial conditions.

The extensive nature and specific character of this material makes it possible and essential to introduce specialization of the engineers and physicists in the field of atomic power and applied nuclear physics, and particularly the separation of problems in nuclear power in specialized courses.

It is appropriate to note here that by now the main physical principles of the use of atomic energy have been established, so that the role of the engineer becomes more and more important in the further development of nuclear power. The engineering difficulties connected with the creation of reliably operating power and auxiliary equipment for nuclear installations constitute at present one of the central problems of nuclear power and reactor building.

The introduction of specialization of engineers and physicists in the field of atomic energy, particularly in reactor building and atomic power installations, applied nuclear physics, the use of isotopes and radiation, all have necessitated the creation of specialized courses and corresponding textbooks. In this connection, the Gosatomizdat proposes to publish a series of books under the common title "Principles of Nuclear Power." All these books are based on lecture courses delivered by the authors during many years to students specializing in the field of nuclear power.

Bearing in mind the large relative significance of problems involved in the removal of heat from the nuclear reactors and its conversion into other types of energy, the basic information on thermodynamics and heat transfer have been treated in a separate book – "Applied Thermodynamics and Heat Transfer" (authors, I.I. Novikov and K.D. Voskresenskiy).

The content of the second book – "Heat Power for Nuclear Installations" (author, B.V. Petunin) comprises an analysis of the operating cycles used in atomic power installations, and also a description of the principal apparatus, primarily heat exchangers, constituting modern atomic power installations. The book gives all information necessary for the design of the indicated apparatus.

In the third book – "Calculation and Design of Atomic Reactors" (authors, Ya.V. Shevelev and A.Ya. Kramerov) are developed the operating principles and constructions of atomic reactors, methods of physical and thermal design of reactors, and the principles of the construction of atomic reactors of various types.

The general editor of the first two books is Corresponding Member of the USSR Academy of Sciences I.I. Novikov, while that of the third is Academician of the Ukrainian Academy of Sciences A.I. Leypunskiy.

The series "Principles of Nuclear Power" will contain several additional books in addition to those listed.

I.I. NOVIKOV

Part One

THERMODYNAMICS

I. I. Novikov

Chapter 1

FUNDAMENTAL CONCEPTS

§1. SCOPE AND METHOD OF THERMODYNAMICS

Thermodynamics, the science of energy and its properties, investigates the laws governing the conversion of energy in physical, chemical, and other processes, including the conversion of heat into work in heat engines. It is based on two experimentally established laws, which are called the first and second laws of thermodynamics.

The first law of thermodynamics is an application of a universal law of nature, namely the law of transformation and conservation of energy, to heat phenomena.

The second law of thermodynamics characterizes the direction of the macroscopic processes which occur in the surrounding nature. Because it reflects the peculiarities of these processes only, the second law of thermodynamics has a more limited significance compared with the first.

The task of thermodynamics consists of investigating the macroscopic properties of real bodies in the equilibrium state and processes occurring with bodies.

In technical thermodynamics, the laws of thermodynamics are applied to processes involving the mutual conversion of heat and work. The main problem of technical thermodynamics consists of providing the foundations for the theory of heat engines.

§2. THERMODYNAMIC QUANTITIES

Heat and Work

Various bodies transfer to one another certain amounts of energy when they interact; the state of each of the interacting bodies is then accordingly changed. The transfer of energy from one body to another can be realized in two ways.

The first method of energy transfer is well known from mechanics and consists of the performance of work by one body on another. In this case the increase in the energy of the body is equal to the work performed on it by the other body. We shall henceforth denote the work by L. It is customary to assume that the work performed by the body is positive and the work performed on a body (obtained by the latter) is negative. The unit of work in the technical system of units is the kilogram meter while in the physical system it is the erg. The transfer of energy to a body by performing work on it is always connected with a change in the external conditions in which the body is situated, and particularly the displacement of the body or of its individual parts as a unit.

The second method of energy transfer is not connected with the performance of work by one body on another, and consists of a direct transfer of energy from a hotter body to a colder one. That part of the energy which is transferred to a body from a hotter body directly (as a result of thermal contact between them) is called the heat received by the body.

The quantity of heat is usually denoted by Q. The heat obtained by a body is considered positive and the heat given up by the body negative. The unit of the quantity of heat is the kilocalorie (large calorie) which is equal in practice* to the quantity of heat necessary to heat one kilogram of water from 19.5 to 20.5°C at normal atmospheric

pressure.

The amount of heat and the amount of work are measures of the energy transferred to the body in the form of work and in the form of heat. The amount of work performed by the body, and the amount of heat obtained by the body depend on those conditions under which the state of the body is changed, i.e., on the character of the process or on the path of the transition of the heat from the initial state to the final one.

Thermodynamic System

The object of study in thermodynamics is various thermodynamic systems. A thermodynamic system refers to an aggregate of material bodies which are in mechanical and thermal interaction with one another and with the surrounding (external) medium.

A thermodynamic system which cannot exchange heat with the surrounding medium is called thermally insulated or adiabatically insulated. An example of a thermally insulated system is a gas in a vessel the walls of which are covered with an ideal thermal insulation. Such an ideal thermally insulated shell is called an adiabatic shell.

Any system contained in an adiabatic and absolutely rigid shell, which makes impossible both thermal and mechanical interaction with the surrounding medium, is called closed or insulated. A system which has identical properties in all of its parts is called uniform. A system which consists of several homogeneous but different parts, separated from one another by separation surfaces, is called heterogeneous. The different parts of which this system consists are called phases. In accordance with the three forms of the aggregate state of matter, a distinction is made between the gaseous, liquid, and solid (crystalline) phases. A system having no internal separation surfaces and consisting of one phase is called homogeneous.

The macroscopic quantities characterizing the state of the system as a whole are called the thermodynamic parameters. From the totality of the large number of thermodynamic parameters it is necessary to single out especially those parameters, with the aid of which the processes of mutual conversion of heat and work are described. They have been called the thermal parameters. The principal thermal parameters are the temperature, the pressure, and the volume of the body. The volume of the body is denoted by V, the pressure by p, the empirical temperature by t, and the absolute temperature by T, with $T = t + 273.16^{\circ}C$.

Thermodynamic Equilibrium

If the state of a thermodynamic system does not change with time (i.e., the properties of the system, compared at two different instants of time, are the same), it is said that the system is in thermodynamic equilibrium, or simply in equilibrium.

In the state of thermodynamic equilibrium the transfer of heat from some parts of the system to others and the mechanical displacement of the individual parts of the system relative to one another are zero, i.e., thermal and mechanical equilibrium prevail.

The state of thermodynamic equilibrium of an uninsulated system interacting with the surrounding medium is defined uniquely by specifying the external conditions – the mechanical parameters and the temperature of the surrounding medium. Any system which is in invariant external conditions will sooner or later enter into the state of thermodynamic equilibrium, no matter what its initial state may be. A system cannot go spontaneously out of the equilibrium state (the principle of nonself-violation of the thermodynamic equilibrium).

§3. EQUATION OF STATE

It is not the task of thermodynamics to establish analytic rela-

tionships for specific properties of real bodies (for example, the connection between the pressure, volume, and temperature of a uniform body, the dependence of the specific heat on the temperature, etc.). These relationships are established either experimentally or (where possible) theoretically, for example on the basis of the kinetic theory of matter. Thermodynamics observes, however, many limitations in the properties of bodies and by the same token makes it possible to formulate certain general laws in the behavior of real bodies. One of these general laws, pertaining to uniform bodies, is the following.

The macroscopic state of any uniform body* in equilibrium is determined only by two independent parameters. The latter can be taken as any pair of thermal parameters: the volume V and the temperature of the body t, or the pressure p and the temperature t, etc. All the macroscopic or thermodynamic quantities which characterize the equilibrium state of a body can be represented as functions of these two parameters.

Inasmuch as a simple increase or decrease of the amount of matter at constant temperature and density cannot change the pressure of a uniform body which is in equilibrium, it is possible to choose as one of the independent parameters of state not only the volume of the body V, but also its molar volume $V_\mu = V/M$, equal to the volume of one kilogram-mole, or the specific volume** $v = V/G$, where G is the weight of the body in kilograms and $M = G/\mu$ the number of moles in a given amount of matter, with μ the molecular weight.

Thus, the pressure of a uniform body p in the state of equilibrium can be represented in the form of a certain function of the temperature t and of the volume of the body (specific v, molar V_μ, or total volume V).

The equation

$$p = f(t, V), \qquad (1.1)$$

which relates the pressure p, temperature t, and the volume V of a uniform body in the equilibrium state is called the thermal equation of state of the body.

The connection between the parameters p, t, and V for a given uniform body in the state of equilibrium can be represented graphically by plotting Eq. (1.1) in the space having coordinates p, t, and V. In this three-dimensional thermodynamic space, the equation of state of each body is characterized by a certain surface, called the thermodynamic surface of the given body.

Equation of State of Gases

The form of the equation of state, i.e., the form of the function $p = f(t, V)$, is determined, as was already indicated, by experiment or is established on the basis of the kinetic theory of matter.

Ideal gas. There are no intermolecular action forces in an ideal gas, and the molecules themselves are considered as material points which have no spatial extents. In these cases the equation of state has the form

$$p = \frac{G}{\mu} kN_A \cdot 10^3 \frac{T}{V}. \qquad (1.2)$$

The product of the Boltzmann constant k by the number of molecules per kilogram-mole of matter, which is equal to the Avogadro number N_A multiplied by 10^3, is called the universal gas constant and denoted by μR

$$\mu R = kN_A \cdot 10^3.$$

The numerical value of the universal gas constant is

$$\mu R = (1.37 \cdot 10^{-16})/(981 \cdot 10^5) = 6.0228 \cdot 10^{23} \cdot 10^3 \text{ erg/mole} \cdot \text{degree} =$$

$$= 848 \text{ kgm/kg} \cdot \text{mole} \cdot \text{degree}.$$

The quantity R, which is equal to the ratio of the universal gas

constant μR to the molecular weight μ of the matter is called the gas constant for 1 kg of a given matter. The values of the gas constant R for different substances are shown in Table 1.

TABLE 1

Value of Gas Constant for Different Substances

1 Вещество	2 Химическая формула	3 Молекулярный вес	Газовая постоянная, 4 кгм/кг·град
5 Водород . . .	H_2	2,016	420,60
6 Гелий	He	3,99	212,00
7 Метан	CH_4	16,08	52,90
8 Аммиак . . .	NH_3	17,03	49,79
9 Вода	H_2O	18,016	47,07
10 Азот	N_2	28,02	30,26
11 Окись углерода	CO	28,00	30,29
12 Воздух	—	28,95	29,27
13 Кислород . . .	O_2	32,00	26,50
14 Двуокись углерода	CO_2	44,00	19,27

1) Substance; 2) chemical formula; 3) molecular weight; 4) gas constant, kgm/kg·degree; 5) hydrogen; 6) helium; 7) methane; 8) ammonia; 9) water; 10) nitrogen; 11) carbon monoxide; 12) air; 13) oxygen; 14) carbon dioxide.

If we use the quantity R, then Eq. (1.2) can be rewritten in the form

$$pV = GRT. \qquad (1.3)$$

Equation (1.3) is called the Clapeyron-Mendeleyev equation for an ideal gas.

In the case of a mixture of several ideal gases, each of the gases behaves as if there were no other gases in the mixture, owing to the lack of interaction between the molecules. Therefore the pressure of each of the gases comprising the mixture will equal the pressure which this gas would have were it to be in the same amount, volume, and temperature as in the mixture

$$p^{(j)} = G^{(j)} R^{(j)} \frac{T}{V}. \qquad (1.4)$$

The pressure $p^{(j)}$ is called the partial pressure of the given (j-th) gas; V is the total volume occupied by the gas mixture and T is

the absolute temperature of the mixture.

The total pressure of a mixture of ideal gases is equal to the sum of the partial pressures of the individual gases comprising the mixture (Dalton's law)

$$p = \Sigma p^{(j)}. \qquad (1.5)$$

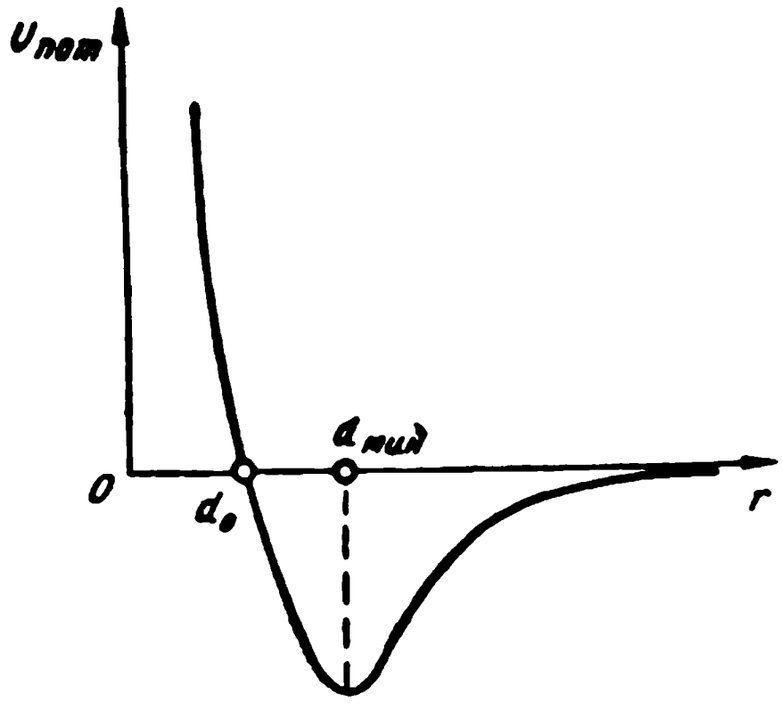

Fig. 1. Dependence of the potential energy of interaction of two molecules on the distance between their centers.

Real gas. Gas molecules have actually finite although very small geometrical dimensions and are attracted to one another by forces which increase the smaller the average distance between the molecules, i.e., the smaller the specific volume of the gas. Figure 1 shows the dependence of the potential energy of interaction of two molecules u on the distance r between the centers of the molecules. The force acting on each of the molecules is equal to $-\partial u/\partial r$. At very small distances between molecules, tremendous repulsive forces develop, as a result of which the volume of the gas cannot be decreased below a certain limit the value of which is b for 1 kg.

Owing to the interaction between the molecules, the properties of the real gases deviate from those of ideal ones; consequently, the Clapeyron-Mendeleyev equation is applicable to real gases only at very large rarefaction, or, what is the same, at low gas density. The mutual attraction forces prevent the scattering of the gas molecules and consequently act in the same direction as the external pressure p, retaining the gas in a definite volume. As a result of the action of these forces, the so-called internal or molecular pressure p_{mol} is produced:

$$p_{mol} = a/v^2,$$

where a is some coefficient.

The equation of state of a real gas, with account of the forces acting between the molecules per kilogram of gas, has the following form:

$$\left(p+\frac{a}{v^2}\right)(v-b)=RT. \qquad (1.6)$$

The quantities a and b can be regarded in first approximation as being certain constants that depend on the nature of the substance. They are calculated from the formulas

$$a=\frac{27}{64}R^2\frac{T_к^2}{p_к};\quad b=\frac{1}{8}R\frac{T_к}{p_к}. \qquad (1.7)$$

where p_k is the critical pressure of the given substance and T_k its absolute critical temperature.

The critical temperature and pressure have a simple physical meaning: mutual transformations of the liquid and gas phases, particularly the liquefaction of gas, are possible only under the condition $t < t_k$, $p < p_k$.

The critical pressure, temperature, and volume p_k, T_k, and v_k represent the values of these parameters at the critical point of the substance, defined by the conditions $(\partial p/\partial v)_t = 0$ and $(\partial^2 p/\partial v^2)_t = 0$. The critical parameters are the most important thermodynamic characteristics of the substance, expressing the effect of the action of the molecular forces in generalized quantitative form.

Equation (1.6) is called the Van der Waals equation for a real gas. It is more accurate than the Clapeyron-Mendeleyev equation, but is still approximate and is used only in a limited range of states.

In order to improve the Van der Waals equation and to increase its accuracy, various corrections to it were proposed, as a result of which many new equations were obtained. Some of these equations of state turn out to be so accurate (within a limited range of states),

that they were used to compile tables of the thermodynamic properties of substances.

The development of the kinetic theory of matter has led recently to the establishment of a general equation of state for real gases

$$pv = RT\left[1 - \sum \frac{\nu}{\nu+1}\frac{B_\nu}{v^\nu}\right]. \qquad (1.8)$$

In this equation, the coefficients B_ν at the powers of $\underline{v}$ in the right part of the equation, called the virial coefficients, are expressed in terms of the potential energy $\underline{u}$ of interaction of two molecules of the given gas and the temperature T, and can be calculated if one knows the functional dependence of u(r) on the distance $\underline{r}$ between the molecules (see Fig. 1). The Van der Waals equation is a particular case of this general equation of state, which is obtained by neglecting in it all the terms in the right half of (1.6) containing $\underline{v}$ to a higher power than the second, something that can always be done provided the gas density is not too high.

Without dwelling here on a detailed analysis of Eq. (1.8), we shall merely point out that it is almost unsuitable for practical use, for in order to ensure the required accuracy it is necessary to retain in the right half of this equation a considerable number of terms and this makes it assume a cumbersome form. In addition, the calculation of the virial coefficients is not feasible in all cases, since the exact expression for the potential energy u(r) of the interaction between two molecules is unknown for many substances. Therefore in calculating the thermodynamic properties of various substances and in setting up thermodynamic tables (or diagrams), use is usually made of experimental data which are used either directly or to derive empirical formulas (equations) on the basis of which all calculations are carried out.

If there is a mixture of several real gases, then the total pressure of the mixture can be determined with sufficient degree of accuracy from the formula

$$p = \sum p^{(j)}\left(t, \frac{V}{\mu^{(j)}M}\right)z^{(j)}, \qquad (1.9)$$

where $p^{(j)}$ is the pressure of the j-th gas, the amount of which is M moles (M is the number of moles of mixture, equal to $\Sigma M^{(j)}$) and occupies at a temperature t the same volume V as the entire mixture (or having a specific volume $v = V/\mu^{(j)}M$ and a temperature t), while $z^{(j)}$ is the molar concentration of the j-th gas, equal to $M^{(j)}/M$.

Thermodynamic Similarity

In place of the variables v, p, and T we introduce new variables $\varphi = v/v_k$, $\pi = p/p_k$, and $\tau = T/T_k$ and we call them, respectively, the reduced volume, pressure, and temperature. Then, using Expression (1.8) for a and b, it is easy to transform the Van der Waals equation to the following form:

$$\left(\pi + \frac{3}{\varphi^2}\right)(3\varphi - 1) = 8\tau. \qquad (1.10)$$

Equation (1.10) is called the reduced Van der Waals equation of state. This equation contains dimensionless parameters π, φ, and τ and numerical coefficients. The equation contains no other quantities characterizing the given substance. It follows therefore that the reduced equation is the same for the most varied substances, provided they obey the Van der Waals equation.

The states of two or several substances in which they have identical parameters π, φ, and τ are called the corresponding states.

If substances satisfy one and the same reduced equation of state and have identical two out of three reduced parameters π, φ, and τ, then the third reduced parameter will also be the same for them, i.e., the substances are in corresponding states. This statement is called

the <u>law of corresponding states</u>.

Substances obeying the law of corresponding states and satisfying one and the same reduced equation of state are called <u>thermodynamically similar substances</u>.*

The law of corresponding states must not be regarded as a consequence of the Van der Waals equation. Any equation of state which is valid for all or for only several substances in which there are contained in addition to the universal constants (for example μR, N_A, $\underline{k}$) only two individual constants characterizing the nature of the substance and varying on going from one substance to the other can be transformed into a reduced equation of state $\pi = f(\tau, \varphi)$. This equation, being applicable to all the substances under consideration, is a mathematical expression of the law of corresponding states for the given group of substances.

The law of corresponding states pertains not only to the connection between the parameters π, τ, and φ but also extends to other thermodynamic quantities. In particular, this law is valid for the states of phase equilibrium between the liquid and vapor phase, and therefore the phase equilibrium curve should have the same form in reduced coordinates for thermodynamically similar substances. On the other hand, the law of corresponding states can be applied to various processes of conversion of mechanical energy into heat and of heat transfer from one body to another. In all these processes the thermodynamically similar substances will behave in identical fashion and will satisfy the same equations containing the dimensionless parameters π, φ, τ, and $C_{V_\infty}/\mu R$, where C_{V_∞} is the molar specific heat of the substance in the gaseous state as $p \to 0$.

With the aid of the law of corresponding states we can establish the following important relationships for such properties of substances

as the surface tension σ, the viscosity η, and the heat conduction λ:

$$\sigma = \sqrt[3]{\mu R T_k p_k^2}\, B(\tau); \tag{1.11}$$

$$\eta = \sqrt[6]{\frac{\mu^3 p_k^4}{g^3 \mu R T_k}}\, E\left(\pi, \tau, \frac{C_{v_\infty}}{\mu R}\right); \tag{1.12}$$

$$\lambda = \sqrt[6]{\frac{g^3 \mu R^5 p_k^4}{\mu^3 T_k}}\, D\left(\pi, \tau, \frac{C_{v_\infty}}{\mu R}\right). \tag{1.13}$$

For all thermodynamically similar substances the functions B, E, and D are the same. They are universal functions of the dimensionless parameters.

The general relationships (1.11)-(1.13) make it possible to determine, from the known properties of one of the substances, the properties of the other thermodynamically similar substances under the same conditions, i.e., for equal values of the reduced parameters.

For the physical properties of substances, equations of the type (1.11)-(1.13) can be derived also from the following simple considerations. According to the law of corresponding states, any of the quantities characterizing the physical properties of a body should be equal to the product of a factor having the same dimensionality as the property under consideration and made up of the main physical parameters of the given body: the critical pressure p_k or the critical volume v_k, the critical temperature T_k, the quantity μ/g (the ratio of the molecular weight μ to the acceleration due to gravity g), the universal gas constant μR multiplied by a dimensionless function of the dimensionless parameters π, τ or τ, φ and $C_{V_\infty}/\mu R$. Combining p_k, T_k, μ/g, and μR in such a way as to obtain the dimensionality of the quantity of interest to us, we can obtain expressions for the dimensional factor of any of the relations of the type (1.11)-(1.13).

§4. THE THERMODYNAMIC PROCESS

If the external conditions in which a thermodynamic system* is situated are changed, then the state of the system will also change.

Therefore the thermodynamic parameters which characterize the state of the system will have different values at different instants of time. The sequence of variation of the state of the system comprises the thermodynamic process.

Any process in which the state of the body changes represents a deviation from the equilibrium state. Violation of the equilibrium leads to occurrence inside the body of processes which oppose the deviation from the state of equilibrium. These internal processes, which compensate for the violation of equilibrium and which tend to restore it, are the elementary processes of energy exchange upon collision of the molecules.

A process which is so slow (quasistatic) that equilibrium can be established in practice in the system at any instant of time (or a state very close to equilibrium) is a quasiequilibrium process. The degree of approximation of this process to a strictly equilibrium process will be the greater, the smaller the rate of change of the state of the system. In the limit we have an infinitely slow process, which is a completely equilibrium process and represents an aggregate of equilibrium states through which the system passes in succession.

If the state of the system at each instant of the process is not an equilibrium state, then such a process involving the variation of a state is called nonequilibrium. In a nonequilibrium state, the parameters of the system are in general not determined uniquely by the external conditions. Therefore in order to describe a nonequilibrium process it is necessary to specify, in addition to the external conditions, also one or several internal parameters (for example the density distribution).

The distinguishing features of the character of each of the processes wherein the state of the body is varied are determined by those

specific physical conditions under which the given process proceeds. The simplest processes are as follows: isothermal process, characterized by a constant body temperature during the entire time of the process (t = const); isobaric process, occurring under conditions of equal pressure on the body (p = const); isochoric process, in which the volume of the body V remains unchanged during the entire process (V = = const); adiabatic process, when the body is placed in a thermally insulated shell which permits no heat exchange between the body and the surrounding medium.

In general form, a thermodynamic process is characterized by the condition X = const, where X is one of the external or internal parameters of the system or some function of these parameters.

Among the thermodynamic processes, particular interest attaches to so-called closed or cyclic processes, in which the system, after going through a series of successive states, returns to the initial or original state. A cyclic process is also called a cycle.

Equilibrium processes in the variation of the body admit of a direct graphic representation. In fact, any equilibrium process involving the variation of the state of the body represents an aggregate of equilibrium states, through which the body goes successively, so that at any instant of time the thermal parameters of the body p, t, and V have fully defined values, each of them in the equilibrium state. Because of this, each of the states of the body in the equilibrium process can be represented in thermodynamic space in the form of a point with coordinates equal to the values of the pressure and temperature of the volume of the body at the given instant. The totality of these points forms in the thermodynamic space a continuous line which depicts graphically the equilibrium process. The form of this line depends on the law governing the variation of the state of the body.

Nonequilibrium processes, and particularly fast processes, in which the thermal parameters of the body (all or some) are different in different parts of the body and therefore have no value which is defined for the entire body, cannot be represented graphically in the form of a line in the space p, t, V. These processes are sometimes arbitrarily denoted by a dashed curve.

§5. ANALYTIC EXPRESSION FOR THE WORK AND HEAT OF A PROCESS

Work of a Process

If an infinitesimally small expansion of the system occurs in an external medium which has everywhere one and the same pressure p' then the increase in the volume of the system V by an amount dV is accompanied by an elementary work $dL = p'dV$, which is performed by the system on the surrounding medium. This work is called the work of the change in volume.

In an equilibrium process, the pressure of the external medium p' is equal to the pressure of the body p ($p' = p$), and therefore the expression for dL has in this case the form $dL = pdV$.

The expression for the work L performed by the system at finite and not infinitesimally small change of its volume from a value V_1 to a value V_2 will have the following form:

$$L = \int_{V_1}^{V_2} p'dV. \qquad (1.14)$$

where the integral is taken along the path of transition of the system from the initial state to the final one, i.e., over all states of the system during the process 1-2. In an equilibrium process the work L performed by the system can be expressed as:

$$L = \int_{V_1}^{V_2} pdV.$$

The work L depends on the path along which the system goes over

from state 1 into state 2, i.e., it is a function of the processes and not of the state. This is clearly seen, in particular, from the expression for the work of the equilibrium process. The pressure p depends in accordance with Eq. (1.1) not only on the volume V of the body, but also on its temperature t. Therefore on going over from one and the same initial state 1 into one and the same final state 2, the pressure p at the intermediate point will have different values, depending on the temperature, which changes differently in different processes. Consequently, the work of the process L will also be different.

The work performed by the system against external forces upon increase in the volume is called the expansion work, and the work performed by the external forces on the system upon decrease in volume is called the contraction work. The expansion work has a positive sign and that of contraction a negative sign. The work of the system per kilogram of weight is called the specific work.

It must be recalled that the work performed by the system during the course of the variation of its state does not equal zero only when the process results in a displacement of bodies that are external relative to the system. In the case of expansion of a body in vacuum, when there is no displacement of external bodies at all, the work is equal to zero, as follows also from (1.14), inasmuch as in the case of expansion in the vacuum the pressure p' on the boundaries between the body and the external medium is equal to zero.

The work performed by the system does not always reduce merely to the work involved in changing the volume, and consequently is not always expressed by Formula (1.13). Thus, for example, if a liquid contained in a vessel is made to move by means of a stirrer rotated from the outside, then the work performed by the external source on the liquid has a finite value, numerically equal to the increment of the kin-

etic energy of the liquid, whereas the integral $\int p'dV$ is equal to zero, owing to the constancy of the volume V of the liquid. Work done against electric, magnetic, and other forces is likewise unaccompanied by a change in volume. In the most general case the work L performed by the system is equal to the sum of the work of change in volume $\int p'dV$ and the work L_V which is not connected with a change in the volume V_2 of the system

$$L = \int_{V_1}^{V_2} p'dV + L_V.$$

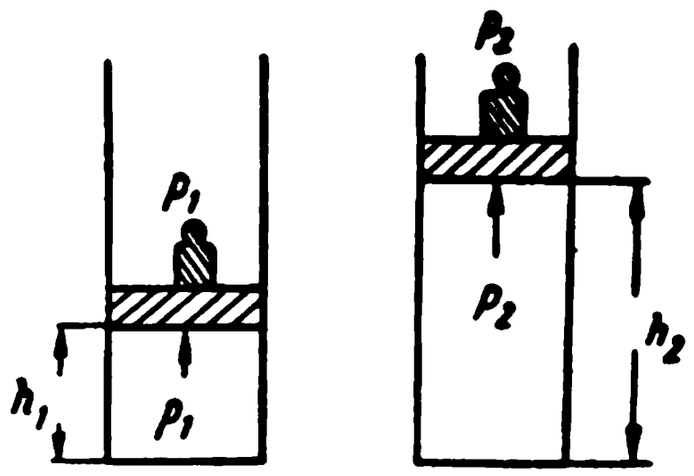

Fig. 2. Simplest thermodynamic system.

Let us calculate now the useful external work which is performed by the system on a certain external object when the state of the system changes. Let the system under consideration be a gas in a cylinder under a piston (Fig. 2). The piston is loaded with a weight P so that the external pressure on the gas is equal to $p' = P/\Omega$, where Ω is the area of the piston. In the initial state the pressure and the volume of the gas and of the surrounding medium are, respectively, p_1, V_1 and p'_1, V'_1, while in the final state they are p_2, V_2 and p'_2, V'_2. Let us assume that the gas has expanded from a volume V_1 to a volume $V_2 > V_1$; at the same time the piston with the weight has moved from a height h_1 to a height $h_2 > h_1$. As a result of the expansion the gas performed work $\int_{V_1}^{V_2} p'dV + L_V$, and the potential energy of the weight increased by an amount $P_2h_2 - P_1h_1$, equal to $p'_2V_2 - p'_1V_1$ (inasmuch as $h\Omega = V$). The difference between the work performed by the gas and the increment in the potential energy is precisely the useful work L' which was performed on the external object as a result of the given process:

$$L' = \int_{V_1}^{V_2} p'dV + L_V - (p'_2V_2 - p'_1V_1). \qquad (1.15)$$

Since

$$\int_{V_1}^{V_2} p'dV = -\int_{p'_1}^{p'_2} Vdp' + p'_2V_2 - p'_1V_1,$$

the useful external work is

$$L' = -\int_{p'_1}^{p'_2} Vdp' + L_V. \qquad (1.16)$$

The quantity $p'_2V_2 - p'_1V_1$ is called the pushing work. Each of the terms ($-p'_1V_1$ or p'_2V_2) represents the work that must be performed in order to introduce a body with volume V in an external medium which is at a pressure p'.

Graphic Representation of the Work of an Equilibrium Process

It follows from Expression (1.14) that in the case of an equilibrium process the work involved in the change in the volume of the body can be represented graphically in the pV plane by the area contained

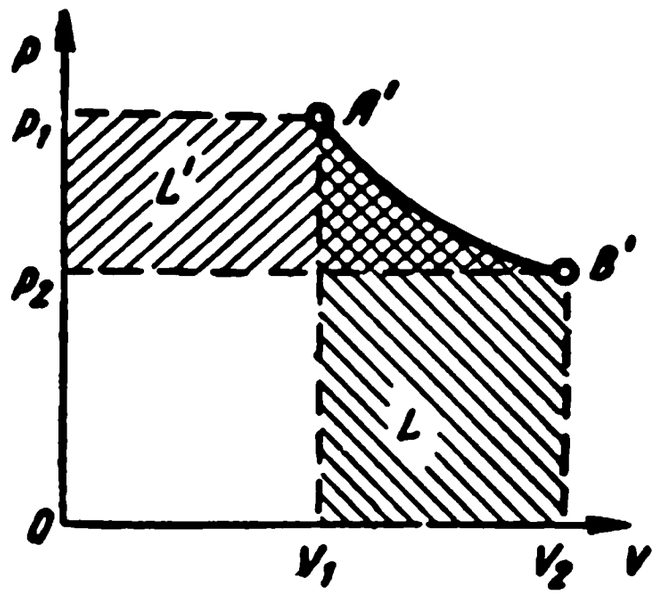

Fig. 3. Graphic representation in the pV plane of the work of an equilibrium process wherein the state of the body is changed.

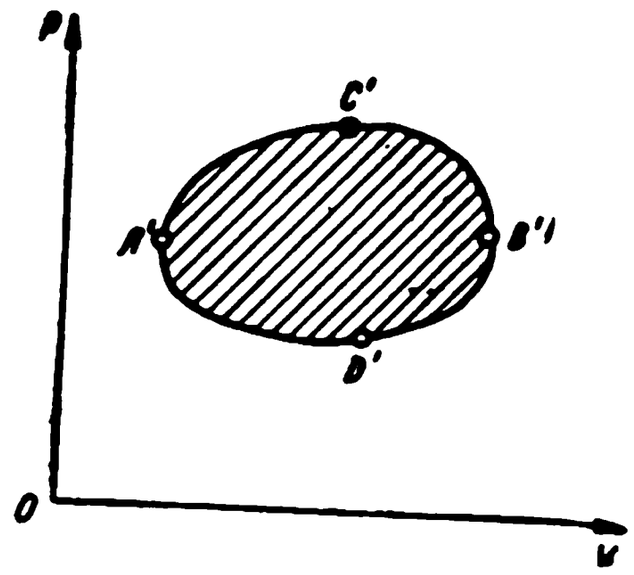

Fig. 4. Graphic representation of the work of a cyclic process (a cycle).

between the curve of the process and the volume axis OV_1 (Fig. 3). The useful external work L', which is equal to $\int_{p'_1}^{p'_2} Vdp$, is the area $A'B'p_2p_1$, contained between the process curve and the pressure axis Op.

If the body goes through a cyclic process (a cycle), represented in the pV plane by a closed curve A'C'B'D'A' (Fig. 4), then the work L performed by the body per cycle will be equal to the area bounded by the process line (the area of the cycle). If the cyclic process is carried out in a clockwise direction the work will be positive, while if the opposite direction is used it will be negative. The useful external work L' performed by the body during one cycle is also given by the area of the cycle. Thus, L' = L.

Heat of the Process. Specific Heat

Along with performance of work, any thermodynamic process is accompanied by the supply (or removal) of heat to the body from the surrounding medium. The supply of heat to the body leads to a change in

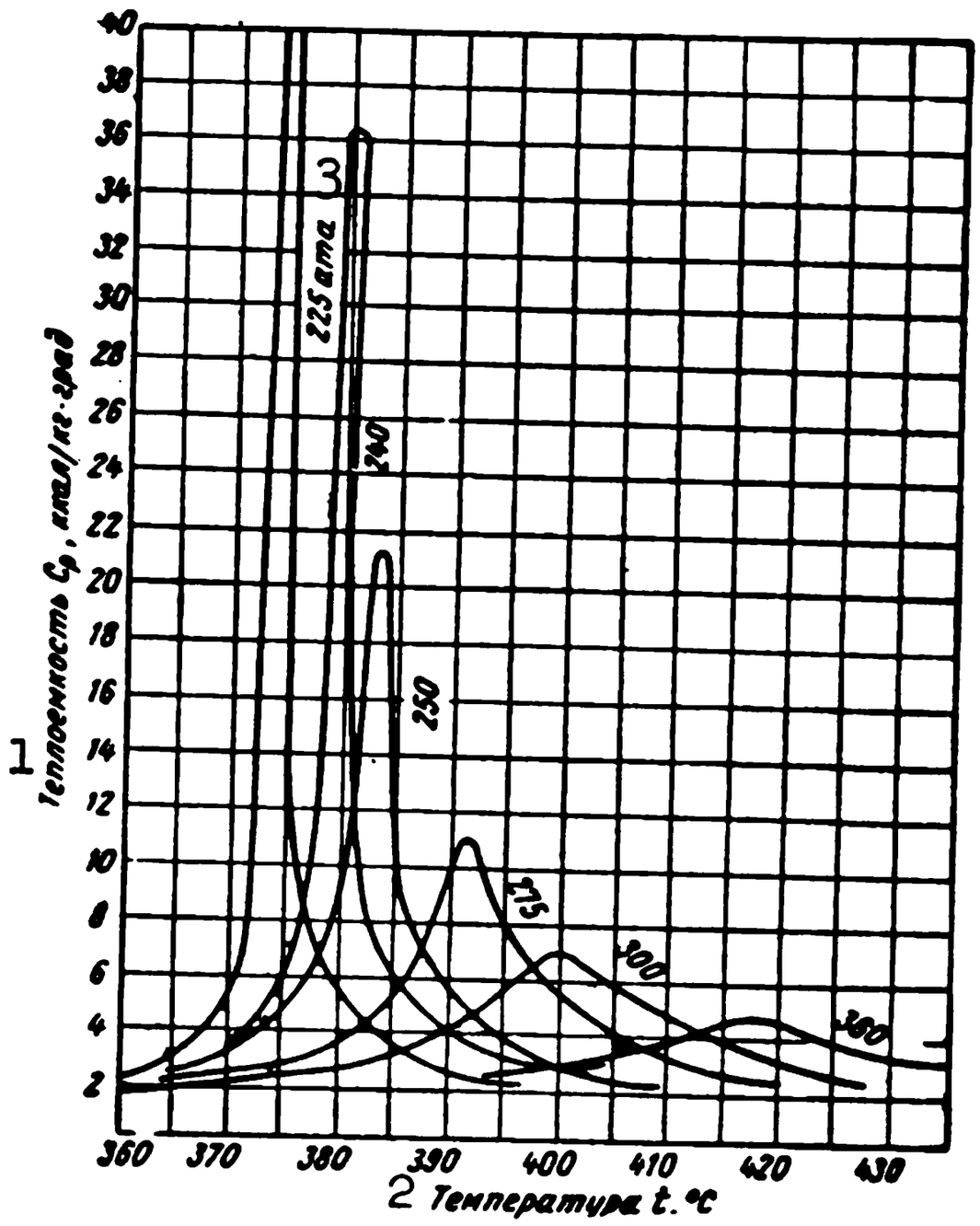

Fig. 5. Character of the dependence of C_p on the temperature and on the pressure for water vapor. 1) Specific heat C_p, kcal/kg·degree; 2) temperature $\underline{t}$, °C; 3) atm abs.

its temperature, the change being dependent on the external conditions in which the change of the state of the body takes place, i.e., on the character of the process. The ratio of the amount of heat dQ obtained by the body in the case of an infinitesimally small change in its state to the resultant temperature change dt of the body is called the <u>specific heat of the body</u> C_X in the given process

$$C_x = \left(\frac{\partial Q}{\partial t}\right)_x. \qquad (1.17)$$

The specific heat is numerically equal to the amount of heat which must be imparted to the body in order to change its temperature by 1° under the given conditions. The specific heat C_X depends on the conditions under which the process of heating the body occurs. This dependence is indicated in Eq. (1.17) by the subscript X of the partial derivative $\partial Q/\partial t$, which denotes that the change in the temperature of the system occurs in a completely defined and equilibrium manner: at a constant value of the parameter X. The specific heat is usually measured in kilocalories per degree centigrade; sometimes the specific heat is expressed in units of work per degree centigrade.

Inasmuch as $dQ = C_X dt$ in accordance with Eq. (1.17), the total amount of heat Q obtained by the body in a given process of variation of the state of the body is determined by the expression

$$Q = \int_{t_1}^{t_2} C_X dt, \qquad (1.18)$$

where the integral is taken over the path of the process (along the curve whose equation is X = const) from the initial state 1 to the specified final state 2. It follows therefore that the amount of heat Q is a function of the process and not of the state of the body.

Of particular significance in thermodynamics are the specific heat at constant volume

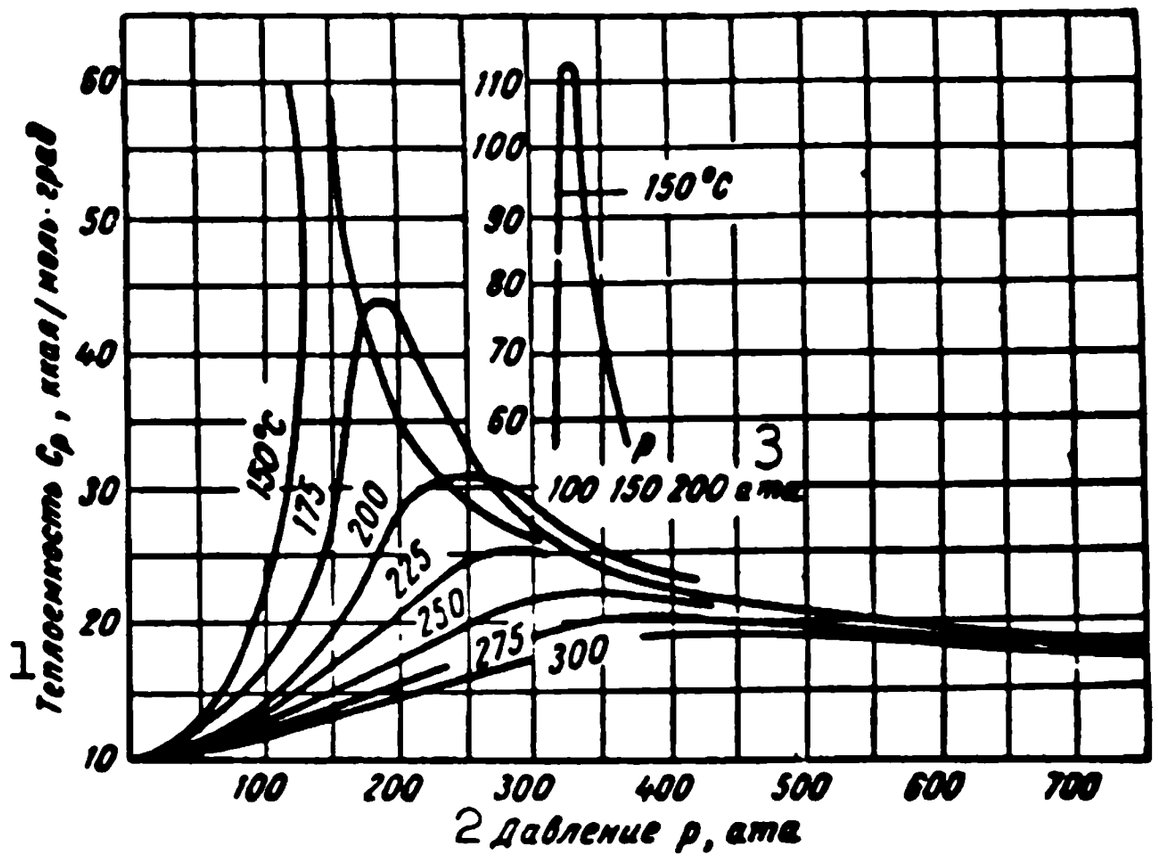

Fig. 6. Character of the dependence of C_p on the temperature and on the pressure for ammonia. 1) Specific heat C_p, kcal/mole·degree; 2) pressure p, atm abs; 3) atm abs.

$$C_v = \left(\frac{\partial Q}{\partial t}\right)_v \qquad (1.19)$$

and the specific heat at constant pressure

$$C_p = \left(\frac{\partial Q}{\partial t}\right)_p. \qquad (1.20)$$

Equations (1.17)-(1.20) determine the value of the specific heat at given values of the parameters of the state t, V and t, p. These values of C are called the true specific heats.

The true specific heats are functions of the thermal parameters of the body. For a homogeneous body, the true specific heats are functions of any two of three parameters. The character of the dependence of C_p on t and on p is shown in Fig. 5 for water vapor and in Fig. 6 for ammonia.

In addition to the true specific heats, one considers also in technical thermodynamics the so-called average specific heats C_x^{sr}. The average specific heat of a given process in the temperature interval from t_1 to t_2 is determined by the following equation:

$$C_X^{cp}\Big|_{t_1}^{t_2} = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} C_X dt. \qquad (1.21)$$

§6. REVERSIBLE AND IRREVERSIBLE PROCESSES

Reversible processes refers to processes which can occur in both the forward and backward direction and furthermore in such a way that in the reverse process (i.e., on returning to the initial state), the system passes through the same state as in the direct process, but in the reverse sequence without the occurrence of any residual final changes in the system itself or in the surrounding medium.

Processes which do not satisfy this condition are called irreversible. In such a process the system cannot return to the initial state, either along the same path that it covered prior to reaching the final state or along some other circuit, without additional external action (without forcing).

In a reversible process, the system can return to the initial state without additional external action, while an isolated system can return there without any external action whatever. A reversible system, in particular, is characterized by the fact that the work performed during the course of this process is perfectly sufficient in order to return the system to the initial state under the same external conditions.

It follows therefore that a measure of the irreversibility of the process may be the value of the additional external action (for example, work) which must be performed in order to return the body to the initial state or the magnitude of the final changes which remain in the external medium after the body returns to the initial state.

If the body is not in equilibrium, then any process involving the change of the state of the body will be irreversible. An example of this is a nonequilibrium process involving the change of the volume of the

body, in which the pressure of the body and the pressure of the surrounding medium are always different, inasmuch as the work performed as a result of the process will be insufficient to return the body to the initial state. These processes include the expansion of bodies in vacuum, expansion or contraction during friction, etc. The irreversibility of the first of these processes is due to the fact that when the body expands in vacuum the work is equal to zero, while to compress the body to the initial state positive work must be performed. Likewise irreversible is any process in which there is no thermal equilibrium (the temperatures of the bodies are different), inasmuch as the process of heat transfer at a finite temperature difference cannot be reversed since the backward transition of the heat from the body with the lower temperature to the body with the higher temperature is impossible without the performance of external work.

Consequently, <u>any nonequilibrium process involving the change of the state of the body is always irreversible</u>. Irreversible processes differ in the degree of irreversibility; in some it can be larger and in others smaller.

In some irreversible processes the state of the body at each point of the process differs relatively little from equilibrium, and is characterized by definite values of the thermodynamic parameters, in analogy with the situation prevailing in equilibrium processes. Such processes are called <u>externally irreversible</u>.

An example of an externally irreversible process is the change in the state of the body during heat exchange with a body having a high temperature. In spite of the irreversible character of the heat exchange (owing to the finite potential difference), the states of the bodies that exchange heat differ little from equilibrium and can be regarded at each point of the process as equilibrium states. Thus, one

can assume with a great degree of approximation that externally irreversible processes are internally in equilibrium and can be represented graphically.

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[Footnotes]

5 With accuracy to 0.02%.

8 It is assumed here that there is no external force field acting on the body at all, for otherwise the body could not have the same density everywhere.

8 · This is obvious for when G = const the values of $\underline{v}$ and V or V_μ differ only by a constant factor.

15 The criterion of thermodynamic similarity of different substances is that these substances must belong to one and the same class of chemical compound and that their critical coefficients, i.e., the ratios RT_k/p_kv_k, must be equal.

16 It is assumed here that the system is not insulated from the surrounding medium.

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[List of Transliterated Symbols]

11 пот = pot = potentisial'nyy = potential

11 мин = min = minimal'nyy = minimum

11 мол = mol = molekulyarnyy = molecular

12 к = k = kriticheskiy = critical

25 ср = sr = sredniy = average

Chapter 2

FIRST LAW OF THERMODYNAMICS

§1. NATURE OF FIRST LAW OF THERMODYNAMICS. LAW OF CONSERVATION AND CONVERSION OF ENERGY

The first law of thermodynamics is an application of the law of conservation and conversion of energy, which is the most general universal law of nature, to thermal phenomena.

There are many qualitatively different forms of energy, for example the kinetic energy connected with the motion of a body as a whole, electric energy connected with the motion of electric charges, the internal energy connected with molecular motion, etc. As a result of interaction between bodies, energy of a given type can be converted or transformed into any other type of energy and this conversion proceeds always in such a way that the total amount of energy does not change, i.e., <u>energy cannot be destroyed or created</u> (the law of conservation and conversion of energy). It follows therefore that if a thermodynamic system has performed as a result of a process 1-2 work L' and has obtained at the same time from the surrounding bodies (external medium) an amount of heat Q, then the change of energy of the given system $E_2 - E_1$ amounts to

$$E_2 - E_1 = -L' + Q. \qquad (2.1)$$

On the basis of the law of conservation and conversion of energy it is possible to establish exact quantitative relationships between individual forms of energy. Indeed, if different forms of energy are taken in amounts such that each of them causes separately one and the

same change in the state of the given system, then these amounts of energy of different forms will be equivalent by virtue of their mutual convertibility. Thus, in particular, it is possible to determine by experiment the mechanical equivalent (the amount of mechanical work) of a given amount of energy of some other form.

Thus, if an amount of heat Q is converted into work, then

$$L = (1/A)Q, \qquad (2.2)$$

where A = 426.45 kgm/kcal is the mechanical equivalent of a unit of heat; 1/A is the thermal equivalent of the unit of work.

The concept of energy is inseparably connected with the motion of matter: energy is a physical measure of the motion of matter. The difference between different types of energy is due to the qualitative difference in the specific forms of motion of the material bodies. Mutual conversion of energy of bodies reflects the unbounded possibility of motion of becoming transformed from one form to another. Consequently, conservation of energy is evidence that the motion of the material world cannot be destroyed.

F. Engels called the law of conservation and conversion of energy a great fundamental law of motion, an absolute law of nature, while V.I. Lenin, emphasizing the principal philosophical significance of the law of conservation and conversion of energy, indicated that this law is a statement of the main premises of materialism. This is precisely why the law of conservation and conversion of energy has been subjected from the instant of its discovery to our days to vicious attack on the part of reactionary idealistic philosophy.

The law of conservation and conversion of energy is a further development and concretization of the universal law of conservation of matter and motion of M.V. Lomonosov. The justification and development of this law have occupied Hess (1840), Joule (1840), Mayer (1842), and

Helmholz (1847).

Modern science has broadened our concepts of energy and established an inseparable link between energy and mass of a body. At the present time there is an established law, according to which mass and energy are interrelated: to each change in the mass of the body there corresponds a fully defined change in its energy, numerically equal to the product of the change in mass by the square of the velocity of light* c^2 and, conversely, any change in the energy of the body is connected with a change in its mass.

§2. CONVERSION OF HEAT INTO WORK

Mutual Conversions of Heat and Work in Cyclic Processes

Let us consider a cyclic process wherein the state of a thermodynamic system is changed in that it goes over from an initial state 1 to a certain state 2 along the path 1a2 (Fig. 7), and from state 2 to the initial state 1 along the path 2b1; each of these transitions can be either reversible or irreversible.

From the first law of thermodynamics it follows that in a cyclic process the difference between the amount of heat $Q = \oint dQ$ obtained by the system from the surrounding medium and the work $L = \oint dL$ performed by it should be equal to zero**:

$$Q - AL = 0.$$

If the difference $Q - AL$ were not equal to zero and were, for example, less than zero, this would mean that as a result of the cyclic process the system upon returning to the initial state has performed more work than the heat absorbed. If now a certain part of this work, numerically equal to Q, were to be converted again into heat and given up to the surrounding medium, then the latter would by the same token return to its initial state. Consequently, the system and the surrounding medium would not have any residual changes after the end of the

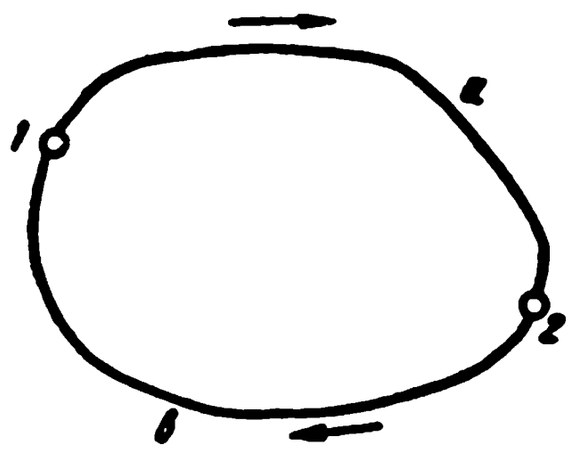

Fig. 7. Cyclic process of variation in the state of bodies.

cyclic process and would be in the initial state, but in spite of this a certain amount of positive work would be obtained.

By letting the system perform such a cyclic process many times, we could obtain any amount of positive work without consuming any amount of heat whatever, i.e., from nothing, which is in contradiction with the law of conservation of energy. Therefore the previously made assumption that in a cyclic process the difference between the absorbed heat Q and the work L performed by the system is smaller than zero is in error. The difference Q – AL can likewise not be larger than zero. Thus, it follows from the first law of thermodynamics that

$$\oint dQ - A\oint dL = 0. \qquad (2.3)$$

This means that in any cyclic process the difference between the amount of heat absorbed by the system and the work performed by the system is equal to zero or (what is the same) the work performed by the system as a result of the cyclic process is equal to the amount of heat which it received from the outside.

It follows therefore that to produce a perpetual motion engine of the first kind, which would perform useful work without any change in the state of the surrounding bodies, is impossible. Indeed, if the state of the working medium of the engine is the same at the start and at the end of the cycle, then in order to restore the initial state of the surrounding bodies it is necessary to impart to them an amount of heat which is exactly equal to the work performed by the engine.

It is seen from Eq. (2.3) that when $Q > 0$ we also have $L > 0$, i.e., if the body acquires heat during the course of the cycle, it must perform work. In this case the heat is converted into work con-

sisting in the fact that the body, after obtaining a certain amount of energy in the form of heat from the surrounding medium, gives up the same amount of energy in the form of work. If $Q < 0$ then also $L < 0$, i.e., in order for the body to give up heat to the surrounding medium work must be performed on the body. In this case the work is converted into heat.

In an unclosed process the heat obtained by the body from the surrounding medium is not completely converted into work. Some part of the heat obtained by the body goes to increase the energy stored in the body – its internal energy.

It follows further from Eq. (2.3) that for any change in the state of the system from a certain initial state 1 to a final state 2 the quantity $Q_{1-2} - AL_{1-2}$ has one and the same value and does not depend on the character of the process 1-2 or on the path of the transition from state 1 into state 2:

$$Q_{1-2} - AL_{1-2} = \text{idem.} \qquad (2.4)$$

The fundamental equation (2.1), and also Eqs. (2.3) and (2.4), are valid for both reversible and irreversible processes (they have the same general significance).

§3. INTERNAL ENERGY

Total and Internal Energy of a Body

Any thermodynamic system, including any of the real bodies, will have in any state a certain energy E, which represents the sum of the kinetic energy of motion of the system as a whole E_{kin}, the potential energy E_{pot} due to a definite position of the entire system or its macroscopic parts in some external force field (for example, in the force of gravity, in an electric or magnetic field, etc.), and a so-called internal energy U, which is not connected with the motion of the entire system as a whole or with the presence of an external force

field and consists of the proper energy of the individual parts comprising the system under consideration

$$E = E_{kin} + E_{pot} + U.$$

When the system is unique (there is no surrounding medium) the total energy of the system coincides with its internal energy $E = U$.

Change in Internal Energy upon Change in the State of the Body

In various changes of state, a thermodynamic system obtains from the surrounding medium or, conversely, gives up to the medium energy in the form of heat and mechanical work, as a result of which the internal energy of the system U changes. The change in internal energy of the system for an infinitesimally small process dU at $E = U$ consists, in accordance with Eq. (2.1), of the amount of heat dQ obtained by the system from the external medium and the work dL performed by the system and taken with the opposite sign:

$$dU = dQ - AdL. \qquad (2.5)$$

As shown in §2 of Chapter 2, for any cyclic process the integral $\oint(dQ - AdL)$, and consequently the change in the internal energy of the system, is equal to zero:

$$\oint dU = 0.$$

It follows hence that the increment in the internal energy of the system dU becomes a complete differential and the internal energy U becomes a function of the state of the system.

The change in the internal energy of the system $U_2 - U_1$ when the system goes from state 1 into state 2 amounts to, in accord with Eq. (2.5),

$$U_2 - U_1 = Q_{1-2} - AL_{1-2}. \qquad (2.6)$$

In an isochoric process ($V = \text{const}$) the work consumed in the change of volume $L = \oint p'dV$ is equal to zero. Consequently, according to Eq. (2.5), if there is no work connected with the change in volume,

then

$$Q_V = U_2 - U_1.$$

Consequently, the amount of heat obtained by the body in an isochoric process is numerically equal to the difference in the values of the internal energy at the end and at the beginning of the process.

Properties of Internal Energy

Internal energy is the energy reserve of the body. In each of the states it has a fully defined value, i.e., it is a function of the state. From the molecular point of view the internal energy of any body, which represents an aggregate of a large number of particles (molecules, atoms, ions, etc.), is the energy of all the particles comprising the body and is equal to the sum of their kinetic and potential energies.

Unlike the internal energy U, the amount of heat Q and the mechanical work L are not functions of the state but functions of the processes occurring in the system. Their magnitude depends on the path along which the transition is made from the initial state to the given state. Therefore, for example, it is meaningless to speak of the amount of heat possessed by a body in a given state, inasmuch as the amount of heat can have any value depending on how the body reached the given state. Mathematically this is expressed by the fact that infinitesimally small amounts of heat dQ and of mechanical work dL are not total differentials. To the contrary, the difference dQ – AdL is a total differential, equal to the differential of the internal energy dU.

Since the amount of heat and the mechanical work are not functions of the state, the internal energy of the system cannot be subdivided into thermal and mechanical components. Only when the state of the system changes can the change in the energy of the system be subdivided into the work performed by the system and the amount of heat

obtained by it. Such a subdivision is not determined uniquely by the initial and final states of the system, but depends on the character of the process which occurs in the system.

Heat and work are forms of energy transfer; they are inseparably connected with the process of the variation of the state and represent functions of the process that occurs in the system.

In order to calculate the absolute magnitude of the internal energy it is necessary to integrate Eq. (2.5). As a result of the integration, the expression for U will contain a certain integration constant U^o:

$$U = \int(dQ - AdL) + U^o, \qquad (2.7)$$

i.e., the internal energy is determined accurate to a constant component. Since in thermodynamics we are interested principally not in the absolute value of the internal energy but in the changes resulting in it from the processes that occur in the system, the value of the integration constant U^o is of no importance in the solution of many thermodynamic problems.

It is demonstrated in kinetic theory of matter that U^o is the so-called zero-point energy — the energy of the internal atomic motions at absolute zero temperature.

For a complicated system consisting of several parts (several bodies), the internal energy is equal to the sum of the internal energies of the individual parts, provided we can neglect the energies of interaction of these parts with each other (the latter is connected with the presence of boundaries or with the presence of separation surfaces between the bodies and represents their surface energy)

$$U = \Sigma U_j. \qquad (2.8)$$

Inasmuch as the internal energy is additive, for a homogeneous body it will be equal to the product of the number of moles $M = G/\mu$ by the

molar internal energy U_μ or the product of the weight of the body G by the specific internal energy $\underline{u}$

$$U = (G/\mu)U_\mu = Gu.$$

The equilibrium state of a homogeneous body is determined by two independent thermal parameters. Consequently, in the equilibrium state the internal energy of a homogeneous body will be a function of any two thermal parameters $\underline{p}$, $\underline{t}$, or V.

Internal Energy of an Ideal and Van der Waals Gas

The internal energy of an ideal gas is simplest to express. In this case the potential energy of interaction between the molecules in the formula for U is zero, i.e., $(\partial U/\partial V)_t = 0$. Consequently, U is a function of the temperature only.

At low temperatures, when the energy of the oscillatory motion of the atoms and molecules can be disregarded as being too small and the energy of each of the rotational motions of the molecule can be assumed equal to kT/2 (on the basis of the equipartition law) the internal energy of 1 kg-mole of an ideal gas will have the following form:

$$U_\mu = \frac{3+\delta_{vr}}{2} A\mu RT + U_\mu^\circ, \qquad (2.9)$$

where δ_{vr} is the number of rotational degrees of freedom of a molecule of a given gas.

The internal energy of a Van der Waals gas differs from the internal energy of an ideal gas by the value of the potential energy of the interaction of the molecule

$$U_\mu = \frac{3+\delta_{vr}}{2} A\mu RT - \frac{a\mu^2}{V_\mu} + U_\mu^\circ, \qquad (2.10)$$

where $\underline{a}$ is the constant of the Van der Waals equation.

§4. ENTHALPY

For a closed process L = L', and since $\oint(dQ - AdL) = 0$, we have also $\oint(dQ - AdL') = 0$. We have verified earlier that dQ − AdL is a

total differential, and consequently $dQ - AdL'$ is also a total differential of some function of state, which we denote by I:

$$dI = dQ - AdL'. \qquad (2.11)$$

In the case of a reversible process $dQ = dU + ApdV$ and $L' = -\int Vdp$. Hence

$$dI = dU + Ad(pV)$$

or

$$I = U + ApV. \qquad (2.12)$$

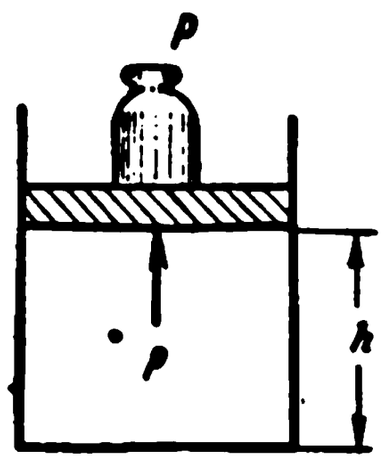

Fig. 8. Gas under pressure in a cylinder under a piston.

The quantity I is called <u>enthalpy</u>, <u>the heat function of the body</u>, or the <u>heat content</u>. The latter name is less correct, since, as was already explained in §3 of Chapter 2, it is meaningless to speak of the "heat content" in some body.

In order to introduce a body with volume V into an external medium that has everywhere an equal pressure p' it is necessary to perform work to displace an equal volume of the medium; this work is equal to p'V. It is transmitted to the surrounding medium and comprises the potential energy of the latter; the higher the pressure of the medium the larger its value. Therefore, if the body is in an external medium with pressure p', then any state of the body is connected with a certain energy E* equal to U + Ap'V. This is easily illustrated by means of the following example. Assume that we have a cylinder with a certain amount of gas under its piston (Fig. 8). In order to balance the gas pressure p it is necessary to load the piston with a weight $P = p\Omega$ (Ω is the area of the piston). The total energy E* of the system under consideration will be equal to the sum of the internal energy of the gas U and the potential energy of the weight $Ph = pV$, i.e., $E^* = I$.

This makes clear the physical meaning of enthalpy: the enthalpy

of the body is the energy connected with the given state of the body.

With respect to the expanded system (body plus surrounding medium), enthalpy plays the same role as internal energy does for an isolated body. The change in enthalpy in processes that occur at constant pressure is equal to the amount of heat Q obtained by the system:

$$Q_p = I_2 - I_1.$$

Indeed, according to the first law of thermodynamics

$$dQ = dU + ApdV.$$

But according to Eq. (2.14)

$$dU + ApdV = dI - AVdp,$$

therefore

$$dQ = dI - AVdp, \qquad (2.13)$$

from which it follows that $dQ = dI$ when p is constant.

If the system is thermally insulated, $dQ = 0$ (there is no heat exchange between the system and the surrounding medium), then dI is, equal to zero in processes that occur in such a system at constant pressure [see Eq. (2.13)]. This means that the enthalpy of a thermally insulated system remains constant for an isobaric process.

Inasmuch as the internal energy U is determined from thermodynamic relations accurate to a certain constant, the same constant will be contained, in accordance with (2.12), also in the expression for the enthalpy I. In thermodynamics this constant is chosen arbitrarily. Thus, it is assumed for example that water at a temperature of $0^{\circ}C$ at its saturated vapor pressure has $I = 0$. Frequently the enthalpy of a gas in the ideal state (as $p \to 0$) is also assumed equal to zero at $0^{\circ}C$.

Like the internal energy, the enthalpy is an additive quantity. The additivity of the enthalpy follows from the additivity of the internal energy U and of the volume V and follows directly from (2.12). For a homogeneous body, the enthalpy can be represented in the form of

the product of the molar enthalpy I_μ by the number of moles $M = G/\mu$ contained in the given body or in the form of the product of the specific enthalpy $\underline{i}$ by the weight of the body G in kilograms

$$I = \frac{G}{\mu} I_\mu = Gi,$$

where I_μ and $\underline{i}$ are functions of any two of the three parameters $\underline{p}$, $\underline{t}$, and V. Since $Q = I_2 - I_1$ when $\underline{p}$ is constant and $Q = U_2 - U_1$ when V is constant, obviously the enthalpy in isobaric processes plays the same role as the internal energy U in isochoric processes.

It is particularly convenient to use enthalpy in those cases when the main variables (independent parameters) determining the state of the system are chosen to be not the volume and the temperature, but the pressure and the temperature.

Enthalpy of an ideal gas. The analytic expression for the enthalpy of any body can be determined if one knows the internal energy and the equation of state of this body. Thus, for example, if we use the expression given in §3 of Chapter 2 for the internal energy of an ideal gas and recognize that $pV_\mu = \mu RT$, we obtain that the molar enthalpy of an ideal gas at not too high temperatures, when the vibration energy of the atoms in molecules can be neglected, will have the following form

$$I_\mu = \frac{5 + \delta_{вр}}{2} \mu RT + U^0_\mu. \qquad (2.14)$$

§5. ANALYTIC EXPRESSION FOR THE FIRST LAW OF THERMODYNAMICS

According to the first law of thermodynamics, the amount of heat dQ obtained by the system is equal to the sum of the increments of the internal energy of the system dU and of the work done in changing the volume dL or the sum of the increments of the enthalpy dI and the useful external work dL' performed by the system

$$dQ = dU + AdL,$$

$$dQ = dI + AdL'.$$

These equations are valid for both reversible and irreversible processes and represent an analytic expression of the first law of thermodynamics for an infinitesimally small process.

If the work performed by the body is connected only with the change in the volume of the body, so that $dL = p'dV$ and $dL' = -Vdp'$, where p' is the external pressure (the pressure of the surrounding medium), then (2.5) and (2.11) can be rewritten in the form

$$dQ = dU + Ap'dV, \qquad (2.15)$$

$$dQ = dI - AVdp'. \qquad (2.16)$$

If the system performs along with the work done in changing the volume also work not connected with the change in volume, the right halves of the equations for dQ will contain the additional term dL_V:

$$dQ = dU + Ap'dV + AdL_V, \qquad (2.17)$$

$$dQ = dI - AVdp' + AdL_V. \qquad (2.18)$$

§6. SPECIFIC HEAT

Analytic Expression for the Specific Heats C_V and C_p

If the analytic expressions for the internal energy and for the enthalpy of the body are known as functions of the parameters of state, then it is possible to determine with the aid of the first law of thermodynamics the values of the specific heats of a body at constant volume $C_V = (\partial Q/\partial t)_V$ and at constant pressure $C_p = (\partial Q/\partial t)_p$ along with their dependence on the parameters of the state.

For this purpose we consider an equilibrium process wherein the body is heated and we choose as the independent parameters characterizing the state of the system the volume V and the temperature t. Since the number of independent parameters is chosen equal to two, all the deductions that follow pertain in particular to a uniform body.

The elementary amount of heat dQ obtained by the system in the

case of an infinitesimally small equilibrium process assumes, in accordance with Eq. (2.17) with $dL_V = 0$, the following form

$$dQ = \left(\frac{\partial U}{\partial t}\right)_V^{dt} + \left[Ap + \left(\frac{\partial U}{\partial V}\right)_t\right] dV.$$

From the expression obtained it follows that at constant volume ($dV = 0$) we have

$$dQ = \left(\frac{\partial U}{\partial t}\right)_V^{dt}.$$

and consequently, the specific heat at constant volume is

$$C_V = \left(\frac{\partial U}{\partial t}\right)_V. \qquad (2.19)$$

Thus, the specific heat C_V of a body of constant volume is equal to the partial derivative of the internal energy (regarded as a function of t and V) with respect to the temperature t at V = const and is a function of t and V.

The specific heat C_p at constant pressure has a somewhat more complicated analytic expression

$$C_p = \left(\frac{\partial U}{\partial t}\right)_V + \left[Ap + \left(\frac{\partial V}{\partial t}\right)_t\right] \left(\frac{\partial V}{\partial t}\right)_p$$

or

$$C_p = C_V + \left[Ap + \left(\frac{\partial U}{\partial V}\right)_t\right] \left(\frac{\partial V}{\partial t}\right)_p. \qquad (2.20)$$

A simpler expression for the specific heat C_p can be obtained by choosing as the independent parameters not the volume V and the temperature t, but the pressure p and the temperature t. Then we get from (2.16)

$$dQ = \left(\frac{\partial I}{\partial t}\right)_p dt - \left[AV - \left(\frac{\partial I}{\partial p}\right)_t\right] dp,$$

from which it follows that when p = const we get

$$dQ = \left(\frac{\partial I}{\partial t}\right)_p dt$$

and accordingly

$$C_p = \left(\frac{\partial I}{\partial t}\right)_p. \qquad (2.21)$$

The specific heat C_p of the body at constant pressure is equal to the partial derivative of the enthalpy with respect to the temperature t at p = const and is the function of p and t.

It will become clear from what follows that the specific heats C_V and C_p of all bodies are positive, with $C_p \geq C_V$.

Specific Heat of the Body in a Process X = const

The specific heat C_X of a body in a process characterized by the constancy of some parameter X is by definition equal to $(\partial Q/\partial t)_X$ and can be represented in the form

$$\left.\begin{aligned} C_x &= C_V + \left[Ap + \left(\frac{\partial U}{\partial V}\right)_t\right]\left(\frac{\partial V}{\partial t}\right)_x, \\ C_x &= C_p - \left[AV + \left(\frac{\partial I}{\partial p}\right)_t\right]\left(\frac{\partial p}{\partial t}\right)_x. \end{aligned}\right\} \qquad (2.22)$$

The specific heat of a body in any process wherein $(\partial V/\partial t)_X$ is not equal to zero is "the specific heat while the body performs work of expansion."

Depending on the character of the process, the specific heat C_X can be either positive or negative. Since C_V and C_p are positive, C_X will be negative only if $(\partial V/\partial t)_X < 0$, i.e., if the increase in the temperature of the body is accompanied by a decrease in its volume, and consequently by the performance of work on the body. In order for the quantity C_X to be negative, it is necessary that the work involved in changing the absolute magnitude of the body exceed the change in the internal energy of the body upon heating. Then it is possible to heat the body not only without supplying any heat from the outside but, to the contrary, some of the work performed on the body is in itself released in the form of heat.

In order to calculate the specific heat of the body C_X for an arbitrary process, it is necessary to know not only the equation of the

process itself, but also the specific heat C_V or C_p of the given body and its equation of state.

It is seen from (2.22) that the specific heat C_X of any system consisting of several independent parts is equal to the sum of the specific heats of the individual parts, owing to the additivity of U and V.

From this it follows in particular that the specific heat of a uniform body is proportional to the number of moles M contained in the given body and to its weight G. Consequently, along with the specific heat of the entire body it is possible to introduce the molar and specific heats, which represent the specific heats of one kilogram-mole or one kilogram of the body, respectively. We shall henceforth denote the molar specific heats by $C_{X\mu}$, and the specific ones by c_X:

$$C_x = \frac{G}{\mu} C_{x\mu} = G c_x.$$

In addition to the specific heat c_X per unit weight, one sometimes introduces also the so-called specific heat per unit volume c'_X, which is the specific heat of the amount of matter which occupies under normal conditions (760 mm Hg and $0^\circ C$) a volume equal to 1 m^3.

The specific heats per unit volume, per mole, and per unit weight are connected by the following obvious relation:

$$c'_x = \frac{C_{x\mu}}{V_{\mu норм}} = \frac{c_x}{v_{норм}}.$$

Line of a Process with Absorption and Release of Heat

Assume that we have a uniform body, the state of which is represented by a point 1 on the pV plane (Fig. 9). We draw through the point 1 a line representing a reversible adiabatic process — the adiabat ab. When the state of the body lies along the adiabat ab, the amount of heat obtained by the body is equal to zero ($Q_{ab} = 0$). If the state of the body changes in reversible fashion from the point 1 to the point lying under the reversible adiabat (for example to the point

2), then the body gives up an amount of heat $Q_{1-2} < 0$. If the state changes to the point 3, which lies above the adiabat, then the body absorbs heat $Q_{1-3} > 0$ (under the condition that $(\partial V/\partial t)_p > 0$).

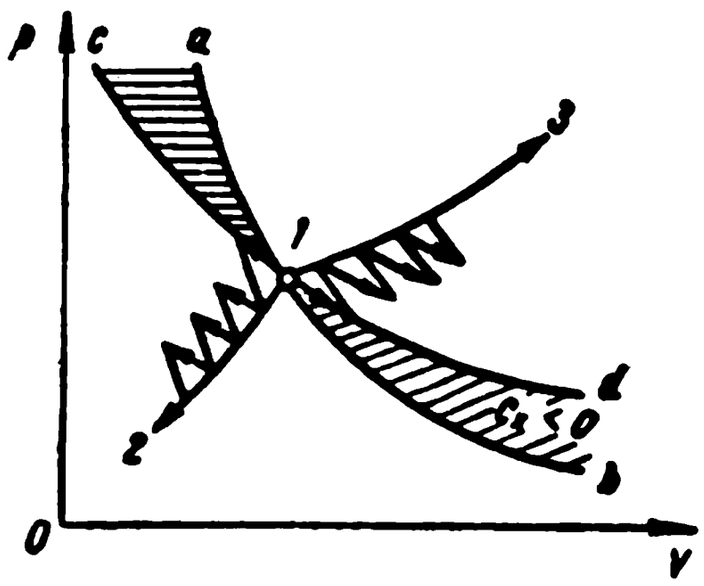

Fig. 9. Location of the lines of a process with absorption and release of heat on the pV plane.

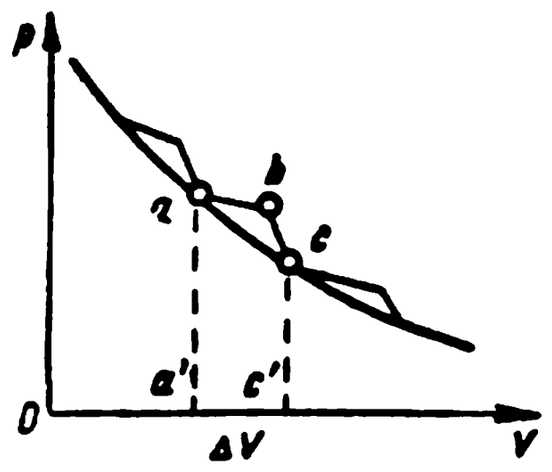

Fig. 10. Subdivision of an arbitrary process into an aggregate of many small isothermal and adiabatic processes.

In order to verify this, let us consider first an isothermal process cld. In the case of an infinitesimally small isothermal process, the internal energy of the body changes by an amount $(\partial U/\partial V)_t\, dV$ and the amount of heat dQ obtained by the body is equal to $[Ap + (\partial U/\partial V)_t]dV$. It will become clear from what follows that when $(\partial V/\partial t)_p > 0$ the sum $Ap + (\partial U/\partial V)_t$ is positive and when $(\partial V/\partial t)_p < 0$ it is negative. Therefore if $(\partial V/\partial t)_p > 0$ then in the case of isothermal expansion the body absorbs heat $Q_{1-d} > 0$ and in the case of isothermal contraction it gives up heat $Q_{1-c} < 0$.

On the other hand, any arbitrary process can be regarded as an aggregate of an infinitely large number of infinitesimally small isothermal and adiabatic processes (or portions) alternating with one another (Fig. 10).

In fact, in the case of the elementary process ac the body performs work dL_{ac}, numerically equal to the area acc'a', and obtains an amount of heat $dQ_{ac} = dU_{ac} + AdL_{ac}$, where $dU_{ac} = U_c - U_a$. In the case of adiabatic-isothermal transition from a to c (path abc), the work performed is dL_{abc} and is numerically equal to the area abcc'a, which is larger than the work of the process ac by the size of the area abca, namely $dL_{abc} = dL_{ac}$ + area abca, and the heat absorbed is $dQ_{abc} =$

$= dU_{ac} + dL_{ac}$ + area abca $= dQ_{ac}$ + area abca.

When the section ac is infinitesimally small, the area abca is an infinitesimally small quantity of second order of smallness compared with the area acc'a', from which it follows, accurate to infinitesimally small quantities of second order, that $dL_{ac} = dL_{abc}$, $dQ_{ac} = dQ_{abc}$, i.e., any reversible infinitesimally small process can be represented in the form of an aggregate of infinitesimally small reversible isothermal and adiabatic processes.

Let us assume that in the body under consideration we have in the vicinity of the point 1 $(\partial V/\partial t)_p > 0$. Then any process which is represented by an arbitrary line 1-2 (see Fig. 9) drawn below the adiabat ab contains when expanded into elementary isothermal and adiabatic processes isothermal portions corresponding to the compression of the body only. In each of these portions $dQ < 0$ and therefore for the entire process as a whole $Q_{1-2} < 0$. To the contrary, on all the isothermal portions of the line 1-3, which is above the adiabat $dQ > 0$, the body expands, so that for this process as a whole $Q_{1-3} > 0$.

Thus, a reversible process in which the state of the body changes is accompanied by absorption of heat if the line of the process is located to the right of the reversible adiabat drawn to the point 1, which represents the initial state of the body,* and by release of heat if the line of the process is located to the left of the adiabat.

The temperature of the body along the line of the process can either increase or decrease. When $(\partial V/\partial t)_p > 0$, the higher the isotherm, the higher its temperature. Along the line 1-3, which passes to the right of the isotherm cd, the temperature increases, $t_3 > t_1$, and along the line 1-2, which passes below the isotherm cd, the temperature decreases. When the state of the body changes along the line 1-2 we have $dQ < 0$ and $dt < 0$, so that the specific heat of the process

1-2 will have a positive sign; the same sign of the specific heat will apply on the line 1-3, where $dQ > 0$ and $dt > 0$ (when $(\partial V/\partial t)_p < 0$ all the foregoing conclusions are reversed).

Along any of the lines passing between the adiabat and the isotherm (this region is shaded in Fig. 9), dQ and dt have opposite signs and the specific heat will be negative.

Specific Heat of an Ideal Gas

The simplest form of the dependence of the specific heats on the parameters of the state is possessed by an ideal gas. In this case $(\partial U/\partial V)_t = 0$ and consequently the specific heats C_V and C_p are the functions of the temperature only.

Differentiating Eq. (2.9) with respect to T, we obtain the following value for the molar specific heat of an ideal gas at constant volume:

$$C_{V\mu} = \frac{3 + \delta_{ap}}{2} A\mu R. \qquad (2.23)$$

Substituting into (2.22) the values of the derivatives $(\partial U/\partial V)_t = 0$ and $(\partial V/\partial t)_p = \mu R/p$, we obtain

$$C_{p\mu} - C_{V\mu} = A\mu R,$$

or

$$C_{p\mu} - C_{V\mu} = A\mu R. \qquad (2.24)$$

Thus, the difference between the molar specific heats $C_{p\mu}$ and $C_{V\mu}$ of an ideal gas is equal to the universal gas constant $A\mu R$, i.e., 848/426.4 = 1.986 kcal/degree.

Using the previously given value of $C_{V\mu}$, we obtain

$$C_{p\mu} = \frac{5 + \delta_{ap}}{2} A\mu R. \qquad (2.25)$$

Equations (2.23) and (2.25) are approximate and valid only in that region of temperatures where the vibration degrees of freedom of the molecular motion are not yet excited and the rotation degrees of

freedom are fully realized. At high temperatures it is necessary to take into account the energy of the vibrations of the atoms in the molecule, so that the expressions for C_V and C_p of an ideal gas will contain the vibrational component of the specific heat C_{kol}, which is an increasing function of the temperature.

It follows from the formulas for C_V and C_p that the specific heats of polyatomic ideal gases depend only on the temperature, but not on the pressure of the gas. The specific heat of a monatomic ideal gas, in which $\delta_{vr} = 0$ and $\delta_{kol} = 0$, does not depend on the temperature and is a constant quantity.

Using the expression for C_V and C_p, we obtain that the ratio of the specific heats C_p/C_V, which we henceforth denote by κ, will have the form

$$\kappa = \frac{5 + \delta_{sp}}{3 + \delta_{sp}}.$$

For monatomic gases, for which $\delta_{vr} = 0$, we have $\kappa = 5/3$; for diatomic gases, for which $\delta_{vr} = 2$, we have $\kappa = 7/5$; for polyatomic gases, where $\delta_{vr} = 3$, $\kappa = 4/3$ (except for gases with many molecules, for which $\delta_{vr} = 2$). In diatomic and polyatomic ideal gases the numerical value of the ratio of the specific heats C_p and C_V decreases somewhat (usually insignificantly) with increasing temperature.

§7. APPLICATION OF THE FIRST LAW OF THERMODYNAMICS TO THE FLOW OF GAS OR LIQUID

Adiabatic Flow

Let us assume that we have a flow of moving gas (or liquid), with respect to which it is known that it is stationary, i.e., the pressure, temperature, and flow velocity at each point of the stream do not vary with time. An example of such a flow is steady flow of gas or liquid in a pipe.

We assume first that the stationary flow under consideration is

adiabatic, i.e., thermally insulated. The flow with constant velocity can be either reversible or irreversible. When the gas goes from one section to another, its parameters change from values p_1, t_1, v_1 to values p_2, t_2, v_2, respectively, and the flow velocity changes from w_1 to w_2.

The increment in kinetic energy per kilogram of gas $(w_2^2/2g) - (w_1^2/2g)$ is the available specific useful work l' which can be performed by the moving gas on the external work object. Inasmuch as from (2.20) we have $Adl' = dq - di$, we get for adiabatic flow

$$A\left(\frac{w_2^2}{2g} - \frac{w_1^2}{2g}\right) = i_1 - i_2. \qquad (2.26)$$

or

$$i_2 + A\frac{w_2^2}{2g} = i_1 + A\frac{w_1^2}{2g}.$$

Consequently, <u>for a thermally insulated stationary flow of gas (or liquid) the sum of the specific enthalpy i and the specific kinetic energy $w^2/2g$ remains constant</u>.

Equation (2.26) is valid for gases and liquids without viscosity, as well as for viscous gases and liquids, the flow of which is accompanied by the action of friction forces. Thus, this equation is valid for both reversible and irreversible thermally insulated flows.

<u>Flow with Performance of External Technical Work and with Supply of Heat</u>

If the gas (or liquid) performs useful work L'_m on an external object of work during the course of its flow, for example rotates a rotor of some engine (this work is also called <u>technical work</u>), then Eq. (2.26), which expresses the law of conservation of energy, will have in this case the following form (provided the potential energy of the flowing gas or liquid is equal to zero)

$$A\left(\frac{w_2^2}{2g} - \frac{w_1^2}{2g}\right) + Al'_m = i_1 - i_2.$$

or

$$i_2 + A\frac{w_2^2}{2g} = i_1 + A\frac{w_1^2}{2g} - Al'_m. \qquad (2.27)$$

Here $l'_m = L'_m/G$ is the external technical work per kilogram of gas along the path 1-2.

If an amount of heat Q is delivered to the flowing gas (or liquid) along the path 1-2, then on the basis of the general equation (2.20) we have

$$A\left(\frac{w_2^2}{2g} - \frac{w_1^2}{2g}\right) + Al'_m = q + i_1 - i_2,$$

or

$$i_2 + A\frac{w_2^2}{2g} = i_1 + A\frac{w_1^2}{2g} + q - Al'_m, \qquad (2.28)$$

where q = Q/G is the amount of heat supplied to 1 kg of the flowing body along the path 1-2. By the amount of heat q in Eq. (2.28) is meant both the heat obtained by the flowing gas (or liquid) from the external medium by heat exchange with the latter and the heat released in the stream by internal heat sources, for example as a result of combustion of part of the gas. Thus, q is the general or summary amount of heat obtained by 1 kg of flowing gas along the path 1-2. The heat of friction q_{tr} is not contained in the quantity q. Indeed, to overcome the friction forces it is necessary to consume work l_{tr}. Since the work against the friction forces goes over completely into heat, the amount of heat q_{tr} released per kilogram of flowing gas is equal to l_{tr}. In order to take into account the influence of the friction on the flow of the gas (or liquid), it would be necessary to substitute in the right half of (2.28) the values of l_{tr} and q_{tr}. As a result of the equivalence of these two quantities, they cancel each other and thus they drop out of the equation. This explains why Eq. (2.28), and also Eq. (2.26), are valid both for reversible flows, i.e.,

not accompanied by friction forces, and for flows which are accompanied by friction and have one and the same form for both cases.

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[Footnotes]

31 We recall that the mass of the body is equal to the mass of the body at rest, m_0, divided by $\sqrt{1 - (w^2/c^2)}$, where w is the velocity of motion of the body.

31 In technical thermodynamics Q is usually measured in kilocalories and L in kilogram meters; this explains the appearance of the factor A in front of L. According to the standard GOST 8550-57, all thermal quantities must be expressed in absolute kilojoules; in this case there will be no factor A in front of Q.

46 This pertains in particular to an irreversible process with absorption of heat. Indeed, for the cyclic process abca (see Fig. 10), regardless of whether the elementary process ac is reversible or not, we have $dQ_{abc} - dQ_{ac} = dL_{abca}$, since dL^{obr}_{abca} = area abca, and $dL^{neobr}_{abca} < dL^{obr}_{abca}$, we get $dQ_{abc} - dQ^{neobr}_{ac} <$ area abca, i.e., dQ^{neobr}_{ac} and dQ_{abc} are equivalent. Consequently, any process accompanied by the absorption of heat moves above the reversible adiabat drawn through the initial point.

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[List of Transliterated Symbols]

33 кин = kin = kineticheskiy = kinetic

33 пот = pot = potentsial'nyy = potential

37 вр = vr = vrashchatel'nyy = rotational

44 норм = norm = normal'nyy = normal

48 кол = kol = kolebatel'nyy = vibrational

50 тр = tr = treniye = friction

51 обр = obr = obratimyy = reversible

51 необр = neobr = neobratimyy = irreversible

Chapter 3

SECOND LAW OF THERMODYNAMICS

§1. ESSENCE OF THE SECOND LAW OF THERMODYNAMICS

The second law of thermodynamics determines the direction in which macroscopic processes proceed. It is expressed by means of a set of premises which generalize experiments and which pertain, first, to the states of equilibrium of the thermodynamic systems and, second, to processes occurring in these systems.

First Formulation of the Second Law of Thermodynamics

Starting from the first law of thermodynamics alone, it is possible to assume that any conceivable process which does not contradict the law of energy conservation can occur in nature. Thus, for example, one could assume that in the case of heat exchange between two bodies of different temperature the heat can flow both from the body with the higher temperature to the body with the lower temperature and conversely from the body with the lower temperature to the body with the higher temperature. The only limitation imposed on this process by the first law of thermodynamics is the requirement that the amount of heat given up by the first body and received by the second (in the absence of useful external work) be equal.

The question of the direction in which heat actually flows in the example considered above of heat exchange between two bodies and also in other more complicated cases is answered by the second law of thermodynamics, according to which heat can flow spontaneously only from a body with a higher temperature toward a body with a lower tem-

perature; uncompensated transfer of heat from a body with a lower temperature to a body with a higher temperature is impossible.

From this statement, which is one of the formulations of the second law of thermodynamics, it follows that it is impossible by any means to cause heat to move from a less heated body to a more heated one such that all the other bodies participating in the process return after the end of the process to their initial state, i.e., without the appearance of some residual "compensation" changes in the surrounding bodies (for example, without the consumption of work or without the realization of some other equivalent process). To the contrary, heat can proceed spontaneously from a hotter body to a colder one, i.e., even if other bodies participate during this process, these bodies can return to their initial states after the end of the process. This means that the process of heat exchange at a finite temperature difference is a strictly unilateral irreversible process.

Second Formulation of the Second Law of Thermodynamics

The second law of thermodynamics leads to the following important conclusion concerning the properties of heat engines.

It is impossible to realize a continuously moving engine such that in each of its cycles positive work is done only by cooling one body, without simultaneous occurrence of some other changes in the system or in the surrounding bodies.

This general conclusion can be regarded as a new formulation of the second law of thermodynamics.

It is easy to verify that the first and second formulations of the second law do not contradict each other. Indeed, if it were possible to obtain positive work by cooling only one single source of heat, then, by converting the work obtained into heat at a temperature higher than the source temperature, we would transport the heat to a

higher temperature without any residual changes in the state of the bodies participating in the process, something we already know to be impossible.

From the second formulation of the second law of thermodynamics for thermally uniform systems there follow three important consequences, the first two of which pertain to isothermal processes and the third to adiabatic ones.

1. <u>In a cyclic isothermal process performed by a system, the work $\oint_{t=\text{const}} dL$ can be only consumed (if the process is irreversible), but not created; in the limiting case of a reversible process the work is equal to zero.</u>

To verify the correctness of this statement, let us consider a cyclic isothermal process by varying the states of some thermally uniform system in which a constant temperature is maintained by one single source of heat. Since the work of the system in the cyclic process is equal to the amount of heat obtained by the system, then if this work is positive it is accomplished at the expense of cooling the existing single source of heat, which in accordance with the second law of thermodynamics is impossible. Consequently, this work is either equal to zero or is negative. In a reversible process the work must equal zero, for were it to be negative, then upon reversal of the process (something which is always possible in view of the reversibility of the process) the work would become positive, which contradicts the second law of thermodynamics. For irreversible processes the work has a negative sign, i.e., it is consumed.

It follows therefore that the operating cycle of a heat engine in the presence of a single source is always performed in a counterclockwise direction (Fig. 11).

2. <u>In a reversible cyclic isothermal process the total amount of</u>

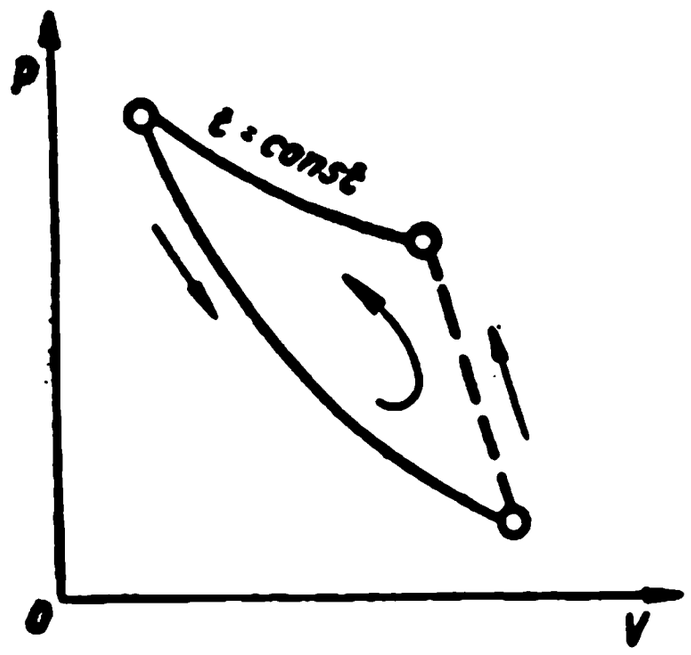

Fig. 11. Cycle of heat engine in the presence of a single source of heat.

heat $\oint_{t=\text{const}} dQ'$ obtained by the system is equal to zero; in an irreversible process

$$\oint_{t=\text{const}} dQ < 0.$$

Indeed, as was shown earlier, the work $\oint_{t=\text{const}} dL$ performed by the system in a reversible cyclic isothermal process is equal to zero. But according to Formula (2.6) for any cyclic process we have $\oint dQ = A\oint aL$.

Therefore if $\oint_{t=\text{const}} dL = 0$, we also have $\oint_{t=\text{const}} dQ = 0$.

If a cyclic isothermal process is irreversible, then $\oint_{t=\text{const}} dL < 0$

and accordingly $\oint_{t=\text{const}} dQ < 0$.

3. There exist those states of a thermally uniform system which cannot be reached by adiabatic transition.

Indeed, were an adiabatic transition from a given state 1 into any other state possible, including a transition to the state 2 with the same temperature but with a different volume, in which the system can be transferred reversibly from the initial state 1 isothermally upon removal of a certain amount of heat, then by performing a cyclic process consisting of the adiabatic transition 1-2 and the reversible isothermal transition 2-1 with supply of heat $-Q_{1-2}$ we would obtain a work L, consisting of the work $U_1 - U_2$ of the adiabatic process and the work $-Q_{1-2} - (U_1 - U_2)$ of the isothermal process

$$L = -Q_{1-2} - (U_1 - U_2) + (U_1 - U_2) = -Q_{1-2},$$

which would be positive, for by assumption $-Q_{1-2} > 0$. Consequently, as a result of this cyclic process positive work would be performed in the presence of only one source of heat, which is impossible in accord-

ance with the second law of thermodynamics; consequently, an adiabatic transition of a system from a given state into any other state is impossible.

Perpetual Motion of the Second Kind

In spite of the principal difference between the first and second laws of thermodynamics, they can be formulated in some common fashion.

The first law of thermodynamics states that it is impossible to create work from nothing, i.e., it is impossible to create a perpetual-motion engine which would perform useful work without an external source of energy.

The second law of thermodynamics states that it is impossible to obtain continuously work from bodies that are in thermal equilibrium, in spite of the fact that these bodies have a definite reserve of internal energy.

An engine that could perform work by cooling bodies of equal temperature must also be regarded as a perpetual-motion engine (more accurately, almost perpetual), since the energy reserves in the bodies surrounding us, which are in a state close to thermal equilibrium, are practically inexhaustible. Therefore such an engine is called a perpetual-motion engine of the second kind. Consequently, the second law of thermodynamics can be formulated also to read: a perpetual-motion engine of the second kind is not realizable.

A comparison of perpetual motions of the first and second kind makes even clearer the principal differences between the two laws of thermodynamics. The formulation of the first law of thermodynamics can be reversed, i.e., we can state that work can be neither produced from nothing nor destroyed without trace. The formulation of the second law of thermodynamics in the form of a statement of the impossibility of producing a perpetual-motion engine of the second kind does not admit

of such a reversal. In other words, complete reversal of the processes involved in the conversion of work into heat is impossible. In order to explain this, let us consider a process in which all the work performed by the external forces goes over into heat, which is then given up to a certain body. The device in which work would be continuously converted into heat is in its action the direct opposite of the perpetual motion of the second kind and therefore can always be realized. Such a device, for example, is the Joule instrument for the determination of the mechanical equivalent of heat. In this instrument a falling weight rotates a stirrer, which is situated inside a reservoir with liquid, as a result of which the energy of the falling weight is imparted to the liquid in the form of heat of friction and causes it to be heated. To reverse this process, i.e., to lift the weight to the initial height by suitable cooling of the liquid and furthermore in such a way that no changes are produced in the surrounding bodies is impossible, for this would denote the realization of a perpetual-motion engine of the second kind.

In the example considered here the mechanical work was converted into the heat by the forces of friction. Therefore the result obtained can be formulated in still another manner: <u>it is impossible to reverse completely a process wherein heat is produced by friction</u>.

<u>Heat Engine</u>

In order to perform work continuously, it is necessary to have at least two bodies of different temperature – two sources of heat. However, a temperature difference alone is not sufficient to realize the conversion of heat into work. For example, if two bodies of different temperature are simply brought in contact with each other, then the heat will flow from the hotter body to the colder one without performing any useful external work. In order to realize a heat engine which

performs work continuously, it is necessary to produce some cyclic process between the bodies of different temperature (and furthermore in a clockwise direction). For this purpose we need still another body. The auxiliary body, which performs during the time of operation of the heat engine a multiply repeating cyclic process consisting, in the case of two heat sources, of alternating isothermal and adiabatic processes, is called the working medium. In one of the parts of the cyclic process the working medium expands, thereby performing useful positive work L_{rassh} due to the heat obtained from the hotter body, which is therefore called the heat source, and partially due to its internal energy. In the second portion of the cycle the working medium returns to its initial state, thereby giving rise to the continuity of engine operation. To return the working medium to the initial state it is necessary to perform on it compression work L_{szh}, part of which is given up by the working medium to the colder body or the heat sink in the form of heat, while the other part goes to restore the initial internal energy of the working medium. Work in this last part of the cyclic process is performed at the expense of some fraction of the positive work obtained during the first part of the cycle. The difference between the work of expansion L_{rassh} and the work of compression L_{szh} of the working medium constitutes the positive useful work L of the engine during one cycle transferred to the external object of work by means of a special mechanism

$$L' = L_{rassh} - L_{szh}.$$

In order for the useful work per cycle to be as large as possible, the working medium is preferably chosen to be (at least during the expansion stage) a gas or vapor having the ability of greatly increasing in volume and consequently the ability of performing the maximum work of expansion under the given conditions.

According to the first law of thermodynamics [see Eq. (2.3)] the useful external work of the heat engine per cycle, independently of whether the processes in the engine are reversible or not, will be equal to the difference between the amount of heat Q_1 given up to the working medium by the hot body and the amount of heat Q_2 obtained from the working medium by the colder body

$$AL = Q_1 - Q_2.$$

From an examination of the working process of a heat engine we see that the heat given up by the hotter body is not completely converted into work; some fraction of the work is transferred to the colder body via the working medium. The transition of the heat from the hotter bodies to the colder ones as a result of the action of a heat engine and the deviations in the states of the bodies from their initial values resulting from this transition are precisely those changes in the bodies that participate in the process which in accordance with the second law of thermodynamics unavoidably accompany any cyclic process, whether it be reversible or not, wherein heat is converted into work. The equality of Q_2 to zero would denote that work is performed only at the expense of cooling a single source of heat, namely the hotter body. Thus, the conversion of heat into work is always accompanied by a compensating transition of a certain amount of heat from the hotter body to the colder one.

In the scheme of a continuously acting heat engine considered above, the working medium repeats periodically one and the same cyclic process. In the cycles of real engines, the working medium is usually replenished periodically, i.e., is replaced by an equal amount of "fresh" substance which is in the same state. From the thermodynamic point of view the replacement of a working substance can be regarded as the return of the substance spent in the engine to its initial

state. Therefore a cycle in which the working medium is replenished does not differ in principle at all from a cycle using the same working medium and in the analysis of the various heat engines the replenishment of the working substance can be disregarded.

Significance of the Second Law of Thermodynamics

The significance of the second law of thermodynamics to the theory of heat engines lies in the fact that it determines the degree of useful utilization of the heat in the engine. Inasmuch as only part of the heat supplied can be converted into work in heat engines, the useful action, and consequently also the economy of the engine is characterized by the ratio of the amount of heat converted into useful work to the entire heat supplied. This ratio is called the effective efficiency of the engine. Its limiting value is established on the basis of the second law of thermodynamics.

Maximum work is performed by the engine in the case of a reversible process; in real irreversible processes the work performed always has a smaller value. From the second law of thermodynamics it follows, as will be shown below, that there exists a special criterion of irreversibility, with the aid of which it is possible to analyze the operation of each engine quantitatively and by the same token obtain ways of realizing a working process in the most effective and economical manner.

With the aid of the second law of thermodynamics it is possible, starting from the known physical properties of the substance, to predict its other properties and to establish quantitative relationships between them. In this lies in principle the great significance of the second law of thermodynamics for the investigation of the physical properties of real bodies.

The second law of thermodynamics leads further to the concept of

thermodynamic temperature and gives a most general and universal method of determining and comparing temperatures of different bodies.

The second law of thermodynamics is based on numerous facts and represents a generalization of experimental data; the physical content of this law, as well as its limits of applicability, become understandable after an examination of the general laws of thermal motion of molecules. The presence of a special thermodynamic principle which determines the laws of conversion of heat into work in heat engines was pointed out by S. Carnot as long ago as 1824.

The second law of thermodynamics was formulated analytically by R. Clausius in 1850 (first formulation) and by W. Thomson (Kelvin) in 1851 (second formulation). The formulation of the second law of thermodynamics in the form of the principle of existence of adiabatically unattainable states belongs to the Russian scientist N.N. Shiller (1900) and to K. Caratheodry (1909). A critical analysis of various formulations of the second law of thermodynamics was realized by T.A. Afanas'yeva-Erenfest (1928).

§2. ENTROPY

Expression for Elementary Quantity of Heat

In §1 of Chapter 3 it was shown that in a cyclic irreversible isothermal process the integral $\oint_{t=\text{const}} dQ$ taken along the line of the process (along the contour of the cycle abcda) vanishes. It follows from this that the quantity dQ can be represented in the form of a product of a temperature function, which we denote by $\psi(t)$ and a differential of some function of the state of the body. It can be shown that $\psi(t)$ is the same for all bodies and, since for an ideal gas it is equal to the absolute temperature, then $\psi(t) = T$ for all other bodies, too.

[Missing text in Russian original] we denote the state function, the differential of which enters into the expression for dQ, by S.

Then

$$dQ = TdS. \qquad (3.1)$$

The state function S of the body is called the entropy.

It follows from (3.1) that the reciprocal of the absolute temperature T is an integrating factor for the elementary amount of heat dQ. The ratio dQ/T is called the elementary reduced heat.

After integrating both halves of (3.1) along some path of transition from the state 1 to the state 2, and assuming that the transition is reversible, we obtain the following expression for the difference of the entropies in states 2 and 1:

$$S_2 - S_1 = \int_1^2 \frac{dQ}{T}. \qquad (3.2)$$

Inasmuch as the entropy is a function of the state, the difference $S_2 - S_1$ will have one and the same value regardless of how the transition from state 1 into state 2 was effected. In a reversible isothermal process Eq. (3.2) for the entropy difference assumes the very simple form

$$S_2 - S_1 = \frac{Q_{1\to 2}}{T}, \qquad (3.3)$$

i.e., the increment in entropy in a reversible isothermal process is equal to the reduced heat.

In a reversible adiabatic process, when dQ = 0 we have

$$S_2 - S_1 = 0. \qquad (3.4)$$

This means that a reversible adiabatic process does not lead to a change in entropy and is therefore also called isentropic.

Absolute Thermodynamic Temperature Scale

The fundamental equation (3.1) indicates that it is possible to

realize a principally new method of measuring the absolute temperature of bodies (except for measuring it with a gas thermometer which contains a highly rarefied gas, i.e., not different from an ideal gas).

In order to determine the ratio of the absolute temperatures of any two bodies it is sufficient to realize between them a reversible cycle with some arbitrary working medium and to measure the quantities of heat Q_1 and Q_2 received and given up by the working medium. The quantity Q_1/Q_2 will give the sought ratio of the absolute temperatures of the two bodies T_1/T_2. Indeed, this cycle will consist of an isothermal process corresponding to the temperature T_1, an adiabatic transition from the temperature T_1 to T_2, an isothermal process at temperature T_2, and a reverse adiabatic transition to the temperature T_1. Inasmuch as in a reversible adiabatic process the entropy remains unchanged, the change in entropy in both isothermal processes will be of the same absolute magnitude and equal to ΔS and $-\Delta S$. Accordingly, the amounts of heat Q_1 and Q_2 will be $T_1\Delta S$ and $T_2\Delta S$, from which it follows that

$$Q_1/Q_2 = T_1/T_2. \qquad (3.5)$$

The measurement results will not depend on the properties of the working medium employed, so that a temperature scale constructed on this basis is <u>universal</u>.

Let us visualize now a cycle between a source of heat with finite temperature T_1 and a heat sink of such low temperature that the amount of heat Q_2 is equal to zero (with $Q_1 \neq 0$). According to (3.5), the absolute temperature of this heat sink should also be equal to zero.

Since the useful work of the cycle AL is equal to $Q_1 - Q_2$, then when $Q_2 = 0$ we have $AL = Q_1$, i.e., were the absolute temperature of the heat sink to be equal to zero, all the heat supplied Q_1 would be completely converted into work. Inasmuch as AL cannot be larger than

Q_1, <u>the zero of absolute temperature represents the lowest of all the possible temperatures and is therefore called the absolute zero of temperature</u>.

<u>Change of Entropy in Irreversible Processes</u>

Let us consider first an irreversible isothermal process. We assume that this is the process 1-2 in the cycle abcda (Fig. 12), in which all other processes are reversible, so that the irreversibility of the cycle abcda is due exclusively to the irreversibility of the section 1-2.

In an irreversible isothermal cycle $\oint\limits_{t=\text{const}} dQ < 0$. Inasmuch as $t = \text{const}$, we also have $\oint\limits_{t=\text{const}} dQ/T < 0$. Therefore for the cycle under consideration we get

$$\int\limits_{1-2} \frac{dQ}{T} + \int\limits_{2bcda1} \frac{dQ}{T} < 0.$$

Since the process 2bcda1 is reversible, the second integral, according to (3.2), is equal to the difference in the entropies $S_1 - S_2$ at the points 1 and 2. Consequently,

$$\int\limits_{1-2} \frac{dQ}{T} + S_1 - S_2 < 0$$

or, denoting the first integral by $\int\limits_1^2 dQ/T$, we get

$$S_2 - S_1 > \int\limits_1^2 \frac{dQ}{T}.$$

The greater the process 1-2 differs from a reversible one, the greater the difference, other conditions being equal, between the values of the increment of the entropy $S_2 - S_1$ and the integral $\int\limits_1^2 dQ/T$.

From the equation obtained for $S_2 - S_1$ it follows that the change in the entropy of a body for any infinitesimally small isothermal process

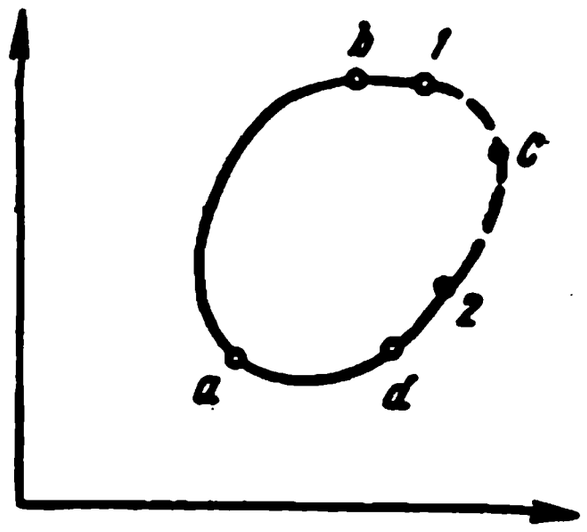

Fig. 12. Cycle with irreversible isothermal process.

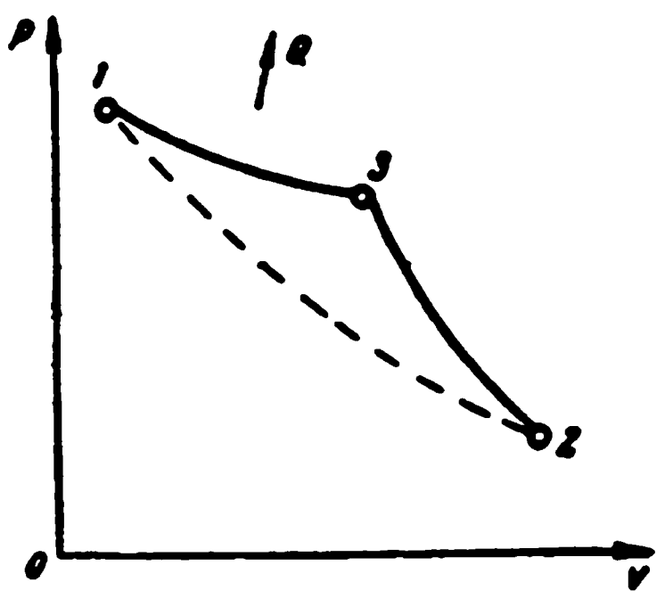

Fig. 13. Cycle with irreversible adiabatic process.

satisfies the relation

$$dS \geq dQ/T, \qquad (3.6)$$

where dQ is the amount of heat obtained by the body from the heat source in an infinitesimally small change of the state of the body; T is the absolute temperature of the heat source; the equal sign pertains to reversible and the "greater" sign pertains to irreversible processes.

Relation (3.6) is valid not only for an isothermal but for any process. In order to verify this, let us consider first an irreversible cyclic process with one source of heat of temperature T_1. This process consists of a reversible isothermal part 3-1, a reversible adiabatic part 2-3, and an irreversible adiabatic part 1-2 (Fig. 13). According to the second law of thermodynamics, in any cycle with one source of heat the useful work and consequently also the amount of heat supplied can be only negative. Therefore, for the cycle 1-2-3-1 under consideration we have $\int_3^1 dQ < 0$ and since $S_3 = S_2$ and $S_1 - S_2 = \int_3^1 dQ/T_1$, we get $S_1 < S_2$, i.e., the entropy of the body at the end of an irreversible adiabatic process is larger than at its start. Consequently, $dS \geq 0$; in the case of a reversible adiabatic process the entropy remains unchanged, while in an irreversible one it increases.

Any infinitesimally small process can be regarded as an aggregate of infinitesimally small isothermal and adiabatic processes. Since for

each of these processes $dS \geq dQ/T$, this relation remains in force also for an arbitrary infinitesimally small process. Therefore Eq. (3.6) holds true for any infinitesimally small process. This means that for any process the change in entropy dS is equal to the ratio dQ/T if the process is reversible, or is larger than this ratio if the process is irreversible.

Increase of entropy in adiabatic expansion of a body in vacuum.

The adiabatic expansion of a body in vacuum is a typical irreversible process and must therefore be accompanied by an increase in entropy. Before we proceed to determine the increment of the entropy in this case, let us recall that the entropy is a function of state and therefore the change in entropy for any transition of the body from one state to another will have one and the same value, equal to the difference between the entropies in these states. It follows therefore that if we know the final state attainable as a result of a real process, then the entropy change due to it can be determined from an examination of some conceivable reversible transition from the initial to the final state. This method of determining the change in entropy in actual irreversible processes is of most general significance.

Let us assume that the expanding body is a gas contained in one part of a thermally insulated vessel with rigid walls and having a temperature t_1 and volume V_1; the second part of the vessel, which is separated from the first by a freely opening (without friction) partition, is evacuated (Fig. 14). If we open the partition, gas will start flowing from one part of the vessel to the other. After a certain time the motion stops and the gas enters into an equilibrium state characterized by a volume V_2, equal to the volume of the entire vessel, and a temperature t_2, different from the initial temperature of the gas t_1. Since the volume of the vessel remains unchanged, owing to the rigidity

of its walls, the work of expansion of the gas, L, is equal to zero and therefore in accordance with (2.6) $U_2 = U_1$, i.e., the internal energy does not change when a gas expands in vacuum.

If many experiments are performed on the expansion of gas in vacuum and the lines U = const are plotted on the tV diagram for different initial values of U (for this purpose it is sufficient to take the investigated body for different initial parameters t_1 and V_1), then it is possible to determine the character of the dependence of the internal energy of the body on the volume and on the temperature. The fact that the temperature of a rarefied gas remains constant upon expansion in vacuum had enabled Joule to establish that the internal energy of a greatly rarefied gas does not depend on the volume occupied by the gas and that consequently the internal energy of an ideal gas is a function of the temperature only.

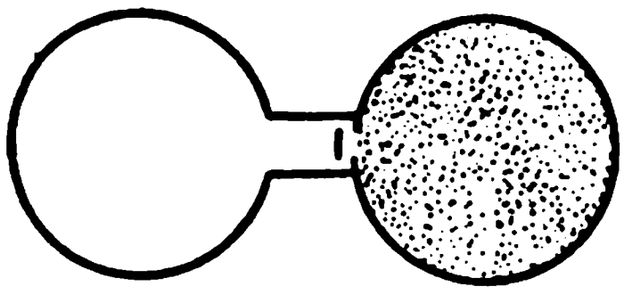

Fig. 14. Diagram of Joule's experiment.

We have verified above that the final state of the body after adiabatic expansion in vacuum is characterized by a volume $V_2 > V_1$ and an internal energy U_2 equal to U_1. It is possible to bring the body to such a state also by means of reversible expansion with constant value U, for which purpose it is necessary to supply to the body an amount of heat

$$dQ = ApdV.$$

It follows therefore that the entropy increment $S_2 - S_1$ in an adiabatic expansion of a body in vacuum amounts to

$$S_2 - S_1 = A \int_{V_1}^{V_2} \frac{p}{T} dV,$$

where the integral is taken for U = const. Since dV is positive in the case of expansion of a body, we get $S_2 - S_1 > 0$.

In the adiabatic expansion of an ideal gas in vacuum we have

$$S_2 - S_1 = AGR \ln \frac{V_2}{V_1}.$$

Increase of entropy in heat exchange. The change of entropy of two bodies in a direct transition of heat from the first more heated body to a second less heated body can be determined in the following manner. For simplicity we assume that both bodies have such high specific heat that the amount of heat Q either given up by them or received by them does not cause a noticeable change in their temperatures. Then the change in the entropy of the first hotter body as a result of giving up an amount of heat Q will be the same as in a reversible isothermal process with temperature equal to the temperature of the first body and amounts to

$$S_2^{(1)} - S_1^{(1)} = -\frac{Q}{T_1}.$$

from which it follows that the entropy of the hotter body decreases during the course of heat exchange.

Bearing in mind that the amount of heat obtained by the second body is equal to the amount of heat Q given up by the first body, we determine in analogous manner the change in entropy of the second colder body:

$$S_2^{(2)} - S_1^{(2)} = \frac{Q}{T_2}.$$

The entropy of the colder body increases.

The change in the entropy of the two bodies, which make up an isolated system, will be equal to the sum of the changes of the entropies of both bodies:

$$S_2 - S_1 = Q\left(\frac{1}{T_2} - \frac{1}{T_1}\right).$$

It follows therefore that the entropy of a system increases in the case of heat exchange with a finite temperature difference.

Increase of entropy in processes with friction. By way of an exam-

ple let us calculate the increment of the entropy of a liquid in the Joule instrument. In this instrument, as is well known, all the external work performed goes over into heat of friction, causing heating of the liquid from a temperature t_1 to t_2. Since this process occurs without a change in the volume of the liquid, the final state can be attained by means of a reversible isochoric heating of the liquid and consequently the increment of the entropy for C_V = const amounts to

$$S_2 - S_1 = G\int_{T_1}^{T_2} \frac{c_V}{T}\,dT = Gc_V \ln \frac{T_2}{T_1}.$$

The temperature difference $t_2 - t_1$ is connected with the external work L' by the equation

$$AL' = Gc_V (t_2 - t_1).$$

It is obvious that the heat of friction Q_{tr}, if released in the body itself and not dissipated on the outside, is completely equivalent to an equal amount of heat obtained by the body from an external source of heat and thus contributes to an increment in the entropy of the body. We shall make use of this result later on, in the analysis of throttling and gas flow.

Increase of entropy in adiabatic mixing (diffusion). Let us consider for simplicity the mixing of two gases which at first have equal pressures $p = p_1$ and temperatures $t = t_1$. Let the first of these gases be in the left half of a vessel and occupy a volume $V_1^{(1)}$ and the second in the right half of the vessel and occupy a volume $V_1^{(2)}$, with

$$V_1^{(1)} + V_1^{(2)} = V,$$

where V is the total volume of the vessel (assumed constant). After mixing, which by assumption occurs without any heat being delivered or removed and without performance of work L_V on the external object, each of the gases extends over the entire volume V and will have a tem-

perature t_2 and pressures $p_2^{(1)}$ and $p_2^{(2)}$, respectively. Since Q = 0 and L = 0, the sum of the internal energies of both gases, equal to Q − AL, remains unchanged

$$U_2^{(1)} + U_2^{(2)} = U_1^{(1)} + U_1^{(2)}.$$

In order to determine the change in the entropy resulting from the mixing, let us imagine the following reversible process, which brings both gases to the same final state. We assume that both gases are separated from each other by two semipermeable partitions $P^{(1)}$ and $P^{(2)}$ (Fig. 15), of which the first is permeable to the first gas but not to the second, and the second is permeable to the second gas and not to the first. Acting on the first partition is the pressure of the second gas $p^{(2)}$, while the second is acted on by the pressure of the first gas $p^{(1)}$. By shifting sufficiently slowly the first partition to the left and the second to the right we can effect a reversible mixing of the two gases, and in order for the final state to be the same as in the irreversible mixing, it is necessary to supply heat to the gases. The amount of heat supplied

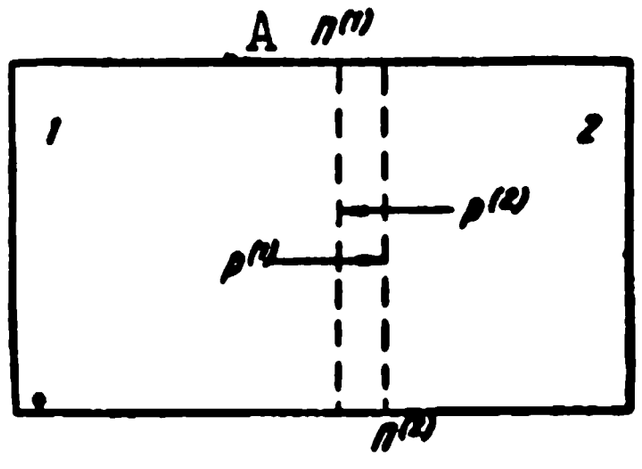

Fig. 15. Diagram of reversible mixing of two gases. A) P.

$$Q^{(1)} = U^{(1)} - U_1^{(1)} + A\int_{v_1^{(1)}}^{v} p^{(1)}dV^{(1)},$$

$$Q^{(2)} = U^{(2)} - U_1^{(2)} + A\int_{v_1^{(2)}}^{v} p^{(2)}dV^{(2)}$$

should be such that the sum $U^{(1)} + U^{(2)}$ remains constant

$$Q^{(1)} + Q^{(2)} = A\left[\int_{v_1^{(1)}}^{v} p^{(1)}\,dV^{(1)} + \int_{v_1^{(2)}}^{v} p^{(2)}dV^{(2)}\right].$$

Since each of the integrals on the right side is larger than zero, the sum $Q^{(1)} + Q^{(2)}$ will have a positive sign. Consequently, the change in entropy $S_2 - S_1$ will also be positive and have a value $\int(dQ^{(1)} + dQ^{(2)})/T$

$$S_2 - S_1 = A\left[\int_{V_1^{(1)}}^{V} \frac{p^{(1)}}{T}\,dV^{(1)} + \int_{V_1^{(2)}}^{V} \frac{p^{(2)}}{T}\,dV^{(2)}\right].$$

Thus, adiabatic mixing of gases is accompanied by an increase in entropy.

The increment in entropy per mole occurring upon mixing of gases having the same initial temperatures and pressures is called the <u>mixing entropy</u> ΔS_{sm}.

The value of the mixing entropy can be readily calculated for ideal gases. In this case the pressure of each of the gases is equal to $p^{(j)} = \mu R^{(j)} M(T/V^{(j)})$ and therefore

$$\Delta S_{см} = A\mu R \sum \frac{M^{(j)}}{\Sigma M^{(j)}} \ln \frac{V}{V_1^{(j)}}.$$

But in the mixing of ideal gases with equal p_1 and t_1 the pressure and temperature do not change, so that the ratio $V/V_1^{(j)}$ is equal to the molecular concentration of the given gas $z^{(j)}$. Hence

$$\Delta S_{см} = -A\mu R \Sigma z^{(j)} \ln z^{(j)}. \qquad (3.7)$$

§3. ANALYTIC EXPRESSION FOR THE SECOND LAW OF THERMODYNAMICS

<u>Physical Meaning and Properties of Entropy</u>

According to the second law of thermodynamics, any thermodynamic system is characterized by a function of state of the system, namely the entropy S, which has the following properties.

The entropy S is a unique function of the state of the system. Along with the parameters <u>p</u>, <u>t</u>, V, U, and I, the entropy S characterizes the state of the system and can be regarded as one of the parameters of state.

In an infinitesimally small change of the state of a thermodynamic system, the change of the entropy of the system is defined by the condition

$$dS \geq dQ/T.$$

Here dS is the total differential of the system entropy; dQ is the amount of heat obtained by the system from the heat source in an infinitesimally small process; T is the absolute temperature of the source, which in a reversible process is equal to the absolute temperature of the system (the equality sign pertains to reversible processes and the "larger" sign pertains to irreversible processes).

The differential entropy of the body dS is the elementary reduced heat dQ/T obtained by the body in the reversible process. It can be also stated that the differential of the entropy is numerically equal to the largest amount of reduced heat which can be obtained by the body from the external source of heat with a temperature equal to the temperature of the body in the initial state, on going over from the state under consideration into an infinitesimally close state.

There are no methods for directly measuring the entropy. The entropy of a body in some state relative to some other initial (or original) state is calculated by summing the reduced heat imparted to the body in a reversible (quasistatic) transition of the body from the initial state to the given state.

If the state of the body under consideration is an equilibrium state, then this calculation needs no explanations. It is somewhat more complicated to calculate the entropy of a body in a nonequilibrium state. Here one starts from the following considerations. Assume that there is some nonequilibrium state of the body. Obviously, by turning on some external force field it is possible to attain that equilibrium state of the given body which will not differ at all from the nonequilibrium state. Thus, for example, the state of a gas with uneven density distribution contained in a vessel, which is not in equilibrium in the absence of an external field, becomes a state of equilibrium under the influence of a corresponding gravitational field.

Consequently, a state which under certain conditions is not in equilibrium may turn out to be in equilibrium under other conditions in the presence of force fields. Therefore, by introducing force fields of different configurations, it is possible to realize in principle a quasistatic transition of the body from an initial state into a given nonequilibrium state and to calculate from the sum of the reduced heat the entropy of the body in the nonequilibrium state, which, as in the equilibrium state, will have a fully defined and furthermore unique value.*

From the expression for the differential of the entropy it follows further that the entropy S of any body is determined accurate to some constant quantity (the integration constant S^0) which represents the entropy of the body at absolute zero:

$$S = \int(dQ/T) + S^0.$$

The numerical value of the constant S^0 cannot be determined on the basis of the first and second laws of thermodynamics alone.

It must be noted that in many important cases interest is attached to the change in the entropy and not in its absolute magnitude. Consequently, the value of S^0 is frequently chosen arbitrarily, from considerations of convenience. In particular, the value of the entropy of liquid water at a temperature of $0^\circ C$ and under the pressure of its own saturated vapor is usually assumed equal to zero; in other substances the entropy is also referred to $0^\circ C$. On the other hand, to calculate processes that are accompanied by a change in the mass of the initial substances and by the formation of new substances characterized by entirely different values of S^0 (for example, in chemical reactions), it is essential to know the exact value of S.

In an isolated system, where any change in the state is adiabatic, the entropy of the system cannot decrease. It either increases or re-

mains constant, $S_2 \geq S_1$. It follows therefore that in an isolated system the only realizable processes are those for which the entropy does not decrease. Processes with $S_2 < S_1$ cannot occur spontaneously in such a system, i.e., they are not spontaneous. An increase in the entropy in an isolated system under a change in state denotes that the processes occurring in the system are irreversible. In fact, inasmuch as the entropy of an isolated system increases, it should decrease in the reverse process, which is impossible. Therefore a process accompanied by an increase in the entropy is always irreversible. In reversible processes the entropy of the isolated system should not change, for only in this case can the process proceed both in the forward and in the reverse directions.

The law governing the variation of the entropy of an isolated system expresses thus the irreversibility and the unilateral nature of the macroscopic processes which occur in real thermodynamic systems and the <u>entropy itself</u> is, first, <u>a criterion of the direction of the real processes which occur in an isolated system</u>; second, <u>it is a measure of the irreversibility</u> of each of these processes or <u>a measure of the irreversibility of an adiabatic process</u>. It follows from the latter that the degree of irreversibility of an adiabatic process is characterized quantitatively by the increase in the entropy as a result of the process.

According to the kinetic theory of matter, the entropy of an isolated system in any state (equilibrium or nonequilibrium) is proportional to the natural logarithm of the probability W of the given state

$$S = Ak \ln W + \text{const.}$$

The entropy S, like the internal energy U, is an additive quantity and therefore the entropy of a complicated system is equal to the sum of the entropies of its independent parts:

$$S = \Sigma S_j.$$

The additivity of the entropy follows directly from the fundamental equation (3.6). Indeed, replacing dQ in this equation by dU + ApdV, we obtain

$$dS = \frac{1}{T} dU + \frac{Ap}{T} dV,$$

Inasmuch as in the case of equilibrium the pressure p and the temperature T in the system are the same everywhere and U and V are additive quantities, S should also be an additive quantity. The entropy retains its additivity property also in the case when the system is thermally nonuniform (when the temperature is different in different parts of the system).

For a uniform body we introduce the molar and specific entropies S_μ and $\underline{s}$, equal, respectively, to the entropy of 1 kg-mole and 1 kg of the given body. Then the total entropy of a body whose weight is G kg is expressed in terms of S_μ and $\underline{s}$ in the following manner:

$$S = \frac{G}{\mu} S_\mu = G\underline{s}$$

Here S_μ and $\underline{s}$ are functions of two independent parameters, for example p and $\underline{t}$ or $\underline{t}$ and V.

Thermodynamic Identity

An analytic expression for the second law of thermodynamics for an infinitesimally small reversible process has, according to (3.6), the form

$$dQ = TdS.$$

Inasmuch as in accordance with the first law of thermodynamics we have for a system whose work goes into a change of volume

$$dQ = dU + ApdV,$$

we obtain, by substituting dQ in (3.6),

$$TdS = dU + ApdV. \qquad (3.8)$$

Equation (3.8) contains only the parameters and functions of state and their differentials. It is called a thermodynamic identity. If U is expressed in terms of I − ApV, then the thermodynamic identity (3.8) assumes the form

$$TdS = dI - AVdp. \qquad (3.9)$$

Let us find the general expression for the entropy S of a system whose state is determined by two independent parameters (for example, for a uniform body).

Rewriting (3.8) and (3.9) in the form

$$dS = \frac{1}{T}\left(\frac{\partial U}{\partial T}\right)_V dT + \frac{1}{T}\left[Ap + \left(\frac{\partial U}{\partial V}\right)_T\right] dV,$$

$$dS = \frac{1}{T}\left(\frac{\partial I}{\partial T}\right)_p dT - \frac{1}{T}\left[AV - \left(\frac{\partial I}{\partial p}\right)_T\right] dp$$

and integrating them between the limits p_0, t_0, V_0 and $\underline{p}$, $\underline{t}$, V along any continuous curve which joins these points (for example, first along an isochore or an isobar and then along an isotherm), we obtain

$$S(T,V) - S(T_0,V_0) = \int_{T_0,V_0}^{T,V_0} \frac{1}{T}\left(\frac{\partial U}{\partial T}\right)_V dT + \int_{T,V_0}^{T,V} \frac{1}{T}\left[Ap + \left(\frac{\partial U}{\partial V}\right)_T\right] dV,$$

$$S(p,T) - (Sp_0,T_0) = \int_{p_0,T_0}^{p_0,T} \frac{1}{T}\left(\frac{\partial I}{\partial T}\right)_p dT - \int_{p_0,T}^{p,T} \frac{1}{T}\left[AV - \left(\frac{\partial I}{\partial p}\right)_T\right] dp.$$

Or, since $(\partial U/\partial T)_V = C_V$, $(\partial I/\partial T)_p = C_p$,

$$S(T,V) - S(T_0,V_0) = \int_{T_0,V_0}^{T,V_0} \frac{C_V}{T} dT + \int_{T,V_0}^{T,V} \frac{1}{T}\left[Ap + \left(\frac{\partial U}{\partial V}\right)_T\right] dV \qquad (3.10)$$

$$S(p,T) - S(p_0,T_0) = \int_{T_0,p_0}^{T,p_0} \frac{C_p}{T} dT - \int_{T,p_0}^{T,p} \left[AV - \left(\frac{\partial I}{\partial p}\right)_T\right] dp, \qquad (3.11)$$

where $S(T_0, V_0)$ and $S(p_0, T_0)$ is the entropy of the body in the initial state.

From the thermodynamic identities (3.8) and (3.9) it follows further that

$$T = \left(\frac{\partial U}{\partial S}\right)_V; \quad p = -\frac{1}{A}\left(\frac{\partial U}{\partial V}\right)_S; \qquad (3.12)$$

$$T = \left(\frac{\partial I}{\partial S}\right)_p; \quad V = \frac{1}{A}\left(\frac{\partial I}{\partial p}\right)_S. \tag{3.13}$$

This means that the absolute temperature T, the pressure p, and the volume V can be represented in the form of partial derivatives of the internal energy U and the enthalpy I at S = const. Since

$$dQ = TdS,$$

and by definition of the specific heat dQ = CdT, the specific heats C_V, C_p, and C_X will be, respectively, equal to

$$C_V = T\left(\frac{\partial S}{\partial T}\right)_V; \tag{3.14}$$

$$C_p = T\left(\frac{\partial S}{\partial T}\right)_p; \tag{3.15}$$

$$C_X = T\left(\frac{\partial S}{\partial T}\right)_X. \tag{3.16}$$

Thus, the specific heat C_X of a process X = const is equal to the product of the absolute temperature T by the partial derivative of the entropy S with respect to the temperature T with X = const.

Entropy of an Ideal Gas

For an ideal gas we have

$$dU = Gc_V dT \text{ and } p = GR\frac{T}{V},$$

Consequently, the entropy S will have the form

$$S(T,V) = G\int_{T_0}^{T}\frac{C_V}{T}dT + AGR\ln\frac{V}{V_0} + S(T_0,V_0).$$

If the specific heat c_V does not depend on the temperature, then

$$S = Gc_V\ln T + AGR\ln V + \text{const}.$$

In p,T coordinates, the equation for the entropy S of an ideal gas with constant specific heat has the following form:

$$S = Gc_p\ln T - AGR\ln p + \text{const}. \tag{3.17}$$

The Ts Diagram

To analyze the conversion of heat into work in systems whose state is determined by two independent parameters, it is particularly con-

venient to use the parameters t and s, regarding all other properties of the body as functions of these parameters.

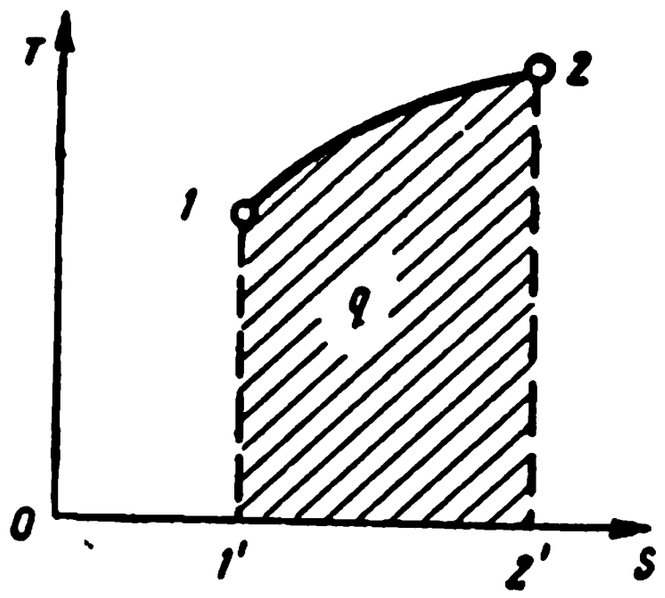

Fig. 16. Graphical representation of the amount of heat obtained by a body on the Ts plane.

If the abscissas represent the values of the entropy s for 1 kg of the body under consideration and the ordinates represent the value of its absolute temperature T, then the equilibrium state of the body is represented on the Ts diagram by a point with coordinates equal to the values of the temperature and entropy of the body in the given state. A reversible process wherein the state of the body changes from the initial state 1 to a certain state 2 is represented on the Ts diagram by a continuous curve passing through points 1 and 2 (Fig. 16). The amount of heat q which 1 kg of the body obtained in a reversible change of state from the point 1 to the point 2 is equal to the integral $\int_1^2 Tds$, taken along the curve of the process from the point 1 to the point 2. But it is seen on Fig. 16 that this integral is equal to the area 122'1' between the curve of the process and the abscissa axis on the section 1-2.

Consequently, the amount of heat obtained by 1 kg of body in a reversible process wherein its state is changed is represented on the Ts diagram by the area contained between the curve of the process and the abscissa axis. If the entropy increases, i.e., $ds > 0$, then also $dq > 0$, while when $ds < 0$ we get $dq < 0$.

The Ts diagram is also called the heat diagram.

A reversible cyclic process is represented on the Ts diagram by a closed curve abcda (Fig. 17). The amount of heat q_1 obtained by the body as its state changes from the extreme left point a to the extreme right point c along the curve abc is equal to $\int_{abc} Tds$ or to the area

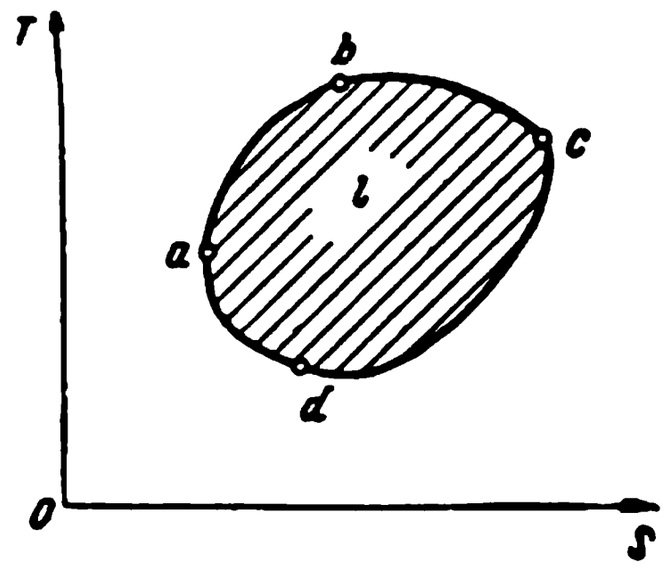

Fig. 17. Reversible cyclic process.

abcc'a', while the amount of heat q_2 given up by the body as its state changes from the point c to the point a, i.e., when the body returns to the initial state a along the path cda, is $-\int_{cda} Tds$ or the area adcc'a'. The difference $q_1 - |q_2|$ between the heat obtained and given up by the body is, in accordance with the first law of thermodynamics, the work Al performed by 1 kg of the body in the cyclic process:

$$Al = \oint Tds.$$

This equation follows directly also from the thermodynamic identity, if we recognize that $\oint dU = 0$. Thus, the work performed by 1 kg of the medium in a reversible cyclic process is numerically equal to the area bounded by the curve of the process (the area of the cycle) on the Ts diagram.

§4. USEFUL EXTERNAL WORK

On the basis of the second law of thermodynamics we can establish a quantitative relation between the work which could be performed by the system under given conditions in a reversible process and the actual work which is performed under the same conditions in the irreversible process.

In §4 of Chapter 2 it was shown that the external useful work L' performed by the body on going over from state 1 into state 2 has the form

$$AL' = Q + I_1 - I_2,$$

where Q is the heat obtained by the body in the process 1-2.

Assume, for example, that it is required to determine the useful external work which can be performed by a body situated in an external medium with constant pressure and temperature equal to p' and t'. There

may be no equilibrium between the body and the surrounding medium, so that in the general case the temperature and the pressure of the body are not equal to the temperature and the pressure of the medium, $p \neq p'$, $t \neq t'$. Because of this inequality, both reversible and irreversible processes can be produced with the body. During each of these processes the body will exchange heat with the surrounding medium and, in addition, it will perform useful work on an external work object. The heat Q obtained by the body from the surrounding medium is $-T'\Delta S'$, where $\Delta S' = S'_2 - S'_1$ is the change in the entropy of the surrounding medium as a result of process 1-2. The outer medium together with the body situated in it represents an adiabatically isolated system, the entropy of which S* is equal to the sum of the entropies of the medium S' and of the body S. Therefore the change in entropy of the surrounding medium is

$$\Delta S' = \Delta S^* - (S_2 - S_1).$$

It follows therefore that

$$Q = -T'(S_1 - S_2) - T'\Delta S^*.$$

Substituting the obtained value of Q into the expression for L' we obtain

$$AL' = I_1 - I_2 - T'(S_1 - S_2) - T'\Delta S^*. \qquad (3.18)$$

According to the second law of thermodynamics, the change in the entropy ΔS^* of an adiabatically isolated system cannot be negative; ΔS^* is either equal to zero (in a reversible process) or is larger than zero (in an irreversible process). Therefore

$$AL' < I_1 - I_2 - T'(S_1 - S_2). \qquad (3.19)$$

In a reversible process, when $\Delta S^* = 0$, the useful external work is maximal:

$$AL'_{макс} = I_1 - I_2 - T'(S_1 - S_2). \qquad (3.20)$$

The maximum work L'_{maks} is also called the <u>theoretical work</u>. In an ir-

reversible process occurring in the same interval of states (with specified initial and final states 1 and 2), we have

$$AL' = AL'_{\text{макс}} - T'\Delta S^*, \qquad (3.21)$$

i.e., <u>the actual useful external work is smaller than the maximum by a positive amount</u> $T'\Delta S^*$. The product of the absolute temperature T' of the surrounding medium by the entropy increment ΔS^* of the entire system as a whole is called the <u>loss of useful work</u> due to the irreversibility of the process. The larger the increment of the entropy of the system (the larger the degree of irreversibility of the process), the smaller the work performed by the system. When the body goes over from a specified initial state into a state of equilibrium with the surrounding medium, the specific maximum useful external work $l'_{maks} = L'_{maks}/G$ is called the <u>work capacity of the body in the given state</u>.

Equation (3.21), which is the fundamental relation between the actual and maximum useful external work, is of most general significance. Using this equation we can determine from the known final values of the entropies of all the bodies participating in a process the difference between the maximum (theoretically available) useful work and the work actually performed in the adiabatically isolated system, i.e., we can determine the loss of work due to the irreversibility of the process. Thus, the loss of work is connected with the fact that some fraction of energy which has work capacity, which in the reversible process could have been converted into useful work, does not go over into work in the irreversible process and is released in the form of heat absorbed subsequently by the bodies with the lowest temperature.

The increase in the useful work [see Eq. (3.20)] of an adiabatically isolated system with increasing degree of irreversibility of the processes (with increasing entropy of the entire system) is sometimes

related with a supposed tendency in nature of all the processes to lead to a "devaluation" or "degradation" of energy. According to this point of view, in the universe, which is considered as an isolated system, the entropy increases with time, as a result of which the possibility of conversion of heat into work decreases, or in other words, the "degradation" of energy takes place. Ultimately there should be established in the universe some final state of absolute thermal equilibrium ("thermal death"), in which all processes stop and energy conversions become unrealizable.

This point of view is completely incorrect. It is connected with an arbitrary extension of the second law of thermodynamics, which is in force for systems of terrestrial dimensions, to the entire universe. To state that only irreversible dissipation of energy occurs continuously in nature is to perpetrate a gross error, contradicting all our knowledge concerning the processes that occur in the universe and the main premises of dialectical materialism concerning the indestructibility of motion and the infinite nature of the process of development of matter.

The antiscientific reactionary theory of the thermal death of the universe, which leads unavoidably to a divine origin and source of the universe, has been subjected to a withering criticism on the part of Engels.

According to Engels, moving matter has an unlimited ability of high-grade transformation from one form into another which is not lost under any circumstance.

The universe, which has no boundaries in either space or time, has an infinite set of different possible states; therefore even after an infinitely long time it cannot go through all the stages, i.e., exhaust them. This means that for the universe as a whole the entropy does not

have a maximum, so that not one of its states is most probable and consequently there can be no final state of the universe at all, as stated by the theory of the thermal death of the universe.

§5. THE NERNST HEAT THEOREM

The study of the properties of various substances at low temperatures close to absolute zero discloses the following important regularity in the behavior of real substances.

In the vicinity of the absolute zero of temperature the entropy of a body in any equilibrium state does not depend on the temperature, volume, and other parameters which characterize the state of the body: $S = S^0$ as $T \to 0$.

This result, which is a generalization of many experiments and which does not follow directly from the first or second laws of thermodynamics, is the content of the Nernst heat theorem. It follows from the heat theorem that near absolute zero the specific heats C_V and C_p, which are equal to $T(\partial S/\partial T)_V$ and $T(\partial S/\partial T)_p$, respectively, vanish as a result of the vanishing of the derivatives $(\partial S/\partial T)_V$ and $(\partial S/\partial T)_p$ as $T \to 0$. In like fashion, when $T \to 0$ the derivative $(\partial V/\partial T)_p$ also vanishes (and consequently also the coefficient of thermal expansion), being equal to the derivative $(\partial S/\partial p)_T$.

This means that no matter what the state of the substance (liquid or solid, a pure substance or a chemical compound), its entropy as $T \to 0$ is one and the same value, provided the substance is in thermal equilibrium in each of these states. Thus, for example, as $T \to 0$ the entropies of the liquid and solid states of any substance are equal and the entropy of a mixture consisting of one mole of substance A and one mole of substance B will be equal to the entropy of one mole of their chemical compound AB (provided the latter exists).

The constancy of the entropy when $T \to 0$ means that in the region

of absolute zero of temperature any of the isotherms coincides with the adiabatic $S = S^0$. Thus, any isothermal system behaves as $T \to 0$ like an adiabatic system and can perform work only as a result of its internal energy, without absorbing any heat from the surrounding bodies and without giving up any heat to them; conversely, any adiabatic system does not differ in this region from an isothermal system. It follows from the latter that it is impossible to reach absolute zero by means of an adiabatic expansion of a body. It is impossible to reach absolute zero likewise by taking heat away from the body, for when $T \to 0$ any body, no matter in what process its state changes, retains its entropy constant, i.e., ceases to give up heat to the surrounding medium.

Planck advanced the hypothesis that <u>at absolute zero the entropy of all substances in the state of equilibrium vanishes</u>: $S^0 = 0$. Since ordinary gases which are at negligibly small pressures condense much earlier, before the temperature $T = 0$ is reached, Planck's statement applies essentially to condensed systems – to solids and liquids (of all bodies, only helium II remains liquid as $T \to 0$, and all others go into the solid state into higher temperatures). In the case when the substance is not in equilibrium at absolute zero, its entropy can have a positive nonvanishing value as $T \to 0$, i.e., $S^0 > 0$.

If we accept Planck's hypothesis, then the heat theorem can be formulated in the following somewhat more general form. <u>Near absolute zero all the thermodynamic quantities characterizing the equilibrium state of the body cease to depend on the temperature.</u> It follows therefore that the partial derivatives of all the thermodynamic functions with respect to the temperature, for example of the entropy, the internal energy, the enthalpy (heat content) and also the pressure and volume, vanish as $T \to 0$. This deduction agrees with experiment, thereby

confirming the correctness of Planck's hypothesis.

On the basis of Planck's hypothesis it is possible to calculate from the known values of the specific heats the absolute values of the thermodynamic functions. Thus, for example, the value of the entropy and enthalpy of a body at a given temperature and pressure is determined by the equations

$$S(p,T)=\int_{p,0}^{p,T}\frac{C_p}{T}\,dT;$$

$$I(p,T)=\int_{p,0}^{p,T}C_p\,dT+I^0,$$

where $I^0 = U^0$ is the enthalpy at $T = 0$ and C_p is the specific heat at a specified pressure. The values of the entropy of some substances calculated in this manner, referred to $p = 1$ atm abs and $t = 298^\circ C$, are listed in Table 2.

TABLE 2

Absolute Entropies of Some Substances

1 Вещество	2 Энтропия, ккал/кг·моль	1 Вещество	2 Энтропия, ккал/кг·моль
H_2	31,211	C (графит) 3	1,3609
O_2	49,003	H_2O (жидкость) 4	16,716
CO	47,301	CH_4	44,50
CO_2	51,061		

1) Substance; 2) entropy, kcal/kg-mole; 3) graphite; 4) liquid.

Using the expression for S, we can prove rigorously that absolute zero is unattainable. Let us consider for this purpose reversible adiabatic cooling of a body which, as is well known, leads to the greatest decrease in the temperature, i.e., is the most effective. Let the initial state of the body be p_1, t_1; then its entropy in this state is

$$S_1=\int_{p_1,0}^{p_1,T_1}\frac{C_p}{T}\,dT.$$

In a reversible adiabatic process the entropy of the body does not

change, $S = S_1$, therefore at any point of this process we have

$$\int_{p,0}^{p,T} \frac{C_p}{T}\,dT = \int_{p_1,0}^{p_1,T_1} \frac{C_p}{T}\,dT.$$

Inasmuch as when $T = 0$ the integral in the left half of the equation vanishes, to attain absolute zero it is necessary that the second integral

$$\int_{p_1,0}^{p_1,T} \frac{C_p}{T}\,dT$$

also vanish. This is impossible because C_p for any finite temperature is positive and is not equal to zero. It follows from this that a temperature of absolute zero is unattainable. On this basis, the heat theorem is frequently formulated in the following manner.

<u>There is no method by which to cool a body to absolute zero, i.e., absolute zero is unattainable.</u>

This, however, does not mean that it is impossible to obtain temperatures that are as close as desired to $T = 0$.

Let us note in conclusion that ideal gases do not satisfy the heat theorem. Indeed, for an ideal gas the derivative $(\partial p/\partial T)_V$, which is equal to R/v, does not vanish when $T = 0$ and $v \neq \infty$, as called for by the heat theorem. Likewise, the difference of the specific heats c_p and c_V at $T = 0$ is not equal to zero, as called for by the heat theorem, but to the gas constant R. This failure of the properties of ideal or rarefied gases to satisfy the heat theorem is connected with the fact that the Clapeyron-Mendeleyev equation cannot be employed at low temperatures. At temperatures close to absolute zero rarefied gases obey not the Clapeyron-Mendeleyev equation, but a more complicated equation of state, which takes into account quantum effects (the degeneracy of the gas).

The entropy of any gas in an ideal state can be determined by con-

sidering a successive isobaric transition from the condensed phase of this substance, situated at T = 0 and at a given pressure, to the specified state of the substance in the form of an ultimately rarefied gas. The sum of the changes in the entropy on each of the parts of this transition gives the sought value of the entropy of the gas in the ideal state.

§6. THERMODYNAMIC POTENTIALS

Characteristic Functions

The useful external work L' which can be performed by a system situated in a medium whose pressure p' and temperature t' are constant is determined by the expression

$$AL' < I_1 - I_2 - T'(S_1 - S_2), \qquad (3.22)$$

where the equality sign pertains to reversible processes and the "less than" sign pertains to irreversible ones.

Let us assume that the system under consideration goes through a reversible process in which any two independent parameters of the state have a constant value.*

Let, for example, the volume V and the entropy S of the system remain constant. Then we obtain from (3.22), recognizing that I – Ap'V is equal to U,

$$AL'_{\text{макс}} = U_1 - U_2. \qquad (3.23)$$

At constant p and S we have

$$AL'_{\text{макс}} = I_1 - I_2. \qquad (3.24)$$

At constant t and V we have

$$AL'_{\text{макс}} = (U - TS)_1 - (U - TS)_2.$$

Denoting U – TS by F, we obtain

$$AL'_{\text{макс}} = F_1 - F_2. \qquad (3.25)$$

The quantity F = U – TS is called the free energy. Finally, at constant p and t we have

$$AL'_{\text{макс}} = (I - TS)_1 - (I - TS)_2.$$

Denoting I – TS by Φ, we obtain

$$AL'_{\text{макс}} = \Phi_1 - \Phi_2. \qquad (3.26)$$

The quantity Φ = I – TS is called the isobaric potential.

Inasmuch as U, I, and S are functions of the state of the system, F and Φ will also be functions of state.

In analogy with mechanics, where the work in a field of conservative forces is numerically equal to the difference between the potentials in the initial and final points of the path, the functions U(V, S), I(p, S), F(t, V), and Φ(p, T), the difference in the values of which in two states is the useful external work which could be performed by the system in the case of a reversible transition under suitable conditions from one state into another, have been called the thermodynamic potentials.

Each of the thermodynamic potentials is a function of the state. The quantity –TS, regarded as a function of V and U, is likewise a thermodynamic potential, as can be seen from Eq. (3.22).

In the case of an irreversible process, the equality sign in (3.23)-(3.26) should be replaced by the "less than" sign. It follows from this that for any system which does not perform useful external work (L' = 0), the following laws are typical:

1) in a system with constant volume and entropy, the internal energy cannot increase:

$$(U_2 - U_1)_{V,\,S} \leqslant 0;$$

2) in a system with constant pressure and entropy, the enthalpy cannot increase:

$$(I_2 - I_1)_{p,\,S} \leqslant 0;$$

3) in a system with constant volume and temperature, the free energy cannot increase:

$$(F_2 - F_1)_{V,T} < 0;$$

4) in a system with constant pressure and temperature, the isobaric potential cannot increase:

$$(\Phi_2 - \Phi_1)_{p,T} < 0.$$

Differentiating the functions F = U − TS and Φ = I − TS and using the thermodynamic identities (3.8) and (3.9), we obtain

$$dF = -SdT - Apdv;$$
$$d\Phi = -SdT + AVdp,$$

from which it follows that

$$S = -\left(\frac{\partial F}{\partial T}\right)_V; \quad p = -\frac{1}{A}\left(\frac{\partial F}{\partial V}\right)_T; \tag{3.27}$$

$$S = -\left(\frac{\partial \Phi}{\partial T}\right)_p; \quad V = \frac{1}{A}\left(\frac{\partial \Phi}{\partial p}\right)_T. \tag{3.28}$$

The thermodynamic potentials U(V, S), I(p, S), F(t, V), and Φ(p, t) and the entropy S(V, U) are also called the characteristic functions.

Knowing any of the characteristic functions we can always calculate all other thermodynamic quantities of a given system. For this purpose it is sufficient, as can be seen from Eqs. (3.12)-(3.15), (3.27), and (3.28), to differentiate the characteristic function with respect to the corresponding variables. In particular, the equations (3.27) and (3.28), which determine p as a function of t and V or V as a function of p and t, are precisely equations of state of the body.

Chemical Potential

The thermodynamic potentials U(V, S), I(p, S), F(t, V), and Φ(p, t) are additive quantities. Therefore in a homogeneous system the quantities U, T, F, and Φ should be equal to:

$$U = Gu(v, s);$$
$$I = Gi(p, s);$$
$$F = Gf(t, v);$$
$$\Phi = G\varphi(p, t).$$

The specific isobaric potential φ(p, t), which represents the isobaric potential of a uniform system per unit weight, is called the chemical

potential. It is equal to

$$\left.\begin{aligned} \varphi &= i - sT, \\ \varphi &= u + Apv - sT. \end{aligned}\right\} \tag{3.29}$$

The differential $d\varphi$ of the chemical potential can be reduced with the aid of the thermodynamic identity to the following form:

$$d\varphi = -sdT + Avdp. \tag{3.30}$$

In all the preceding arguments the weight of the body was assumed constant and was expressed in terms of some constant number G. We shall now regard the amount of the body (the weight G) as a variable parameter which can assume different values, the same as all other parameters characterizing the system. Changes in weight are encountered in chemical reactions, when certain substances are formed from others, or in changes of the aggregate state of the substance. When the amount of substance is variable, an additional term, containing the differential of the weight of the body dG, enters into the expression for the differentials of all the thermodynamic potentials. Indeed, differentiating, for example, the equation U = Gu, we obtain

$$dU = Gdu + udG = G(Tds - Apdv) + udG.$$

Since

$$\begin{aligned} Gds &= d(Gs) - sdG = dS - sdG, \\ Gdv &= dV - vdG, \end{aligned}$$

we get

$$dU = TdS - ApdV + (u - sT + Apv)\,dG$$

or in accordance with (3.29)

$$dU = TdS - ApdV + \varphi dG. \tag{3.31}$$

Analogously

$$dI = TdS + AVdp + \varphi dG; \tag{3.32}$$

$$dF = -SdT - ApdV + \varphi dG; \tag{3.33}$$

$$d\Phi = -SdT + AVdp + \varphi dG. \tag{3.34}$$

From Equations (3.31)-(3.34) it follows that

$$\left.\begin{aligned} \varphi &= \left(\frac{\partial U}{\partial G}\right)_{V,S}; \quad \varphi = \left(\frac{\partial I}{\partial G}\right)_{p,S}; \\ \varphi &= \left(\frac{\partial F}{\partial G}\right)_{t,V}; \quad \varphi = \left(\frac{\partial \Phi}{\partial G}\right)_{p,t}. \end{aligned}\right\} \tag{3.35}$$

Thus, the chemical potential is the partial derivative of the thermodynamic potential of the system with respect to the weight of the body at the constant values of the corresponding independent variables (with the chemical potential expressed in terms of different variables). Since any of the differentials dU, dI, dF, and dΦ taken with the negative sign represents the maximum useful work which can be performed by the system situated in a medium with constant pressure and temperature on an external object in the case of an infinitesimally small process characterized by constant values of two corresponding independent variables, the chemical potential is numerically equal to the maximum useful work given up by the system to the outside in the case when the weight of the system is reversibly reduced by unity. As applied to chemical reactions, the chemical potential is the maximum useful work which can be performed by a reagent on an external object when the weight of the medium is decreased by a unit weight, thus completely explaining its name.

§7. THE PARTIAL DIFFERENTIAL EQUATIONS OF THERMODYNAMICS

Using the analytic expression for any of the characteristic functions U(V, S); I(p, S); S(V, U); F(t, V), and Φ(p, t), with the aid of the mathematical operations

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right),$$

$$\left(\frac{\partial x}{\partial y}\right)_z = -\frac{\left(\frac{\partial z}{\partial y}\right)_x}{\left(\frac{\partial z}{\partial x}\right)_y} = -\frac{\left(\frac{\partial x}{\partial z}\right)_y}{\left(\frac{\partial y}{\partial z}\right)_x},$$

$$\left(\frac{\partial f}{\partial x}\right)_y = \left(\frac{\partial f}{\partial z}\right)_y \left(\frac{\partial z}{\partial x}\right)_y$$

we can set up equations containing the derivatives of the characteris-

tic functions. Since the derivatives of the characteristic functions define the physical properties of the substance, the use of such equations is particularly convenient for the analysis of various problems. The equations so obtained are called the partial differential equations of thermodynamics. They establish the quantitative connections between different physical properties of the substance following from the first and second laws of thermodynamics.

The aggregate of differential equations of thermodynamics is the working tool with the aid of which various specific problems in thermodynamics are analyzed; for example, the connection is established between the thermal and caloric quantities, the temperature effect of the throttling process is calculated, etc.

In practice, from among all the possible parameters of state, the ones most frequently used as independent are the parameters p, t, and V, inasmuch as they can be measured directly by experiment. In connection with the use of different independent variables, the need arises for transforming the derivatives of thermodynamic quantities from one set of variables to another. We give below the expressions for the fundamental differential equations in terms of the independent variables t,V and p,t.

Fundamental Derivatives of the Quantities U, I, F, and Φ

The fundamental derivatives of the thermodynamic potentials follow directly from the equations (3.13), (3.14), (3.27), (3.28), and (3.35):

$$\left(\frac{\partial U}{\partial S}\right)_V = T, \left(\frac{\partial U}{\partial V}\right)_S = -Ap, \left(\frac{\partial U}{\partial G}\right)_{V.S} = \varphi, \tag{3.36}$$

$$\left(\frac{\partial I}{\partial S}\right)_p = T, \left(\frac{\partial I}{\partial p}\right)_S = AV, \left(\frac{\partial I}{\partial G}\right)_{p.S} = \varphi, \tag{3.37}$$

$$\left(\frac{\partial F}{\partial V}\right)_{T,} = -Ap, \left(\frac{\partial F}{\partial T}\right)_V = -S, \left(\frac{\partial F}{\partial G}\right)_{T.V} = \varphi, \tag{3.38}$$

$$\left(\frac{\partial \Phi}{\partial p}\right)_T = AV, \left(\frac{\partial \Phi}{\partial T}\right)_p = -S, \left(\frac{\partial F}{\partial G}\right)_{p.T} = \varphi. \tag{3.39}$$

It is necessary to add to these equations also the equations for the specific heats C_V and C_p:

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V; \quad C_V = T\left(\frac{\partial S}{\partial T}\right)_V. \tag{3.40}$$

$$C_p = \left(\frac{\partial I}{\partial T}\right)_p. \quad C_p = T\left(\frac{\partial S}{\partial I}\right)_p. \tag{3.41}$$

Independent Variables t, V

Differentiating the first equation of (3.38) with respect to T with V constant, and the second equation with respect to V with T constant, we obtain

$$\frac{\partial^2 F}{\partial V \partial T} = -A\left(\frac{\partial p}{\partial T}\right)_V$$

and

$$\frac{\partial^2 F}{\partial T \partial V} = -\left(\frac{\partial S}{\partial V}\right)_T.$$

From the equality of the derivatives $\partial^2 F/\partial V \partial T$ and $\partial^2 F/\partial T \partial V$ it follows that

$$\left(\frac{\partial S}{\partial V}\right)_T = A\left(\frac{\partial p}{\partial T}\right)_V. \tag{3.42}$$

Differentiating the expression F = U − TS with respect to V with T constant, we obtain

$$\left(\frac{\partial F}{\partial V}\right)_T = \left(\frac{\partial U}{\partial V}\right)_T - T\left(\frac{\partial S}{\partial V}\right)_T,$$

hence, using (3.38) and (3.42), we obtain

$$\left(\frac{\partial S}{\partial V}\right)_T = \frac{1}{T}\left[Ap + \left(\frac{\partial U}{\partial V}\right)_T\right]. \tag{3.43}$$

$$\left(\frac{\partial U}{\partial V}\right)_T = -A\left[p - T\left(\frac{\partial p}{\partial T}\right)_V\right]. \tag{3.44}$$

Now, differentiating the expression I = U + ApV with respect to V with T constant, we obtain

$$\left(\frac{\partial I}{\partial V}\right)_T = \left(\frac{\partial U}{\partial V}\right)_T + A\left[p + V\left(\frac{\partial p}{\partial V}\right)_T\right],$$

which on the basis of (3.44) yields

$$\left(\frac{\partial I}{\partial V}\right)_T = A\left[T\left(\frac{\partial p}{\partial T}\right)_V + V\left(\frac{\partial p}{\partial V}\right)_T\right]. \tag{3.45}$$

For the derivative $(\partial I/\partial T)_V$ we have

$$\left(\frac{\partial I}{\partial T}\right)_V = \left(\frac{\partial U}{\partial T}\right)_V + AV\left(\frac{\partial p}{\partial T}\right)_V$$

or

$$\left(\frac{\partial I}{\partial T}\right)_V = C_V + AV\left(\frac{\partial p}{\partial T}\right)_V. \qquad (3.46)$$

Let us find now the expression for the derivative $(\partial C_V/\partial V)_T$. For this purpose we differentiate (3.40) with respect to V with $\underline{t}$ constant obtaining

$$\left(\frac{\partial C_V}{\partial V}\right)_T = T\frac{\partial^2 S}{\partial T\,\partial V}$$

Hence, using (3.38), we get

$$\left(\frac{\partial C_V}{\partial V}\right)_T = AT\left(\frac{\partial^2 p}{\partial T^2}\right)_V. \qquad (3.47)$$

Equations (3.43)-(3.47) and also the equations (3.51)-(3.56) that follow, which relate the caloric and thermal quantities, are extensively used for the calculation of the thermal properties of matter from the results of measurement of caloric quantities and the analysis of isothermal processes.

The expression for C_p in the variables $\underline{t}$ and V will have the following form

$$C_p = T\left(\frac{\partial S}{\partial T}\right)_p = T\left[\left(\frac{\partial S}{\partial T}\right)_V + \left(\frac{\partial S}{\partial V}\right)_T\left(\frac{\partial V}{\partial T}\right)_p\right] =$$

$$= C_V - T\left(\frac{\partial S}{\partial V}\right)_T \frac{\left(\frac{\partial p}{\partial T}\right)_V}{\left(\frac{\partial p}{\partial V}\right)_T}.$$

Substituting here the value of $(\partial S/\partial V)_T$ from (3.42), we obtain ultimately

$$C_p = C_V - AT\frac{\left(\frac{\partial p}{\partial T}\right)_V^2}{\left(\frac{\partial p}{\partial V}\right)_T}. \qquad (3.48)$$

From Eq. (3.48) for uniform bodies, for which $(\partial p/\partial V)_T < 0$ always, it follows that $C_p \geq C_V$.

Let us transform now the quantities $(\partial T/\partial V)_S$ and $(\partial T/\partial V)_U$ to the independent variables $\underline{t}$ and V. The partial derivative $(\partial T/\partial V)_S$ is expressed in terms of the derivatives $(\partial S/\partial V)_T$ and $(\partial S/\partial T)_V$ in the following manner

$$\left(\frac{\partial T}{\partial V}\right)_S = -\frac{\left(\frac{\partial S}{\partial V}\right)_T}{\left(\frac{\partial S}{\partial T}\right)_V}.$$

Using (3.40) and (3.42) we obtain

$$\left(\frac{\partial T}{\partial V}\right)_S = -\frac{AT}{C_V}\left(\frac{\partial p}{\partial T}\right)_V. \qquad (3.49)$$

Analogously

$$\left(\frac{\partial T}{\partial V}\right)_U = -\frac{\left(\frac{\partial U}{\partial V}\right)_T}{\left(\frac{\partial U}{\partial T}\right)_V}.$$

from which we obtain on the basis of (3.44)

$$\left(\frac{\partial T}{\partial V}\right)_U = \frac{A\left[p - T\left(\frac{\partial p}{\partial T}\right)_V\right]}{C_V}.$$

Using Expressions (3.45) and (3.46), the identity

$$\left(\frac{\partial T}{\partial V}\right)_I = -\frac{\left(\frac{\partial I}{\partial V}\right)_T}{\left(\frac{\partial I}{\partial T}\right)_p}$$

yields

$$\left(\frac{\partial T}{\partial V}\right)_I = -\frac{A\left[T\left(\frac{\partial p}{\partial T}\right)_V + V\left(\frac{\partial p}{\partial V}\right)_T\right]}{C_V + AV\left(\frac{\partial p}{\partial T}\right)_V}. \qquad (3.50)$$

Independent Variables p and t

Differentiating the first equation of (3.39) with respect to T with p constant, and the second equation with respect to p with T constant, we obtain

$$\left(\frac{\partial S}{\partial p}\right)_T = -A\left(\frac{\partial V}{\partial T}\right)_p. \qquad (3.51)$$

From the expression $\Phi = U + ApV - TS$ we obtain after differentiation

with respect to p with T constant

$$\left(\frac{\partial\Phi}{\partial p}\right)_T = \left(\frac{\partial U}{\partial p}\right)_T + A\left[V + p\left(\frac{\partial V}{\partial p}\right)_T\right] - T\left(\frac{\partial S}{\partial p}\right)_T.$$

Substituting here the value of $(\partial\Phi/\partial p)_T$ from (3.39) and then $(\partial S/\partial p)_T$ from (3.51) we obtain after transformations

$$\left(\frac{\partial S}{\partial p}\right)_T = \frac{1}{T}\left[\left(\frac{\partial U}{\partial p}\right)_T + Ap\left(\frac{\partial V}{\partial p}\right)_T\right]. \qquad (3.52)$$

$$\left(\frac{\partial U}{\partial p}\right)_T = -A\left[T\left(\frac{\partial V}{\partial T}\right)_p + p\left(\frac{\partial V}{\partial p}\right)_T\right]. \qquad (3.53)$$

Differentiating the expression for Φ with respect to T with p constant we obtain

$$\left(\frac{\partial\Phi}{\partial T}\right)_p = \left(\frac{\partial U}{\partial T}\right)_p - T\left(\frac{\partial S}{\partial T}\right)_p - S + Ap\left(\frac{\partial V}{\partial T}\right)_p.$$

Using Eq. (3.39) we get

$$\left(\frac{\partial U}{\partial T}\right)_p = C_p - Ap\left(\frac{\partial V}{\partial T}\right)_p. \qquad (3.54)$$

Differentiating the expression for T with respect to p with T constant we obtain on the basis of (3.53)

$$\left(\frac{\partial I}{\partial p}\right)_T^{\bullet} = A\left[V - T\left(\frac{\partial V}{\partial T}\right)_p\right]. \qquad (3.55)$$

The derivative $(\partial C_p/\partial p)_T$ will have the form

$$\left(\frac{\partial C_p}{\partial p}\right)_T = T\frac{\partial^2 S}{\partial T\,\partial p}.$$

Hence, using (3.38), we get

$$\left(\frac{\partial C_p}{\partial p}\right)_T = -AT\left(\frac{\partial^2 V}{\partial T^2}\right)_p. \qquad (3.56)$$

Equation (3.56) is extensively used for the determination of the dependence of V on T using the results of measurements of the specific heat C_p, and vice versa.

The specific heat C_V in the variables p and t is equal to

$$C_V = T\left[\left(\frac{\partial S}{\partial T}\right)_p + \left(\frac{\partial S}{\partial p}\right)_T\left(\frac{\partial p}{\partial T}\right)_V\right] =$$

$$= C_p - T\left(\frac{\partial S}{\partial p}\right)_T \frac{\left(\frac{\partial V}{\partial T}\right)_p}{\left(\frac{\partial V}{\partial p}\right)_T}.$$

Replacing on the basis of (3.51) the derivative $(\partial S/\partial p)_T$ by $-A(\partial V/\partial T)_p$, we ultimately obtain

$$C_V = C_p + AT\frac{\left(\frac{\partial V}{\partial T}\right)_p^2}{\left(\frac{\partial V}{\partial p}\right)_T}. \qquad (3.57)$$

Let us transform now the derivatives $(\partial T/\partial p)_S$ and $(\partial T/\partial p)_I$ to the independent variables p and t. The derivative $(\partial T/\partial p)_S$ will have the form

$$\left(\frac{\partial T}{\partial p}\right)_S = -\frac{\left(\frac{\partial S}{\partial p}\right)_T}{\left(\frac{\partial S}{\partial T}\right)_p}.$$

Using (3.51), we reduce this expression to the final form

$$\left(\frac{\partial T}{\partial p}\right)_S = A\frac{T}{C_p}\left(\frac{\partial V}{\partial T}\right)_p. \qquad (3.58)$$

The derivative $(\partial T/\partial p)_I$ will be equal to

$$\left(\frac{\partial T}{\partial p}\right)_I = -\frac{\left(\frac{\partial I}{\partial p}\right)_T}{\left(\frac{\partial I}{\partial T}\right)_p}.$$

Substituting here the value of $(\partial I/\partial p)_T$ from (3.55), we obtain

$$\left(\frac{\partial T}{\partial p}\right)_I = \frac{A}{C_p}\left[T\left(\frac{\partial V}{\partial T}\right)_p - V\right]. \qquad (3.59)$$

Thermal Coefficients

The isothermal compressibility or the coefficient of compression of a body at t = const is defined as the quantity

$$\beta_t = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_t. \qquad (3.60)$$

The isothermal compressibility has a positive value for all bodies.

The quantity β_t is usually referred to atmospheric pressure: the volume V is taken at a pressure p = 1 atm and the given temperature.

The adiabatic compressibility of a body refers to the quantity

$$\beta_s = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_S. \qquad (3.61)$$

The adiabatic compressibility, like β_t, is always positive.

If we divide (3.49) by (3.58) we obtain

$$\left(\frac{\partial p}{\partial V}\right)_S = -\frac{C_p}{C_V}\cdot\frac{\left(\frac{\partial p}{\partial T}\right)_V}{\left(\frac{\partial V}{\partial T}\right)_p}$$

or ultimately

$$\left(\frac{\partial p}{\partial V}\right)_S = \frac{C_p}{C_V}\left(\frac{\partial p}{\partial V}\right)_T. \qquad (3.62)$$

It follows therefore that the adiabatic compressibility β_S is C_p/C_V times smaller than the isothermal compressibility β_t.

The coefficient of thermal expansion of a body refers to the quantity α_p, defined by the equation

$$\alpha_p = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p. \qquad (3.63)$$

The coefficient of thermal expansion is usually referred to a temperature 0^oC and a given pressure p.

For liquids the coefficient of thermal expansion has a small near-zero value everywhere except near the critical point. For certain liquids (for example, water with $t < 4^oC$) α_p can have a negative value.

The temperature coefficient of pressure of a body refers to the quantity γ_V, defined by the equation

$$\gamma_V = \frac{1}{p}\left(\frac{\partial p}{\partial T}\right)_V. \qquad (3.64)$$

The temperature coefficient of pressure γ_V is related to the quantities α_p and β_t by the following equation:

$$\gamma_V = -\frac{1}{p}\cdot\frac{\alpha_p}{\beta_t}. \qquad (3.65)$$

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[Footnotes]

73 See Leontovich M.A. Introduction to Thermodynamics. Moscow-Leningrad, Gostekhizdat, 1952.

87 The process wherein a change is produced in the state of a uniform system characterized by the constancy of two parameters must of necessity be a nonequilibrium process; otherwise the state of the system could be defined by specifying these parameters and no processes could occur in the system. In an inhomogeneous system, for example one consisting of two phases of a substance, a reversible process can occur with the parameters p and t constant.

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[List of Transliterated Symbols]

58 расш = rassh = rasshireniye = expansion

58 сж = szh = szhatiye = compression

69 тр = tr = treniye = friction

71 см = sm = smesheniye = mixing

80 макс = maks = maksimal'nyy = maximum

Chapter 4

THERMODYNAMIC PROCESSES AND CYCLES

§1. FUNDAMENTAL THERMODYNAMIC PROCESSES

The change in the state of the body in any reversible process, and also the work performed as a result of the process and the amount of heat absorbed by the body, can be determined if one knows the equation of state and the specific heat C_V or C_p of the body.

In all the calculations that will be carried out from now on, it is assumed that the state of the body is characterized by two independent parameters.

Isothermal Process t = const

A reversible isothermal process of change in the state of any body, for example an isothermal expansion of a gas in a cylinder under a piston, can be realized if there is a sufficiently large source of heat of given temperature. Then, by producing a thermal contact between the working body and the source of heat, we can guarantee a continuous supply of heat from the source to the body during the entire process and thereby maintain the temperature of the body on one and the same constant level.

The pressure p and the volume V of the body in an isothermal variation of the state of the body are variable quantities that assume new values at each point of the process. The connection between p and V in a reversible isothermal process is obtained by substituting in the equation of state $p = f(t, V)$ the given value of t. Then the equation of the isothermal process will have the form

$$p = f(t = \text{const}, V).$$

In the simplest case of an ideal gas, the equation of the isothermal process has, as can be seen from the Clapeyron-Mendeleyev equation, the following form:

$$pV = \text{const},$$

where the constant is equal to GRT.

Knowing the equation of the isothermal process, it is easy to calculate the work of the process. The work of variation of a volume for an equilibrium isothermal process has the form

$$L = \int_{V_1}^{V_2} p(t = \text{const}, V)\, dV.$$

For an ideal gas $p = GRT/V$ and $L = GRT_1 \ln (V_2/V_1) = GRT_1 \ln (p_1/p_2)$ or

$$L = p_1 V_1 \ln \frac{V_2}{V_1}. \qquad (4.1)$$

The useful external work L' in an isothermal change in the state of an ideal gas is equal to L, as can be readily verified. The amount of heat obtained by the body in the isothermal process will have, in accordance with the first law of thermodynamics, the form

$$Q = \int_{V_1}^{V_2} \left(\frac{\partial U}{\partial V}\right)_t dV + AL.$$

For an ideal gas, the internal energy (and also the enthalpy) does not depend on V, $(\partial U/\partial V)_t = 0$, and therefore

$$Q = AL. \qquad (4.2)$$

This means that the work of the isothermal process is numerically equal to the heat supplied. In real substances U increases in isothermal expansion, as a result of which $Q > AL$. The change in entropy in a reversible isothermal process is

$$S_2 - S_1 = \frac{1}{T} \int_{V_1}^{V_2} \left(\frac{\partial U}{\partial V}\right)_t dV + \frac{A}{T} \int_{V_1}^{V_2} p\, dV.$$

For an ideal gas we have

$$S_2 - S_1 = GR \ln \frac{V_2}{V_1}. \qquad (4.3)$$

Isobaric Process p = const

In an isobaric process of changing the state of a body, the pressure p is maintained constant and the temperature t of the body and its volume V change. The constancy of the pressure upon varying the state of a gas in a cylinder with a piston can be realized by maintaining a constant force load on the piston. Since the isobaric process is accompanied by the release or absorption of heat at variable temperature, for a reversible performance of the process it is necessary to have a large (in the limit, infinite) number of heat sources of different temperatures.

We obtain the equation of a reversible isobaric process by substituting in the equation of state the specified value of the pressure p, f(t, V) = const. For an ideal gas it assumes the form V/T = const, where the constant is equal to GR/p.

The work of changing the volume in an isobaric process is expressed by the following equation:

$$L = \int_{V_1}^{V_2} p dV = p_1 (V_2 - V_1). \qquad (4.4)$$

The useful external work L' vanishes when p = const. The heat acquired by the body in an isobaric process is equal to the difference of the enthalpies at the end and at the beginning of the process:

$$Q = I_2 - I_1, \qquad (4.5)$$

or, inasmuch as p = const, we have $I_2 - I_1 = \int_1^2 \left(\frac{\partial I}{\partial T}\right)_p dT$.

$$Q = G \int_{T_1}^{T_2} c_p dT.$$

Part of the delivered heat, equal to $p_1 (V_2 - V_1)$, goes over into the

work of expansion, while the other part goes to increase the internal energy of the body. The change in entropy in an isobaric process has the form

$$S_2 - S_1 = G \int_{T_1}^{T_2} \frac{c_p}{T} dT. \qquad (4.6)$$

Isochoric Process V = const

In an isochoric process the volume V of the body is constant, while the pressure and temperature of the body change. Therefore to ensure strict reversibility of the process it is necessary to have a set of heat sources of different temperature.

The equation of a reversible isochoric process has the form

$$p = f(t,\ V = \text{const}).$$

For an ideal gas p/T = const, where the constant is GR/V.

The work of expansion L is equal to zero in an isochoric process, owing to the constancy of the volume V, while the useful external work is

$$L' = (p_2 - p_1) V_1. \qquad (4.7)$$

The amount of heat delivered to the body is

$$Q = U_2 - U_1 = G \int_{T_1}^{T_2} c_V dT. \qquad (4.8)$$

The entire heat supplied to the body goes to increase the internal energy of the body. The change of entropy in an isochoric process has the form

$$S_2 - S_1 = G \int_{T_1}^{T_2} \frac{c_V}{T} dT. \qquad (4.9)$$

Adiabatic Process

The process of changing the state of a body situated in a thermally insulating shell is called adiabatic. The adiabatic process occurs without heat being transferred from the surrounding medium to the

body or vice versa. At each point of the adiabatic process $dQ = 0$, so that on the basis of (2.5) and (2.11) we have

$$AL = U_1 - U_2,$$
$$AL' = I_1 - I_2,$$

i.e., in the adiabatic process the work necessary to change the volume is equal to the loss of internal energy, while the useful external work is equal to the loss of enthalpy of the body. An adiabatic process of changing the state of the body can be either reversible or irreversible. In a reversible adiabatic process the entropy of the body does not change, while in an irreversible one it increases, $S_2 - S_1 \geq 0$.

Reversible adiabatic process. An adiabatic change in the state of the body can be realized in reversible fashion by placing the body in an adiabatic shell, and maintaining the external pressure strictly equal to the pressure of the body itself while changing the state of the body. The simplest example of a reversible adiabatic process is the expansion (or compression) of a gas situated in a thermally insulated cylinder at sufficiently slow displacement of a suitably loaded piston.

The equation of a reversible adiabatic process for a body whose state is determined by two independent parameters is obtained from the condition $S = \text{const}$, using Eqs. (3.62)

$$\left(\frac{\partial p}{\partial V}\right)_S = \frac{C_p}{C_V}\left(\frac{\partial p}{\partial V}\right)_T$$

and Eq. (3.49)

$$\left(\frac{\partial T}{\partial V}\right)_S = -\frac{AT}{C_V}\left(\frac{\partial p}{\partial T}\right)_V.$$

Dividing the first equation by the second we obtain

$$\left(\frac{\partial p}{\partial T}\right)_S = \frac{C_p}{AT}\cdot\frac{1}{\left(\frac{\partial V}{\partial T}\right)_p}. \qquad (4.10)$$

Analogously we obtain from (4.10) and (3.62) with the aid of (3.48)

$$\left(\frac{\partial T}{\partial V}\right)_S = -\frac{\frac{C_p}{C_V} - 1}{\left(\frac{\partial V}{\partial T}\right)_p}. \qquad (4.11)$$

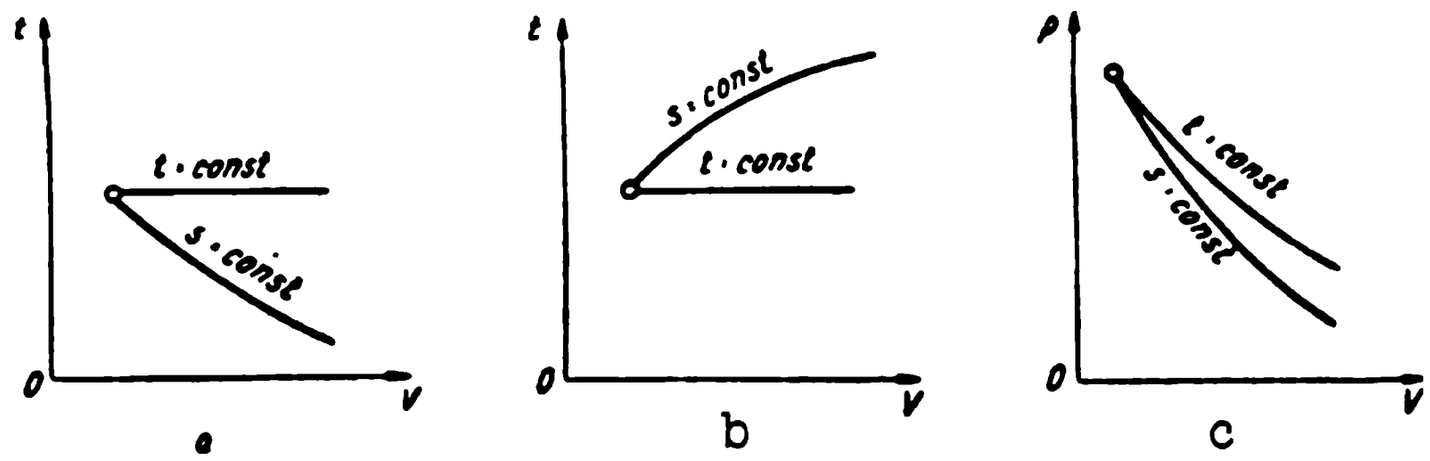

Fig. 18. Relative location of the adiabat and the isotherm on the tV and pV planes.

Inasmuch as $C_p \geq C_V > 0$ for all bodies and the quantity $(\partial p/\partial V)_T$ is negative for homogeneous substances, it follows from (3.62) that the derivative $(\partial p/\partial V)_S$ is also negative and has an absolute magnitude larger than $-(\partial p/\partial V)_T$, meaning that the adiabatic expansion is always accompanied by a decrease in pressure, while adiabatic compression is accompanied by an increase in pressure, whereas the adiabat on the pV plane is steeper than the isotherms drawn from the same point (Fig. 18a, c). It follows from (4.10) and (4.11) that the signs of the derivatives $(\partial p/\partial T)_S$ and $(\partial T/\partial V)_S$ are opposite to the sign of the partial derivative $(\partial V/\partial T)_p$.

In gaseous bodies the derivative $(\partial V/\partial T)_p$ is always positive; therefore the derivative $(\partial T/\partial V)_S$ has a negative sign. This means that adiabatic expansion of the gas leads to a decrease in its temperature or to its cooling, while adiabatic compression, to the contrary, leads to heating (Fig. 18a). This deduction is valid also for liquid, except for those special cases when the derivative $(\partial V/\partial T)_p$ becomes negative (when the liquid does not expand but contracts with heating at p = = const. For water this is observed in the temperature range from 0 to $4^\circ C$). Wherever $(\partial V/\partial T)_p < 0$, adiabatic expansion is accompanied by an increase in temperature and adiabatic compression by a decrease in tem-

perature of the liquid (Fig. 18c).

The differential equation (4.11) can be integrated if one knows the equations of state of the body and analytic expressions for the specific heats C_p and C_V as functions of the temperature and the volume (or pressure). The integration is simplest for an ideal gas.

In this case $(\partial V/\partial T)_p = V/T$ and Eq. (4.11) assumes the form

$$\frac{dT}{T} + (\kappa - 1)\frac{dV}{V} = 0,$$

where κ stands for the ratio C_p/C_V.

To simplify the integration, we assume that κ is independent of the temperature. This assumption is satisfied with sufficient accuracy in the case when the change in temperature as a result of the adiabatic process is not very appreciable. Integrating (4.11) under the assumption κ = const we obtain

$$TV^{\kappa-1} = T_1V_1^{\kappa-1}, \tag{4.12}$$

where T_1 and V_1 are the temperature and the volume of the gas in the initial state. If we change over from the variables T, V to the variables p, T or p, V then the equation of the adiabat assumes the following respective forms:

$$pT^{-\frac{\kappa}{\kappa-1}} = p_1T_1^{-\frac{\kappa}{\kappa-1}}; \tag{4.13}$$

$$pV^{\kappa} = p_1V_1^{\kappa}. \tag{4.14}$$

Equation (4.14) is usually called the Poisson adiabat equation.

From (4.13) and (4.14) it follows, in full agreement with the statements made above, that when the volume is increased the temperature and pressure of the gas decrease.

The work L of changing the volume, performed by the body in a reversible adiabatic process, can be calculated from the known adiabat equation. In particular, for an ideal gas we obtain with the aid of (4.14)

$$L = \int_{V_1}^{V_2} p dV = p_1 V_1^{\kappa} \int_{V_1}^{V_2} \frac{dV}{V^{\kappa}} = \frac{p_1 V_1}{\kappa - 1} \left[1 - \left(\frac{V_1}{V_2} \right)^{\kappa - 1} \right]$$

or

$$L = \frac{p_1 V_1}{\kappa - 1} \left[1 - \left(\frac{p_2}{p_1} \right)^{\frac{\kappa - 1}{\kappa}} \right] \qquad (4.15)$$

Replacing p_1V_1 by GRT_1 and p_2/p_1 by $(T_2/T_1)^{\kappa/(\kappa - 1)}$, we also obtain

$$L = \frac{GR}{\kappa - 1} (T_1 - T_2). \qquad (4.16)$$

Let us calculate now the <u>useful external work</u> L' of the adiabatic process, which is equal to $-\int_{p_1}^{p_2} V dp$. Using the adiabat equation of an ideal gas, we obtain

$$L_0' = \frac{\kappa}{\kappa - 1} p_1 V_1 \left[1 - \left(\frac{p_2}{p_1} \right)^{\frac{\kappa - 1}{\kappa}} \right] = \frac{\kappa}{\kappa - 1} GR (T_1 - T_2). \qquad (4.17)$$

From comparison of L' with L we see that in a reversible adiabatic process the useful external work is κ times larger than the work of changing the volume.

Let us compare a reversible and irreversible expansion of the body from one and the same initial state 1 (p_1, t_1, V_1) to a specified pressure p_2. On the TS diagram a reversible adiabatic process is represented, in accordance with the condition $S_2 = S_1$, by the vertical straight line 1-2 (Fig. 19), while an irreversible process, for which $S_2 > S_1$, is represented by the dashed curve 1-2', which is situated to the right of the straight line 1-2.

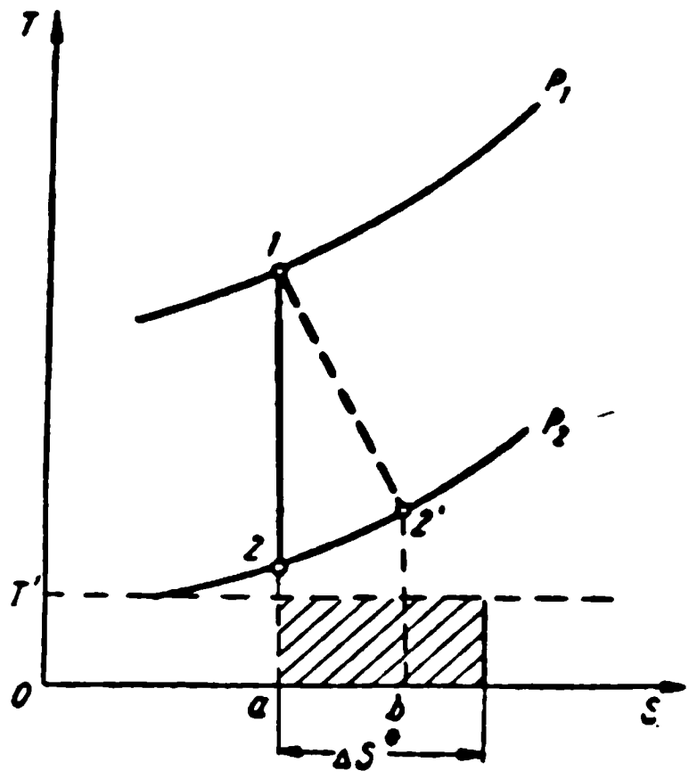

Fig. 19. Plots of reversible and irreversible processes on a TS diagram.

According to (2.11) the useful external work of the reversible process is

$$AL_{1-2}' = I_1 - I_2,$$

and that of the irreversible process is

$$AL'_{1-2} = I_1 - I_2.$$

Inasmuch as points 2 and 2' lie on one and the same isobar, corresponding to the pressure p_2, we have

$$I_{2'} - I_2 = Q_{2-2'}$$

and consequently

$$AL'_{1-2} - AL'_{1-2'} = Q_{2-2'}.$$

Let us consider now an irreversible process 1-2'-2, consisting of an irreversible adiabatic process 1-2' and externally irreversible isobaric process 2'-2. Since the useful external work $-\int Vdp$ of the isobaric process is equal to zero, we get $L'_{2'-2} = 0$ and consequently $L'_{1-2'-2} = L'_{1-2'}$. Processes 1-2 and 1-2'-2 have the same initial and final states and the useful external work is performed in both processes only during the adiabatic expansion. Therefore the general expression (3.21) is applicable to them, according to which

$$AL'_{1-2} - AL'_{1-2'-2} = T'\Delta S^*,$$

and consequently also

$$AL'_{1-2} - AL'_{1-2'} = T'\Delta S^*,$$

where ΔS^* is the increase in the entropy of the entire system as a result of the irreversible process 1-2'-2. ΔS^* is equal to the sum of the increment of the entropy of the body $\Delta S = S_2 - S_1$, which is equal to zero as a result of the isentropic process 1-2, and the increment of the entropy of the surrounding medium $\Delta S'$. The surrounding medium, the temperature of which is assumed constant and equal to T', receives as a result of process 2'-2 a quantity of heat $Q_{2-2'} = I_{2'} - I_2$. Therefore the increment in the entropy of the surrounding medium, and consequently of the entire system, amounts to $Q_{2-2'}/T'$, i.e.,

$$\Delta S^* = \frac{I_{2'} - I_2}{T'}.$$

The loss of useful work $AL'_{1-2} - AL'_{1-2'}$ due to the irreversibility of the adiabatic process 1-2', equal to

$$T'\Delta S^* = I_{2'} - I_2, \qquad (4.18)$$

is represented on the TS diagram by the area of the shaded rectangle. This area is also equal to the area a22'b between the isobaric section 2-2' and the abscissa axis.

Limiting Irreversible Adiabatic Process

A limiting irreversible process refers to a process in which, in spite of the reduction in the pressure of the body and the corresponding increase in the volume, the work of expansion or the useful external work vanishes as a result of the irreversibility of the process. In other words, in limiting irreversible processes the loss of useful energy is equal to the theoretically available work (the latter, for example, is completely consumed in overcoming the friction forces). The entropy increment is maximum in such processes. A limiting irreversible process of the first type (L = 0) is adiabatic expansion of a body in vacuum, as considered in detail in §2 of Chapter 3. Limiting irreversible processes of the second type (L' = 0) include adiabatic throttling.

Adiabatic throttling. Adiabatic throttling refers to an irreversible transition of a body from a pressure p_1 to a pressure p_2 smaller than p_1 without the body acquiring or giving up any heat and without performance of useful work.

Since Q = 0 and L' = 0, it follows from the fundamental equation (2.11)

$$AL' = Q - (I_2 - I_1)$$

that

$$I_2 = I_1. \qquad (4.19)$$

In adiabatic throttling the enthalpy of the body does not change.

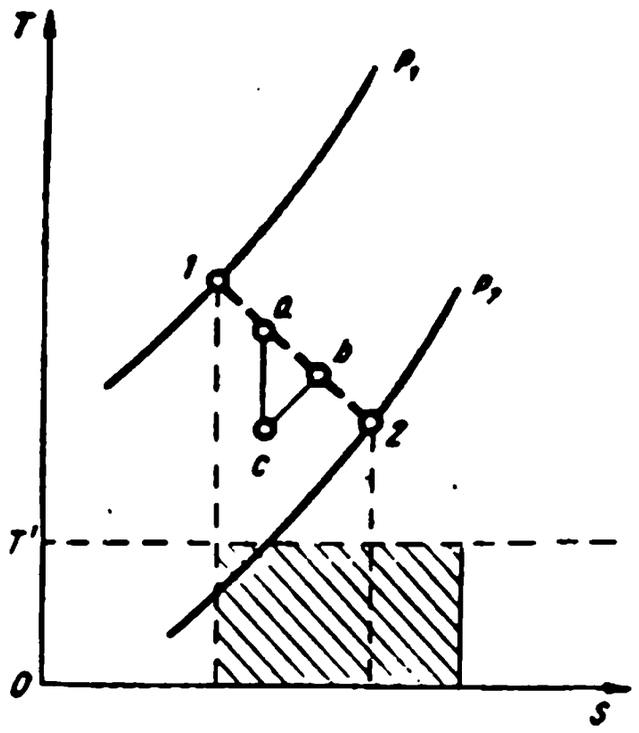

Fig. 20. Plot of the adiabatic throttling process.

In practice throttling can be observed when gases or liquids flow through a valve with small cross section (throttling valve), through an incompletely open valve, through a porous partition, etc.

The adiabatic throttling process can be regarded as an externally irreversible process, so that it is plotted on the TS diagram by a dashed line which coincides (at any rate, at the initial and final points) with the equal-enthalpy line (Fig. 20). From the infinitesimally small triangle abc, the side bc of which lies on the isobar and the side ac on the isentrope, we see that $T(S_b - S_c) = I_b - I_c$, or, since $I_a - I_c = -AV(p_b - p_a)$ and $I_b = I_a$ with $S_c = S_a$,

$$T(S_b - S_a) = -AV(p_b - p_a).$$

This means that the entropy changes connected with the pressure change by the relation

$$TdS = -AVdp.$$

It follows from this that inasmuch as $dS > 0$ we have $dp < 0$. According to Eq. (4.18), the loss of useful work in throttling is $T'\Delta S^*$. (In Fig. 20 it is shown by the shaded area.)

The volume of the gas always increases upon throttling, $V_2 > V_1$. To prove this, let us use Eqs. (3.45), (3.46), and (3.59), from which it follows that

$$\left(\frac{\partial p}{\partial V}\right)_I = \frac{C_p\left(\frac{\partial p}{\partial V}\right)_I}{C_V + AV\left(\frac{\partial p}{\partial T}\right)_V}.$$

Inasmuch as the derivative $(\partial p/\partial T)_V$ is positive in the region of the gaseous state and $(\partial p/\partial V)_T$ is always negative, we get $(\partial p/\partial V)_T < 0$.

It follows therefore that when $dp < 0$ (this always occurs during throttling) we get $dV > 0$. Substituting into the expression for $(\partial p/\partial V)_I$ the value of C_p from (3.48) we verify that in the region of a two-phase state of matter $(\partial p/\partial V)_I$ has a negative sign, so that in this region dV is also larger than zero.

The ratio of the change in temperature to the change in pressure in throttling is called the <u>differential temperature effect of the throttling</u> and is denoted by α_i:

$$\alpha_i = \left(\frac{\partial T}{\partial p}\right)_I.$$

According to (3.59) and (3.50), the differential throttling effect is

$$\left.\begin{aligned} \alpha_i &= A\,\frac{T\left(\frac{\partial V}{\partial T}\right)_p - V}{C_p}, \\ \alpha_i &= -\frac{T\left(\frac{\partial p}{\partial T}\right)_V + V\left(\frac{\partial p}{\partial V}\right)_T}{\frac{C_V}{A}\left(\frac{\partial p}{\partial V}\right)_T - T\left(\frac{\partial p}{\partial T}\right)_V^2}. \end{aligned}\right\} \qquad (4.20)$$

For an ideal gas the temperature effect of throttling is zero. Indeed, according to the Clapeyron-Mendeleyev equation the derivative $(\partial V/\partial T)_p$ is equal to $GR/p = V/T$, and consequently

$$\alpha_i = A\,\frac{T\frac{V}{T} - V}{C_p} = 0.$$

This result can be obtained directly from the condition $I_2 = I_1$ and from the expression for the enthalpy of an ideal gas

$$I = C_p T + I^0.$$

For real gases the temperature throttling effect does not equal zero and can be either positive or negative.

To determine the sign of the temperature throttling effect, let us calculate first the value of α_i at the critical point where $(\partial p/\partial T)_V = 0$ and, as will be made clear later, $(\partial p/\partial T)_V = dp_s/dT$, where p_s is the pressure of the saturated vapor.

From (4.20) we find that α_{ik} is equal to dT/dp_{sk}. But the pressure of a saturated vapor is an increasing function of the temperature; therefore the derivative dT/dp_s is everywhere positive. Consequently, $\alpha_{ik} > 0$, i.e., for all substances the temperature throttling effect at the critical point has a positive value equal to the reciprocal of the slope of the saturated vapor tension for the critical temperature. In other words, throttling of a substance at the critical point or near it always leads to a decrease in the temperature.

In equal fashion the temperature effect of throttling is also positive and equal to the reciprocal of the derivative dp_s/dT in the entire region of two-phase states of matter where $(\partial p/\partial V)_T = 0$ and $(\partial p/\partial T)_V = dp_s/dT$.

Let us assume now that the state of the gas varies along a critical isochore $V(p, T) = V_k$ from the critical point in the direction of rising temperature T. Since the derivative $(\partial V/\partial T)_p$, whose value is $-(\partial p/\partial T)_V/(\partial p/\partial V)_T$, has at the critical point an infinitely large positive value, on moving away from the critical point along the critical isochore the derivative $(\partial V/\partial T)_p$ will decrease, remaining positive all the time. At a certain distance from the critical point the quantity $T(\partial V/\partial T)_p$ becomes equal to V and the temperature throttling effect α_i vanishes according to (4.20). With further increase in the temperature, $T(\partial V/\partial T)_p$ becomes smaller than V and α_i will have a negative sign, i.e., a decrease in pressure during throttling will be accompanied by an increase in temperature. The state in which the temperature throttling effect reverses sign is called <u>the point of inversion</u>.

It is seen from (4.20) that there exists not one but a set of inversion points, determined by one of the following equations:

$$T\left(\frac{\partial V}{\partial T}\right)_p - V = 0, \tag{4.21}$$

$$T\left(\frac{\partial p}{\partial T}\right)_V + V\left(\frac{\partial p}{\partial V}\right)_T = 0. \qquad (4.21)$$

The geometric locus of the points of inversion is a continuous curve called the inversion curve.

Knowing the equation of state of a given substance, we can find with the aid of (4.21) the equation of the inversion curve in explicit form.

Inasmuch as the temperature effect is positive and nowhere equal to zero inside the region of two-phase states, the inversion curve should envelop the boundary curve. The form of the inversion curve for nitrogen is shown in Fig. 21. In the region lying under the inversion curve the temperature effect is positive: throttling is accompanied by cooling, while in the region above the inversion curve the temperature effect is negative: throttling causes heating of the body. The course of the inversion curve offers evidence that one and the same value of the pressure corresponds to two inversion points or two temperatures at which the temperature throttling effect is equal to zero. The first point corresponds to the lower temperature and is called the lower inversion point and the second point is the upper inversion point.

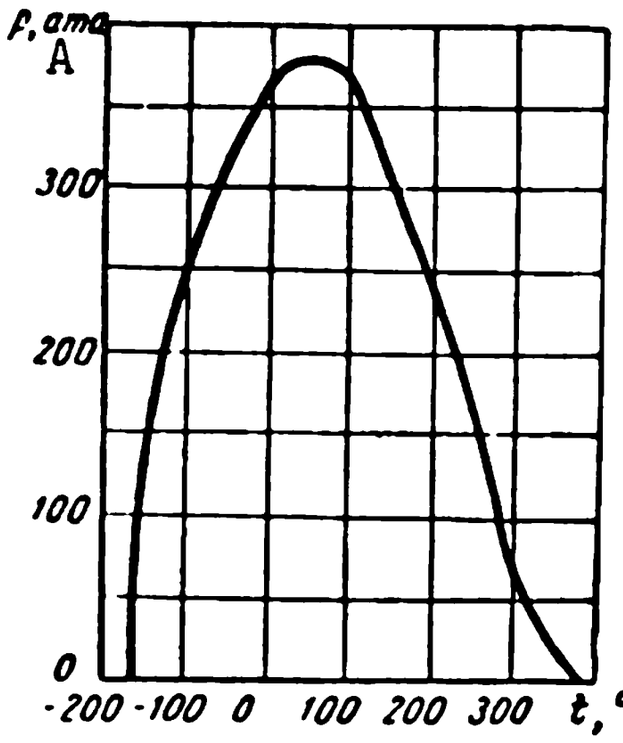

Fig. 21. Inversion curve for nitrogen. A) atm abs.

The value of the temperature throttling effect is completely determined by the properties of the molecular forces. Therefore a study of the throttling process can yield valuable information on the character and magnitude of the molecular forces and properties of real gases.

Production of low temperatures. Cooling of gases upon throttling is used in practice to obtain low temperatures, particularly to liquefy

air and other gases (the Linde method). The initial gas temperature should in this case be lower than the temperature of the upper inversion point; otherwise the gas temperature will not decrease upon throttling, but increase. To obtain low temperatures one uses, in addition to throttling, also adiabatic expansion of the gas with release of external useful work (the method of Claude and Kapitsa). It is interesting to compare the efficiencies of the two methods. It follows from (3.57) that the temperature effect of reversible adiabatic expansion is

$$\alpha_s = A \frac{T\left(\frac{\partial V}{\partial T}\right)_p}{C_p}. \qquad (4.22)$$

Inasmuch as the derivative $(\partial V/\partial T)_p$ is positive in the region of the gaseous state, we have $\alpha_s > 0$, i.e., in a reversible adiabatic expansion of a gas, the latter is always cooled (the gas performs work upon expansion). Unlike throttling, cooling of a gas by reversible adiabatic expansion is also characteristic of an ideal gas. Subtracting the first equation of (4.20) from Eq. (4.22) we obtain

$$\alpha_s - \alpha_i = \frac{AV}{C_p}.$$

from which it follows that in reversible adiabatic expansion the gas becomes cooled more strongly than in throttling.

The comparatively larger cooling of the gas and the constancy of the sign of the temperature effect constitute the principal advantage of reversible adiabatic expansion as a method for cooling as compared with throttling.

Cooling of a body to a temperature lower than the temperature of the surrounding medium entails the consumption of work. Since the body is in its initial state in equilibrium with the surrounding medium, which has a temperature t' and a pressure p', in order to cool the

body it is necessary first to compress it to a pressure $p_1 > p'$ and then cool it by heat exchange with the surrounding medium to a temperature t', after which it is subjected to adiabatic expansion to a pressure p. According to the general equation (3.18), the work that must be performed for this purpose is

$$-AL' = -I(p', t') + I(p, t) + T'[S(p', t') - S(p, t)] + \\ + T'\Delta S^*.$$

Adiabatic Mixing

We have already discussed in §2 of Chapter 3 the mixing of several substances which are at first in different parts of a volume V (the latter being assumed constant). Here we consider two other methods of mixing: mixing in a stream and mixing by filling a constant volume with a stream.

Main computation formulas. Any method of adiabatic mixing is subject to Eq. (2.5) or (2.11), which represent the first law of thermodynamics. Later on for convenience we shall regard the kinetic energy $E_{kin} = Gw^2/2g$ as an independent component of the total energy of the body. In this case Eqs. (2.5) and (2.11) assume the form

$$AL = U_1 - U_2 + A(E_{1кин} - E_{2кин});$$
$$AL' = I_1 - I_2 + A(E_{1кин} - E_{2кин}).$$

If two or several streams of different gases (or liquids) are mixed and no useful work is being done during the mixing, L' = 0, we get from Eq. (2.11)

$$\sum\left(I^{(j)} + G^{(j)}\frac{w^{(j)2}}{2g}\right) = I + G\frac{w^2}{2g}. \qquad (4.23)$$

where I and $\underline{w}$ are the enthalpy and the velocity of the mixture stream and $G = \Sigma G^{(j)}$.

If a constant volume containing one of the gases is filled by a stream of another gas, work of displacement is performed (the introduction of the stream of the first gas into the volume V) equal to

$-p_1^{(1)}V_1^{(1)}$, where $p_1^{(1)}$, $V_1^{(1)}$ are the pressure and volume of the first gas at the entrance into the volume V.

Therefore, we have on the basis of (2.5)

$$U_1^{(1)} + p_1^{(1)}V_1^1 + G^{(1)}\frac{w^{(1)2}}{2g} + U_1^{(2)} = U. \qquad (4.24)$$

The specific values of the internal energy and enthalpy of the gas mixture (including real gases) can be calculated from the general formula

$$\psi\left(t, \frac{V}{G}\right) = \sum \psi^{(j)}\left(t, \frac{V}{\mu^{(j)}M}\right)g^{(j)}. \qquad (4.25)$$

Here ψ is the sought parameter of the mixture (u or i) and $\psi^{(j)}(t, V/\mu^{(j)}M)$ is the value of this parameter for the j-th component when the latter is at a temperature equal to the mixture temperature t and has a specific volume equal to $V/\mu^{(j)}M$, where V is the volume of the mixture; M and G are the total number of moles and the weight of the mixture; $\mu^{(j)}$ and $\mu = G/M$ are the molecular weight of the j-th component and the apparent molecular weight of the mixture; $g^{(j)} = G^{(j)}/G$ is the concentration by weight of the j-th component of the mixture. The pressure p of the mixture is determined by Eq. (1.9).

The specific enthalpy of the mixture can be calculated from the formula

$$s\left(t, \frac{V}{G}\right) = \sum s^{(j)}\left(t, \frac{V}{\mu^{(j)}M}\right)g^{(j)} - \\ - \sum R^{(j)}g^{(j)}\left(\ln g^{(j)} - \ln\frac{\mu^{(j)}}{\mu}\right). \qquad (4.26)$$

The second term in Expression (4.26) represents the value of the mixing energy Δs_{sm} per kilogram of mixture, i.e., the increment in entropy, due to the irreversibility of the process in the adiabatic mixing of several gases having identical initial p and t.

Loss of useful work in mixing and theoretical work of mixture separation. Owing to the irreversible character of the adiabatic mix-

ing, the entropy of the mixture, and consequently the entropy of the entire system consisting of the mixture and the surrounding medium, increases by ΔS_{sm}. According to (3.21) this involves a loss of useful work equal to $T'\Delta S_{sm}$, where T' is the absolute temperature of the surrounding medium.

If it is required to separate the mixture (to separate the pure components), it is necessary to perform for this purpose useful work L' equal to the loss of useful work $T'\Delta S_{sm}$ during mixture. To verify this, let us examine in greater detail the process of separation of a mixture of gases. To separate the mixture we can make use of the difference in the flow velocity of the gases in rarefied state through partitions with small pores. The technical realization of the process reduces in this case to an adiabatic expansion by flow through the porous partitions and subsequent adiabatic compression and cooling of the separated components. Let us assume that the separation process is completely reversible. Then according to (2.11) the external work consumed in separating the mixture amounts to

$$AL' = Q^{(1)} + Q^{(2)} - [I - (I^{(1)} + I^{(2)})]$$

(we neglect the kinetic energy of the streams because it is small).

If the final gas parameters after separation are chosen such that the mixing of their streams results in a mixture having the same parameters as prior to separation, then according to Eq. (4.23) we have

$$I = I^{(1)} + I^{(2)}$$

and consequently

$$AL' = Q^{(1)} + Q^{(2)}.$$

The entropy of the entire system does not change here, owing to the reversibility of the separation process. Since the change in the entropy of the gases during the separation process is ΔS_{razd} and the change in the entropy of the surrounding medium is $-(Q^{(1)} + Q^{(2)})/T'$.

we obtain from the condition S* = const

$$\Delta S_{\text{разд}} - \frac{Q^{(1)} + Q^{(2)}}{T'} = 0.$$

But ΔS_{razd} is equal to the mixing entropy ΔS_{sm} taken with the opposite sign. Indeed, upon adiabatic mixture of the streams of the separated gases we obtain the initial mixture, so that the change in entropy of the gases as a result of the cyclic process "separation-mixing," $\Delta S_{razd} + \Delta S_{sm}$, vanishes, from which follows $\Delta S_{razd} = -\Delta S_{sm}$. The entropy increment of the entire system resulting from this cyclic process will be equal to

$$\Delta S^* = \Delta S - \frac{Q^{(1)} + Q^{(2)}}{T'}.$$

Since $\Delta S = 0$ and $(Q^{(1)} + Q^{(2)})/T' = -\Delta S_{sm}$, we get $\Delta S^* = \Delta S_{sm}$. Replacing $Q^{(1)} + Q^{(2)}$ by $-T'\Delta S'_{sm}$ in the expression for L' we get

$$L'_{\text{разд}} = -T'\Delta S'_{\text{см}}.$$

It is appropriate to note here that the process of mixing in a stream of gases, which is most frequently used in technical applications, is from the thermodynamic point of view completely analogous to the process of adiabatic throttling, inasmuch as the conditions $dQ = 0$, $dL' = 0$, $dI = 0$ are satisfied for both processes.

Polytropic Process

The process X = const, at each point of which the derivative $(\partial Q/\partial t)_X$ has a constant value, is called a polytropic process. In other words, the polytropic process is a process with constant specific heat. Particular cases of a polytropic process are the adiabatic process whose specific heat has a constant value equal to zero, the isothermal process whose specific heat is equal to $\pm\infty$, and also the processes V = const and p = const, provided the specific heat C_V and C_p are constant.

The quantity of heat Q which a body has obtained as a result of

the polytropic process is equal to the product of the specific heat of the process C_x by the temperature difference in the final and initial states

$$Q = C_X(t_1 - t_2).$$

The equation of a reversible polytropic process can be determined by using Expression (2.22) for the specific heat C_x

$$C_X = C_V + \left[Ap + \left(\frac{\partial U}{\partial V}\right)_T\right]\left(\frac{\partial V}{\partial T}\right)_X$$

and the equation of state.

Let us assume for simplicity that the polytropic process is performed by an ideal gas whose specific heat C_V (and consequently also C_p) is constant. Since $(\partial U/\partial V)_T = 0$ for an ideal gas, we get from (2.22)

$$C_X = C_V + Ap\left(\frac{\partial V}{\partial T}\right)_X.$$

Solving this equation with respect to $(\partial V/\partial T)_X$, we obtain

$$\left(\frac{\partial V}{\partial T}\right)_X = \frac{C_X - C_V}{Ap}$$

or, since we have from the Clapeyron-Mendeleyev equation $p = GRT/V$ with $AGR = C_p - C_V$ and $C_X/G = c_X$, where c_X is the per-unit specific heat of the given polytropic process, we get

$$\left(\frac{\partial V}{\partial T}\right)_X = \frac{c_X - c_V}{c_p - c_V} \cdot \frac{V}{T}. \qquad (4.27)$$

The equation obtained is the differential equation of the polytropic process in the variables $\underline{t}$ and V for an ideal gas. This equation can be readily integrated. Indeed, inasmuch as c_X, c_V, and c_p are constant, we have

$$VT^{-\frac{c_X - c_V}{c_p - c_V}} = V_1 T_1^{-\frac{c_X - c_V}{c_p - c_V}}.$$

Replacing the temperature T in this equation by pV/GR, we obtain

$$pV^{\frac{c_p - c_X}{c_V - c_X}} = p_1 V_1^{\frac{c_p - c_X}{c_V - c_X}}. \qquad (4.28)$$

The constant quantity $(c_p - c_x)/(c_v - c_x)$ is called the <u>polytropic exponent</u> and is designated by the letter <u>n</u>:

$$n = \frac{c_p - c_x}{c_v - c_x}. \qquad (4.29)$$

Each of the polytropic processes wherein the state of an ideal gas is changed is characterized by a completely defined value of <u>n</u>. Therefore the specific heat of the polytropic process is advantageously designated by c_n.

According to (4.29), the specific heat c_n of a polytropic process is connected with the specific heats c_V and c_p by the following relation:

$$c_n = \frac{nc_v - c_p}{n-1} = \frac{n-\kappa}{n-1} c_v.$$

We see from this equation that if $1 < n < \kappa$, then $c_n < 0$; when $n = \kappa$ we have $c_n = 0$, when $n < 1$ and $n > k$ we have $c_n > 0$.

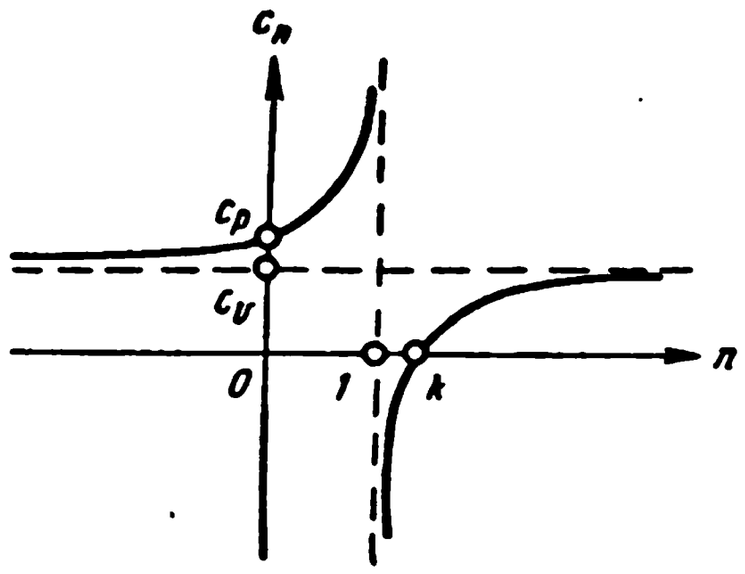

Fig. 22. Character of dependence of the specific heat c_n on the polytropic exponent <u>n</u>.

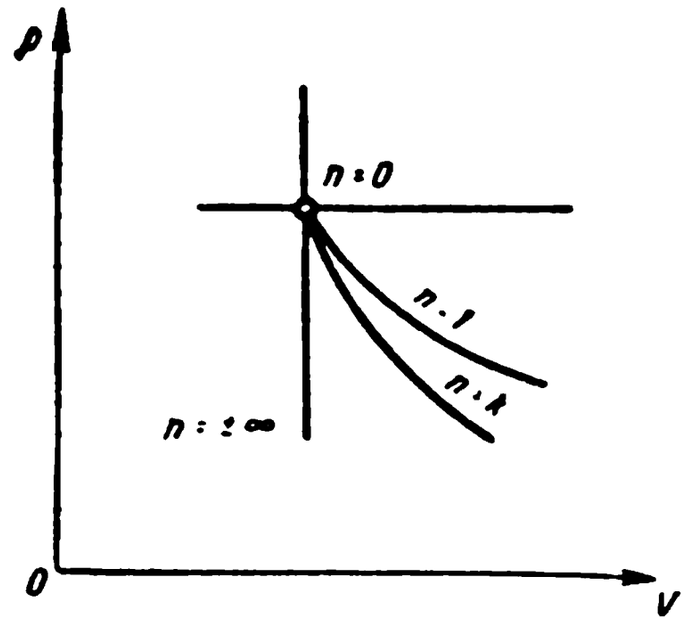

Fig. 23. Form of polytropic curves of an ideal gas with different values of the polytropic exponent <u>n</u>.

Figure 22 shows the character of the dependence of the specific heat c_n on the polytropic exponent <u>n</u>, while Fig. 23 shows in pV coordinates the variation of different polytropic curves of an ideal gas, corresponding to different polytropic exponents from $n = 0$ to $n = \pm\infty$. The isothermal process corresponds to the value $n = 1$, the adiabatic one to $n = \kappa$, the isochoric one to $n = \pm\infty$, and the isobaric process to $n = 0$.

The amount of heat Q received by the body in a polytropic process is, for an ideal gas,

$$Q = G\frac{n-\kappa}{n-1}c_V(T_2 - T_1).$$

Replacing T by pV/GR and bearing in mind that $AR = c_p - c_V$, $p_1V_1^n = p_2V_2^n$, we get

$$Q = -\frac{A(n-\kappa)}{(\kappa-1)(n-1)}p_1V_1\left[1-\left(\frac{p_2}{p_1}\right)^{\frac{n-1}{n}}\right]. \qquad (4.30)$$

From Eqs. (4.29) and (4.30) it follows that in the case of a polytropic expansion of an ideal gas, c_n, Q, and t_2 as functions of $\underline{n}$ have the following values

	c_n	Q	t_2
$0<n<1$	>0	>0	$>t_1$
$1<n<\kappa$	<0	>0	$<t_1$
$n=\kappa$	0	0	$<t_1$
$n>\kappa$	>0	<0	$<t_1$

The work L necessary to change the volume, performed by the body in a reversible polytropic process, has the following form for an ideal gas

$$\left.\begin{aligned} L &= \frac{p_1V_1}{n-1}\left[1-\left(\frac{p_2}{p_1}\right)^{\frac{n-1}{n}}\right], \\ L &= \frac{GR}{n-1}(T_1 - T_2). \end{aligned}\right\} \qquad (4.31)$$

while the useful external work of a polytropic process for an ideal gas is

$$\left.\begin{aligned} L' &= \frac{n}{n-1}p_1V_1\left[1-\left(\frac{p_2}{p_1}\right)^{\frac{n-1}{n}}\right], \\ L' &= \frac{n}{n-1}GR(T_1 - T_2). \end{aligned}\right\} \qquad (4.32)$$

The change in entropy in a polytropic process amounts to

$$S_2 - S_1 = Gc_n \ln\frac{T_2}{T_1}. \qquad (4.33)$$

It is seen from (4.33) that at points of two identical polytropic

curves (having equal n) corresponding to one and the same value of S, the ratio of the absolute temperatures, including also the average temperatures in a given interval of entropy variation, has one and the same value which is constant for the given lines.

§2. REVERSIBLE CYCLES

The Carnot Cycle

The simplest working cycle of the heat engine is a cycle with two sources of heat, the temperatures of which are not equal to each other. By a source of heat is meant a body (or a system of bodies of equal temperature) which can give up heat to other bodies or, on the contrary, which can receive heat from them. It is usually assumed that the temperature of the heat source itself does not change here. The source giving up heat to the working medium is called *the heat supply* and that receiving heat from the working medium is called *the heat receiver*. Usually the heat receiver is the surrounding medium (the atmosphere).

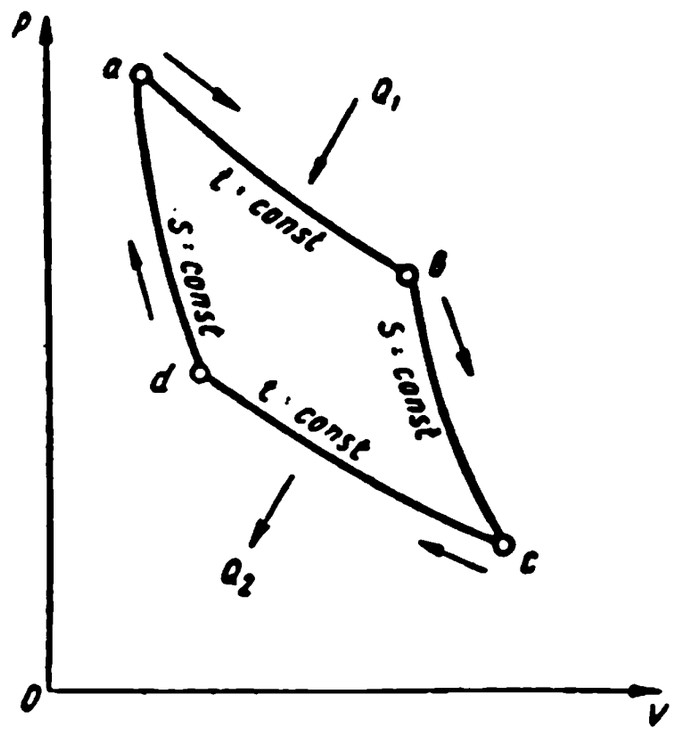

Fig. 24. Reversible cycle with two sources of heat (in pV coordinates).

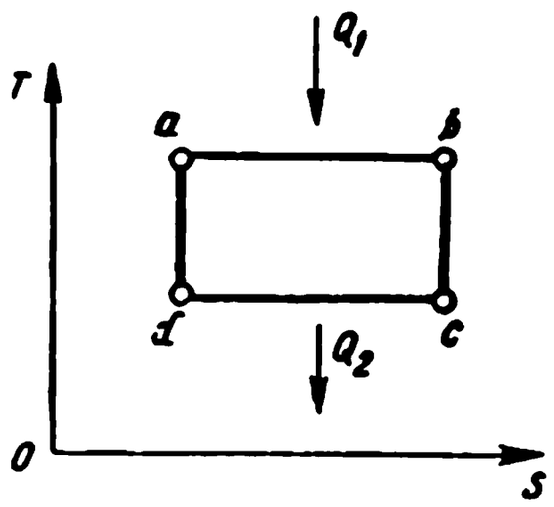

Fig. 25. Reversible cycle with two sources of heat (in TS coordinates).

In the cycle considered here, with two heat sources, the source having the higher temperature t_1 is the heat supply and the source with the lower temperature t_2 is the heat receiver.

Let us explain now how a reversible cycle can be realized with two heat sources. Since the temperature of any of the sources of heat does not change during the cycle, it is clear that in order to carry out the cycle reversibly the heat must be supplied from the sources to the working medium or else removed from the medium isothermally at the heat-source temperatures t_1 and t_2. Since there are no other heat sources, the change in temperature of the working medium from the temperature of the heat supply to the temperature of the heat receiver can occur only adiabatically; further, in order for the process to be reversible, it is necessary that it proceed relatively slowly, in quasistatic manner. Therefore any reversible cycle with two heat sources should consist of four reversible processes – two isothermal and two adiabatic, alternating with each other (Figs. 24 and 25).

The reversible cycle with two sources of heat is called <u>the reversible Carnot cycle</u>.

During one cycle the working medium performs positive work L equal numerically to the area enclosed within the cycle contour abcd. If the weight of the working medium amounts to G kg, then the work performed by 1 kg of the working medium (the specific work of the cycle) will be $l = L/G$ and the amount of heat supplied to 1 kg of working medium is $-q_1 = Q_1/G$.

The work L is performed on the working medium at the expense of the fraction of heat obtained by the working medium from the heat supply. On the upper isotherm, the working medium receives from the heat supply an amount of heat Q_1 which is larger than the amount of heat Q_2 which the working medium gives up to the heat receiver on the lower isotherm; the difference between the heats Q_1 and Q_2, in accordance with the first law of thermodynamics, will be precisely equal to the work L performed during the course of the cycle

$$AL = Q_1 - Q_2;$$
$$Al = q_1 - q_2;$$

where Q_1 and Q_2 are, respectively, equal to

$$Q_1 = T_1(S_b - S_a);$$
$$Q_2 = T_2(S_c - S_d),$$

or, recognizing that $S_c = S_b$, $S_d = S_a$,

$$Q_2 = T_2(S_b - S_a).$$

The reversible Carnot cycle can be performed both in the forward direction (clockwise) and in the opposite direction (counter-clockwise). The direct Carnot cycle, considered above, is called the heat cycle; it is the prototype of the operating cycles of various heat engines. The reverse Carnot cycle is called the refrigeration cycle and is used in heat pumps and refrigeration machinery.

From the expression for the work of the reverse Carnot cycle

$$-AL = -Q_1 + Q_2$$

it follows that in this cycle an amount of heat Q_2 is taken away from the heat source with the lower temperature and heat Q_1, amounting to $Q_2 + AL$ and larger than Q_2, is given up to the heat source with the higher temperature. To realize the reverse cycle it is necessary to consume external work AL equal to the difference $Q_1 - Q_2$.

The degree of profitable utilization of the heat in the reversible heat engine is characterized by the thermal efficiency of the cycle, designated η_t. It represents the ratio of the work L performed during one cycle to the amount of heat Q_1 which has been received during the cycle from the heat source by the working medium:

$$\eta_t = \frac{Al}{Q_1} = \frac{Q_1 - Q_2}{Q_1}.$$

Since $$AL = Q_1 - Q_2, \quad \dot{Q}_1 = T_1(\dot{S}_b - \dot{S}_a), \quad \dot{Q}_2 = \dot{T}_2(S_b - \dot{S}_a),$$

we have

$$\eta_t = \frac{T_1 - T_2}{T_1}. \qquad (4.34)$$

The thermal efficiency of the reversible Carnot cycle is numerically equal to the ratio of the difference in the absolute temperatures of the heat source and heat receiver $(T_1 - T_2)$ to the absolute temperature of the heat source T_1.

It follows further from (4.34) that the thermal efficiency of the reversible Carnot cycle does not depend on the nature of the working body and is a function of the heat-source temperatures only. This statement constitutes the contents of the Carnot theorem.

Arbitrary Reversible Cycle

To realize a reversible Carnot cycle it is necessary to have a minimum number of heat sources of different temperature, two being

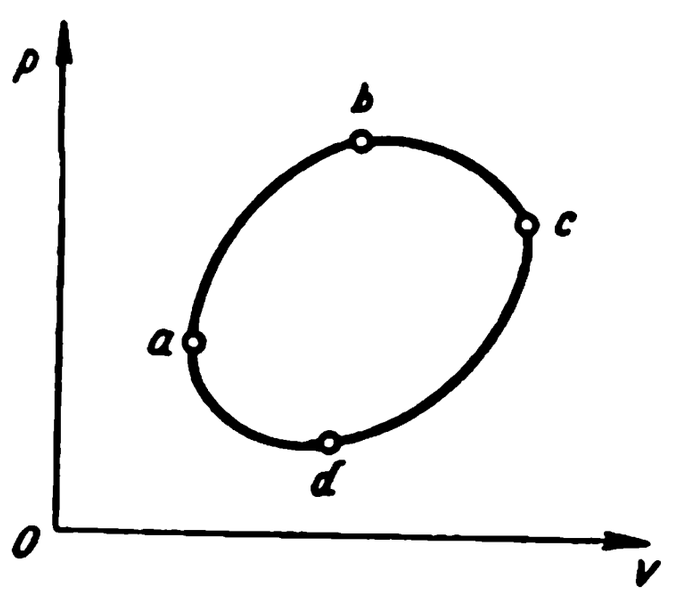

Fig. 26. Arbitrary reversible cycle.

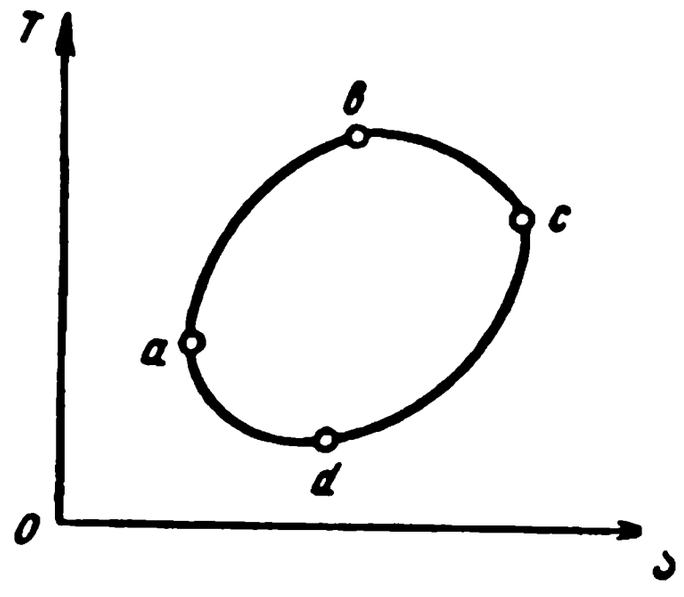

Fig. 27. Heat-supply (abc) and heat-removal (cda) sections.

enough. To realize a reversible cycle of more complicated form, it is necessary to have not two but a set of heat sources of different temperatures. In the most general case of an arbitrary reversible cycle abcd (Figs. 26 and 27), it is necessary to have an infinitely large number of heat sources. The useful work of an arbitrary reversible cycle is

$$AL = Q_1 - Q_2$$

and is numerically equivalent to the area of the cycle abcd. Here Q_1 is the total amount of heat obtained by the working medium from all the heat supplies and Q_2 is the amount of heat given up by the work-

ing medium to all the heat receivers. Q_1 and Q_2 are, respectively, equal to

$$Q_1 = \int_{(A)} dQ = \int_{abc} T\,dS;$$
$$Q_2 = \int_{(B)} dO = \int_{adc} T dS.$$

In the first integral, the integration is over that part of the contour (A) = abc of the cycle abcd which corresponds to the interaction between the working medium and the heat supplies (and where, consequently, dQ > 0 and dS > 0), while in the second integral the integration is over the part of the contour (B) = adc, where the working medium gives up heat to the receivers (with dQ < 0, dS < 0).

The thermal efficiency of the cycle abcd is by definition

$$\eta_t = 1 - \frac{Q_2}{Q_1}$$

or

$$\eta_t = 1 - \frac{\int_{adc} T\,dS}{\int_{abc} T\,dS}. \qquad (4.34)$$

Let us denote the maximum temperature in the cycle abcd by T_{maks} and the minimum temperature by T_{min}. Then, recognizing that

$$\int_{abc} T\,dS < T_{макс}(S_c - S_a),$$

$$\int_{adc} T\,dS > T_{мин.}(S_c - S_a)$$

and consequently

$$\frac{\int_{adc} T\,dS}{\int_{abc} T\,dS} > \frac{T_{мин}}{T_{макс}},$$

we obtain

$$\eta_t < 1 - \frac{T_{мин}}{T_{макс}}. \qquad (4.35)$$

<u>The thermal efficiency of an arbitrary reversible cycle cannot exceed the thermal efficiency of the reversible Carnot cycle produced between</u>

the maximum and minimum temperatures of the given cycle.

Expression (4.34) for the thermal efficiency of an arbitrary reversible cycle can be recast in a form analogous to the thermal efficiency of the Carnot cycle by using the average temperatures of heat supply and heat removal (the average temperatures on the curves abc and adc of the cycle). From the definitions of the average temperatures we have

$$T_{\text{подв}}^{\text{ср}} = \frac{1}{S_c - S_a} \int_{abc} T\,dS,$$

$$T_{\text{отв}}^{\text{ср}} = \frac{1}{S_c - S_a} \int_{adc} T\,dS.$$

Replacing in (4.34) the integrals $\int_{abc} TdS$ and $\int_{adc} TdS$ by the corresponding average temperatures multiplied by the entropy difference $S_c - S_a$, we obtain

$$\eta_t = 1 - \frac{T_{\text{отв}}^{\text{ср}}}{T_{\text{подв}}^{\text{ср}}}. \qquad (4.36)$$

§3. REGENERATION OF HEAT IN CYCLES

Generalized Carnot Cycle

Using the TS diagram, we can readily show that there exist reversible cycles which differ in their configuration from the Carnot cycle, but nonetheless have the same thermal efficiency. Assume that there is a cycle abcd (Fig. 28) consisting of two isotherms ab, dc and two equidistant lines* bc and ad. In this cycle the working medium expands isothermally from the point a to the point b, absorbing thereby an amount of heat $Q_1 = T_1 (S_b - S_a)$ from the heat supply, and then changes its state along the line bc, giving up at the same time a certain amount of heat Q_{bc} numerically equal to the area be'c'cb. To realize a reversible transition of the working medium

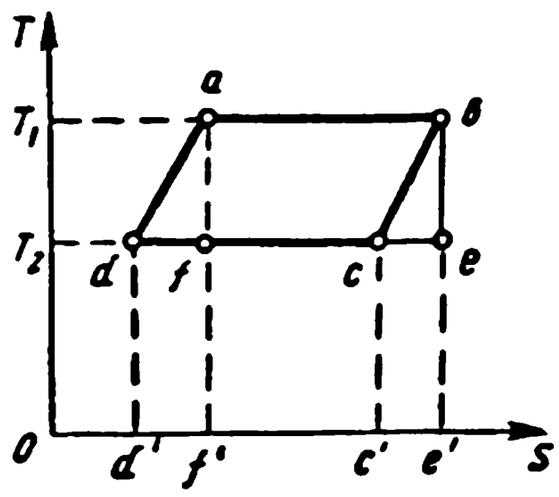

Fig. 28. Generalized Carnot cycle.

from the point b to the point c it is necessary to have a set of heat sources, whose temperatures should comprise a continuous series from T_1 to T_2. These will be called the intermediate sources from now on. Upon reaching the point c, the working medium is isothermally compressed, giving up at the same time an amount of heat $Q_2 = T_2 (S_c - S_d)$ to the heat receiver, after which it returns along the line da to the state a. In the reversible transition from the point d to the point a, the working medium absorbs an amount of heat Q_{da} equal to the area af'd'da. Because the lines bc and ad are equidistant, the areas be'c'cb and af'd'dc are equal to each other and consequently the amounts of heat Q_{bc} and Q_{da} have the same absolute value.

Since the temperatures at the points c and d, and also at the points a and b, have equal values, the heat supplies necessary to realize the reversible transition along the line da can be the same heat sources as were used previously along the line bc as heat receivers. Each of these heat sources gives up to the working medium, as the state of the latter changes along the line da, precisely the same amount of heat which it received previously from the working medium on the line bc and, consequently, it does not give up or receive an excessive amount of heat as a result of the cycle. All these intermediate heat sources are merely regenerators of heat which obtain heat from the working medium in one part of the cycle and then give up the same amount to the working medium on a different part of the cycle. The use of the heat released in one of the parts of the cycle to realize the process on another part of the cycle, at the same temperature, is called regeneration of heat in a cycle.

In the considered cycle abcd with equidistant lines, the real or true sources of heat are only the heat supply with temperature t_1 and the heat receiver with temperature t_2. The work of the cycle is per-

formed at the expense of the heat Q_1 given up by the heat supply to the working medium, part of which, equal to $Q_1 - Q_2$, is transformed into work, while the other part, equal to Q_2, is given up in the form of heat to the heat receiver at a temperature t_2. Indeed,

$$AL = Q_1 - Q_{bc} - Q_2 + Q_{da}$$

or, since $Q_{bc} = Q_{da}$,

$$AL = Q_1 - Q_2.$$

The thermal efficiency of the cycle is

$$\eta_t = 1 - \frac{Q_2}{Q_1},$$

where

$$Q_1 = T_1(S_b - S_a),$$
$$Q_2 = T_2(S_c - S_d).$$

Owing to the fact that the lines bc and ad are equidistant

$$S_b - S_a = S_c - S_d,$$

and therefore

$$\eta_t = \frac{T_1 - T_2}{T_1}.$$

Thus, the thermal efficiency of a reversible cycle realized between two sources of heat with temperatures T_1 and T_2 with transition from the temperature T_1 to T_2 and back along the equidistant lines (with regeneration of heat) is equal to the thermal efficiency of the Carnot cycle.

A reversible cycle consisting of two isothermal and two equidistant lines is called the <u>generalized Carnot cycle</u>.

The ordinary Carnot cycle, owing to the equidistant nature of the adiabatic lines (isentropes), can be regarded as a particular case of the generalized Carnot cycle characterized by a zero value of regenerated amount of heat.

Arbitrary Cycle with Heat Regeneration

Let us consider now an arbitrary reversible cycle abcd (Fig. 29). The working temperature drop and entropy drop in this cycle are, respectively, $T_1 - T_2$ and $S_c - S_a$. Heat is supplied to the working medium on the upper curve abc of the cycle and heat is removed on the lower curve adc. On the sections ak and cr, the heat is supplied and removed at identical temperatures. Because of this, the heat given up by the working medium on the section cr can be used to realize the process on the section ak, i.e., heat regeneration is possible in the cycle. It is seen on Fig. 29 that the process af can be completely realized as a result of the excess heat released on the section f'r, while the process fk can be partially realized by using the heat from section cf'. Therefore the supply of heat to the working medium from external sources must be carried out not starting with the point a, but with the point f, corresponding to the higher temperature. The position of the point f is determined by the condition that the derivative dT/dS have the same value on both curves of the cycle at $T = T_f$. Therefore the amount of heat supplied to the working medium from the heat supplies is

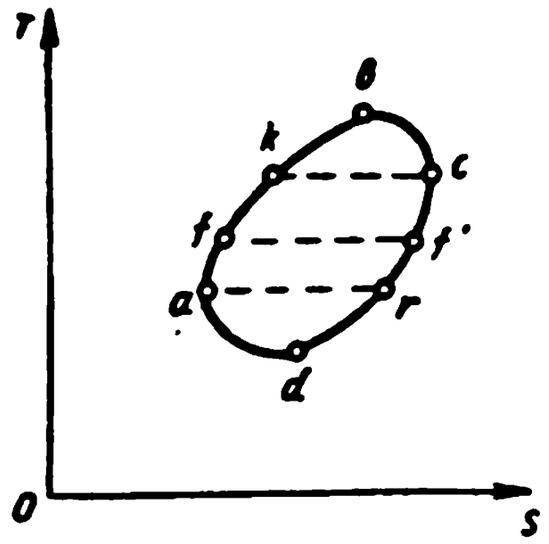

Fig. 29. Decrease in the heat-supply section as a result of regeneration.

$$Q_1 = \int_{abc} TdS - \int_{af} T\,dS - \int_{f'c} T\,dS = \int_{fbf'} TdS,$$

and the amount of heat given up by the working medium to the heat receivers is

$$Q_2 = \int_{adc} TdS - \int_{af} TdS - \int_{f'c} TdS = \int_{fdf'} TdS.$$

The useful work and the thermal efficiency of the cycle are

$$AL = \int_{1bf'} TdS - \int_{1af'} TdS = \oint_{abcd} TdS;$$

$$\eta_{t_{per}} = 1 - \frac{\int_{1af'} TdS}{\int_{1bf'} TdS}. \qquad (4.37)$$

Recognizing that

$$\frac{\int_{1af'} TdS}{\int_{1bf'} TdS} < \frac{\int_{adc} TdS}{\int_{abc} TdS},$$

with

$$\int_{1df'} TdS > T_{мин}(S_{f'} - S_f)$$

and

$$\int_{1bf'} TdS < T_{макс}(S_{f'} - S_f),$$

we obtain

$$\frac{T_{макс} - T_{мин}}{T_{макс}} > \eta_{t_{per}} > \eta_t,$$

i.e., the thermal efficiency of the regenerative cycle is always larger than the efficiency of the cycle without heat regeneration, but smaller than the thermal efficiency of the Carnot cycle realized between the maximum and minimum temperatures of the cycle.

Regeneration of heat in reversible cycles brings each of these cycles closer to the Carnot cycle. Therefore in all cases when this is practically feasible, it is advantageous to improve the cycle by modifying its form in such a way that heat regeneration is possible in it. Thus, for example, in the cycle abcd (Fig. 30) it is necessary for this purpose to change the process bc into the process bc', which is equidistant to ad. Since in both cycles abcd and abc'd the average temperature at which heat is removed is the same and equal to T_2 and the average temperature at which heat is supplied in the cycle abc'd, which is equal to $Q_{ab}/(S_b - S_a)$, is higher than in the cycle abcd [inasmuch as $Q_{ab}/(S_b - S_a) > Q_{dab}/(S_b - S_d)$], the thermal efficiency of

the cycle abc'd will be larger than in the cycle abcd, namely $\eta_{t_{reg}} > \eta_t$.

§4. IRREVERSIBLE CYCLES

Let us consider an irreversible cycle abcd (Fig. 31) with two heat sources having temperatures t_1 and t_2. Were the processes in the cycle reversible, then we could realize a Carnot cycle which, for the same amount of heat supply, would be represented on the TS diagram by the rectangle ABCD. The irreversible cycle abcd differs in shape from the ideal reversible cycle ABCD. As a result of the irreversibility of the processes, the external form of the cycle changes: the cycle becomes compressed (either everywhere or on some portions in a vertical direction), inasmuch as the temperature of the working medium is lower in the irreversible heat exchange than the temperature of the heat supply and is higher than the temperature of the heat receiver, and shifts horizontally to the right owing to the larger increase in entropy of the working medium in the irreversible process. The deformation of the shape of the cycle is all the stronger, the larger the irreversibility of the processes of the cycle. In Fig. 31 the processes of supply and removal of heat are represented by the continuous lines ab and cd, inasmuch as these processes are externally irreversible; the remaining processes of the cycle, being irreversible, are designated by the dashed lines. Let us assume that in the cycles abcd and ABCD the same amount of heat Q_1 is fed to the working medium; then

$$\int_{ab} TdS = T_1(S_B - S_A).$$

The amount of heat given up by the working medium to the heat receiver in the cycle abcd is

$$Q_2 = \int_{dc} TdS = T_2(S_E - S_A).$$

and in the reversible cycle ABCD

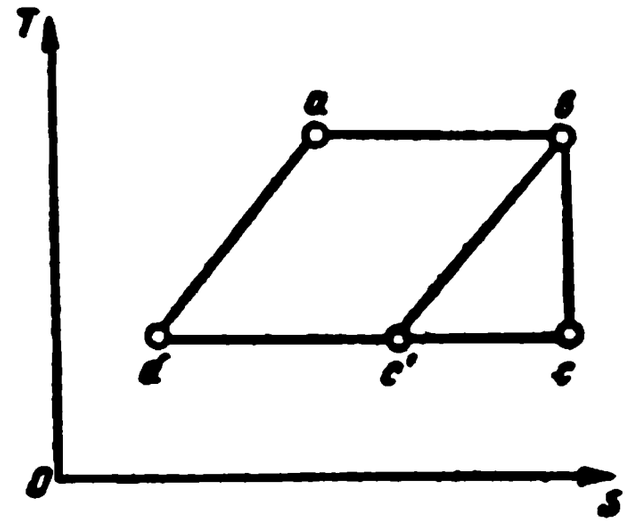

Fig. 30. Change in form of the cycle upon introduction of heat regeneration.

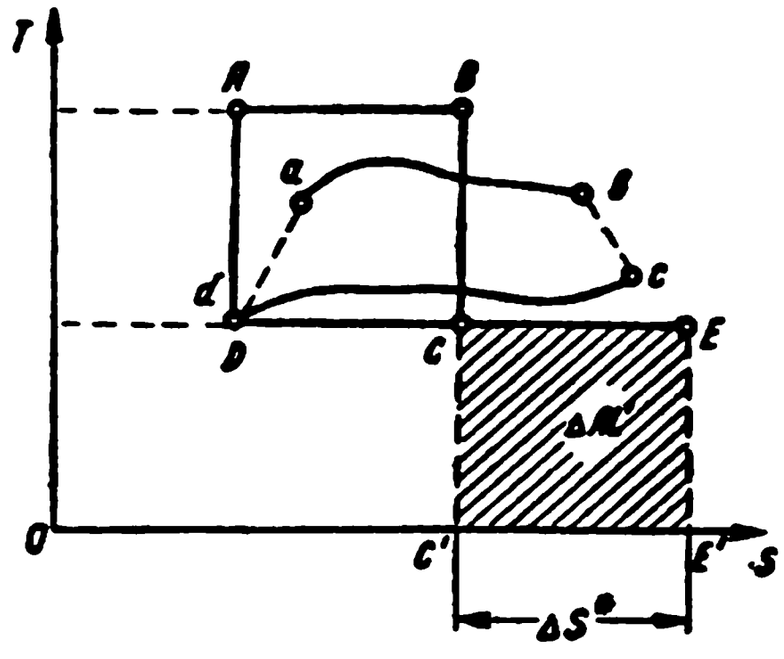

Fig. 31. Irreversible cycle with two sources of heat.

$$Q_{2\,обр} = T_2(S_B - S_A).$$

i.e., Q_2 exceeds $Q_{2\ obr}$ by an amount $T_2\ (S_E - S_C)$.

The difference in entropy at the points E and C represents, as can be readily verified, an increment in the entropy ΔS^* of the entire system as a whole during one cycle. Indeed, ΔS^* is equal to the sum of the changes in the entropy of the heat supply, heat receiver, and the working medium. Inasmuch as the working medium returns to the initial state at the end of the cycle, its change of entropy is zero. The change of entropy of the heat supply is $-Q_1/T_1$, while that of the heat receiver (the surrounding medium) is Q_2/T_2. Therefore

$$\Delta S^* = -\frac{Q_1}{T_1} + \frac{Q_2}{T_2}.$$

Since

$$Q_2 = Q_{2\,обр} + T_2(S_E - S_C),$$

we have

$$\Delta S^* = -\frac{Q_1}{T_1} + \frac{Q_{2\,обр}}{T_2} + S_E - S_C.$$

But in the irreversible cycle ABCD we have

$$-\frac{Q_1}{T_1} + \frac{Q_{2\,обр}}{T_2} = 0$$

and consequently

$$\Delta S^* = S_E - S_C.$$

From this follows the following simple method of graphical determination of the increment of the entropy of the system in any irreversible cycle from the known values of Q_1 and Q_2. A horizontal line (the isotherm $T = T_2$) is drawn in the TS plane at a height T_2 and on it is marked the point D of the initial state of the working medium, as well as the point C of the start of the process of heat removal in the reversible Carnot cycle with the same Q_1. One then determines on this isotherm the point E defined by the condition $T_2 (S_E - S_D) = Q_2$; the length of the segment EC is the sought increment of the entropy of the system, ΔS^*.

This device is of most general significance and can be used in those cases when it is necessary to calculate the entropy increment ΔS^*_j of a system resulting from the irreversibility of some definite process contained in the cycle. Then, assuming all the other processes of the cycle to be reversible, it is easy to determine from the initial and final points of the process under consideration the value of ΔS^*_j. The over-all increment of the system entropy ΔS^* over the entire cycle will be equal to the sum of the increments of entropy due to each of the component processes

$$\Delta S^* = \sum \Delta S^*_j.$$

The useful work $AL = Q_1 - Q_2$ in an irreversible cycle abcd will be

$$AL = Q_1 - Q_{2\,обр} - T_2 (S_E - S_C), \qquad (4.38)$$

i.e.,

$$AL = AL_{макс} - T_2 \Delta S^*$$

in full agreement with the general formula (3.21).

Thus, the useful work of an irreversible cycle is smaller than the work of a reversible cycle by an amount equal to the absolute temperature of the heat receiver (surrounding medium) multiplied by the

increment in entropy of the entire system.

The loss of useful work $\Delta AL' = T_2\Delta S^*$ is represented graphically by the area CEE'C'. Since L_{maks} is in this case the useful work of the reversible Carnot cycle ABCD, equal to $Q_1\,[1 - (T_2/T_1)]$, the expression for L can be represented in the form

$$AL = Q_1\left[1 - \frac{T_2}{T_1}\left(1 + \frac{T_1\Delta S^*}{Q_1}\right)\right]. \qquad (4.39)$$

Then the effective efficiency $\eta_e = \Delta L/Q_1$ of the irreversible cycle abcd is

$$\eta_e = 1 - \frac{T_2}{T_1}\left(1 + \frac{T_1\Delta S^*}{Q_1}\right). \qquad (4.40)$$

It is seen from this equation that the effective efficiency is less than the thermal efficiency of the Carnot cycle by an amount

$$\Delta\eta = \frac{T_2\Delta S^*}{Q_1}.$$

The decrease in cycle efficiency due to the irreversibility is numerically equal to the product of the absolute temperature of the heat receiver by the relative increments of the entropy of the system divided by the amount of heat obtained by the working medium from the heat supply.

Practical Significance of Heat Regeneration

In real heat engines, the greatest deviation from reversibility is connected with the transfer of heat from the heat supply to the working medium, the temperature of which, as a rule, is much lower than the temperature of the heat supply.

The cycle of a real heat engine, in which the process of supply and removal of heat is regarded as irreversible and all the remaining processes are considered reversible (externally irreversible cycle of the given engine), represents the theoretical cycle of the engine.

The harmful influence of the irreversibility of the heat supply

process in the cycle can be decreased by regenerative heating of the working medium.

To verify this, let us consider a heat engine in which the temperature of the heat supply changes during the course of the heat supply. Assume that the externally irreversible cycle of this engine is abcd (Fig. 32); the heat from the supply is received on the line AB, while the heat is supplied to the working medium along the line dab.

In the cycle abcd we have

$$Q_1 = \int_{AB} TdS_{T_0} = \int_{dab} TdS, \qquad Q_2 = \int_{dc} TdS.$$

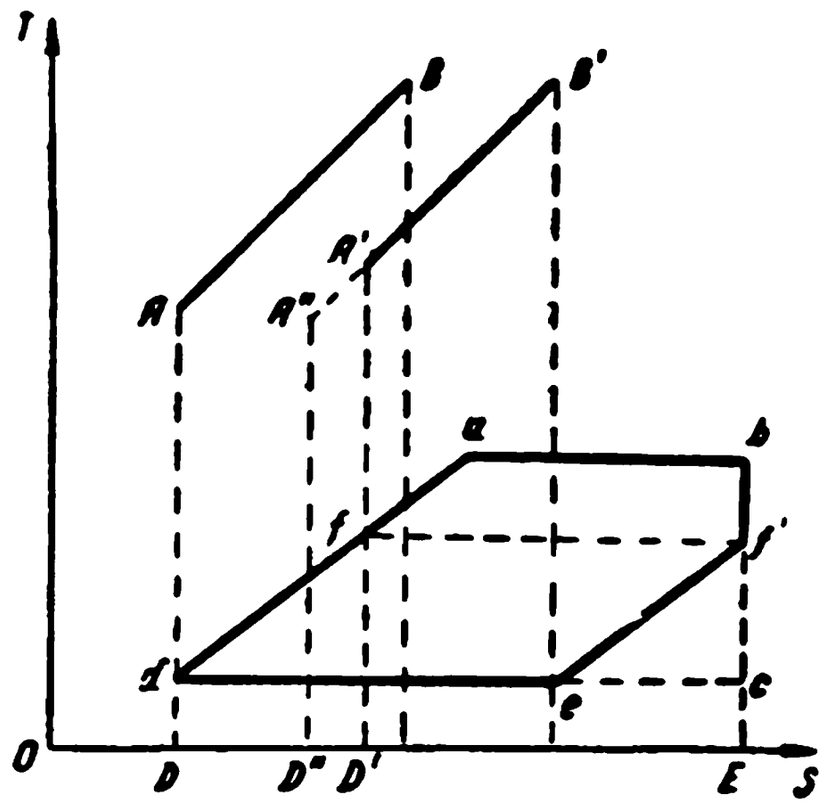

Fig. 32. Regeneration of heat in a cycle with a heat supply of variable temperature.

Let us effect now in the cycle abcd regeneration of heat, for which purpose we replace the portion f'c by the portion f'e. Then, as a result of the heat gathered in the portion f'e we can completely heat the working medium from the point d to the point f. As a result of the regeneration of the heat, the section df of the irreversible heat exchange between the heat supply and the working medium will be completely eliminated; the corresponding amount of heat Q_1 which must be supplied to the working medium from the heat supply decreases by an amount $\int_{df} TdS$ and becomes equal to $Q_{1_{obr}}$ of the fully reversible cycle of the same configuration as the externally reversible cycle.

The foregoing explains the practical significance of heat regeneration. The use of heat regeneration in real heat engines makes it possible to decrease the irreversibility of the cycle due to the finite temperature difference between the heat supply and the working medium

on transfer of heat from the former to the latter. Regenerative heating of the working medium eliminates (in some portions of the cycle completely and in others partially) the irreversible heat exchange and decreases the temperature difference between the heat supply and the working medium. As a result of this, the entropy increment ΔS^* of the system decreases. On the other hand, as a result of the regeneration, a smaller amount of heat Q_1 is necessary. At the same time $(T_2\Delta S^*/Q_1)_{reg} < T_2\Delta S^*/Q_1$, so that the effective efficiency, in accordance with (4.40), increases. Indeed, it is seen from Fig. 32 that

$$Q_1 - Q_{1\,per} = \int_{df} T\,dS = T^{cp}_{df}(S_f - S_d)$$

which represents the difference in areas under the lines AB and AB', or the area A"A'D'D", numerically equal to $T^{sr}_{A"A'}\ (S_{A'} - S_{A"})$. Therefore

$$T^{cp}_{A"A'}(S_{A'} - S_{A"}) = T^{cp}_{df}(S_f - S_d).$$

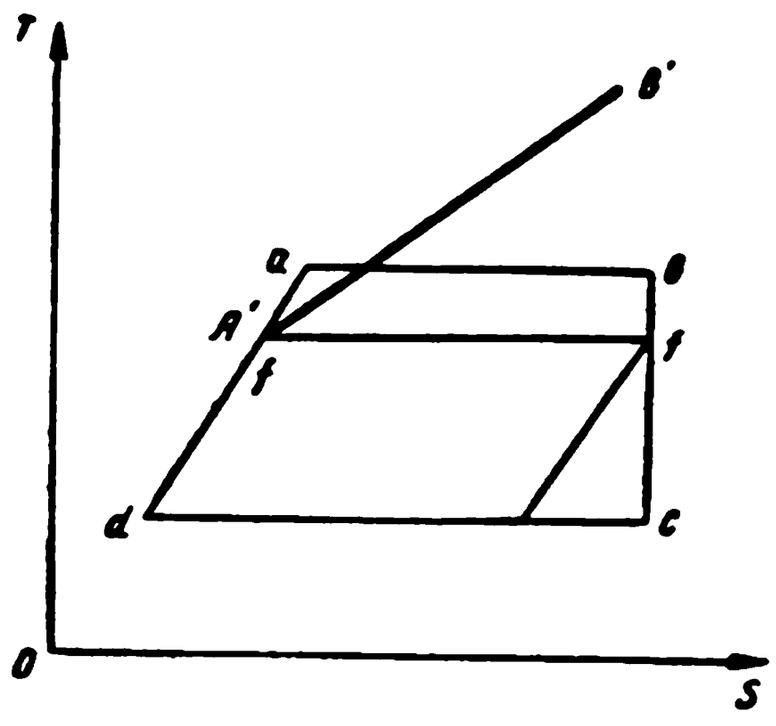

Fig. 33. Decrease in average heat-supply temperature as a result of regeneration at low heat-supply temperatures.

The energy increment of the system ΔS^*_{reg} in the regenerative cycle is $S_E - S_{B'}$. But

$$S_E - S_{B'} = (S_E - S_B) - (S_{B'} - S_B),$$

with

$$S_{B'} - S_B = S_{A'} - S_A = S_f - S_d - \frac{T^{cp}_{df}(S_f - S_d)}{T^{cp}_{A"A'}}.$$

Inasmuch as

$$S_E - S_B = \Delta S^*,$$

we have

$$\Delta S^*_{per} = \Delta S^* + \left(1 - \frac{T^{cp}_{df}}{T^{cp}_{A"A'}}\right)(S_f - S_d).$$

It follows therefore that

$$\left(\frac{T_2\Delta S^*}{Q_1}\right)_{пот} = \frac{T_2\Delta S^* - T_2\left(\frac{1}{T_{дj}^{ср}} - \frac{1}{T_{A'A}^{ср}}\right)T_{дj}^{ср}(S_j - S_д)}{Q_1 - T_{дj}^{ср}(S_j - S_д)}.$$

At sufficiently high temperatures at the point A', when

$$T_{A'A}^{ср} > \frac{T_{дj}^{ср}}{1 - \frac{T_{дj}^{ср}\Delta S^*}{Q_1}} \cdot \left(\frac{T_2\Delta S^*}{Q_1}\right)_{пот}$$

is smaller than $T_2\Delta S^*/Q_1$ and the effective efficiency is increased as a result of the heat regeneration. At relatively low values of $T_{A'}$, the heat regeneration can lead not to an increase but to a decrease in the effective efficiency, resulting from the decrease in temperature on the line ab (Fig. 33), and consequently the reduction of the average temperature of heat supply.

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[Footnote]

127 Equidistant lines are defined as a family of lines satisfying the condition dS/dT = idem when T = idem, i.e., having identical slopes for identical temperatures. In particular, all processes with identical specific heat are represented by equidistant lines.

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[List of Transliterated Symbols]

115 кин = kin = kineticheskiy = kinetic

116 см = sm = smesheniye = mixing

117 разд = razd = razdeleniye = separation

126 макс = maks = maksimal'nyy = maximum

126 мин = min = minimal'nyy = minimum

127 ср = sr = sredniy = average

127 подв = podvod = supply

127 отв = otv = otvod = removal

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[List of Transliterated Symbols]

131 per = reg = regenerativnyy = regenerative

133 обр = obr = obratimyy = reversible

Chapter 5

COMPRESSORS AND PISTON TYPE INTERNAL COMBUSTION ENGINES

§1. COMPRESSORS

A compressor is a continuously acting engine intended for the compression (increase in pressure) of a gas. Depending on the operating principle used for the compression of the gas, compressors are divided into two groups.

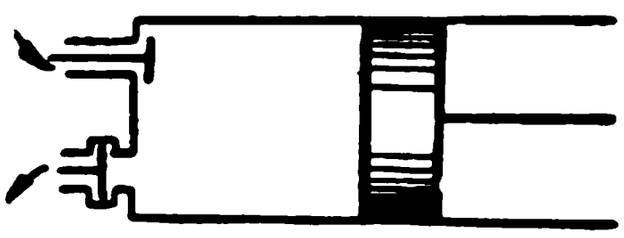

Fig. 34. Diagram of piston compressor.

The first group includes piston compressors (Fig. 34), in which the gas is compressed in a cylinder by means of a piston and after reaching the required pressure is displaced into a high-pressure reservoir.

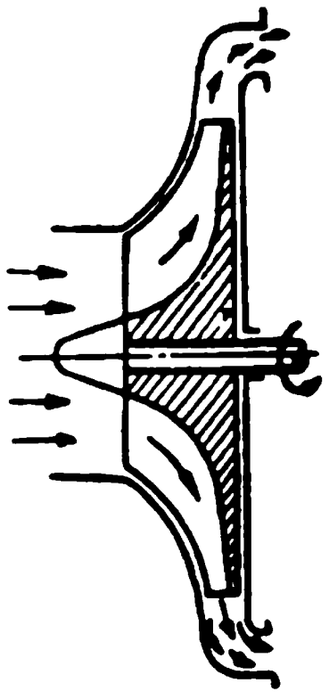

Fig. 35. Diagram of a turbine compressor.

The second group comprises different centrifugal compressors (turbine compressors) and axial compressors (Fig. 35), and also compressors of injection action, in which the compression is effected in two stages. The first stage consists in imparting to the entire gas as a unit a certain velocity and the second consists in converting the kinetic energy of the gas stream into pressure energy.

Compressors of both groups can differ greatly from one another in the structural data and the technical-economic characteristics, but from the thermodynamic point of view the processes occurring in them are perfectly equivalent and are described by the same equations.

. of Compression

The task of thermodynamic analysis is to determine the work which must be expended in order to obtain 1 kg of compressed gas for specified initial and final gas parameters. According to the general equation (2.11), the external work l' which must be expended to compress 1 kg of gas has the form

$$Al' = q_{1-2} - (i_2 - i_1). \qquad (5.1)$$

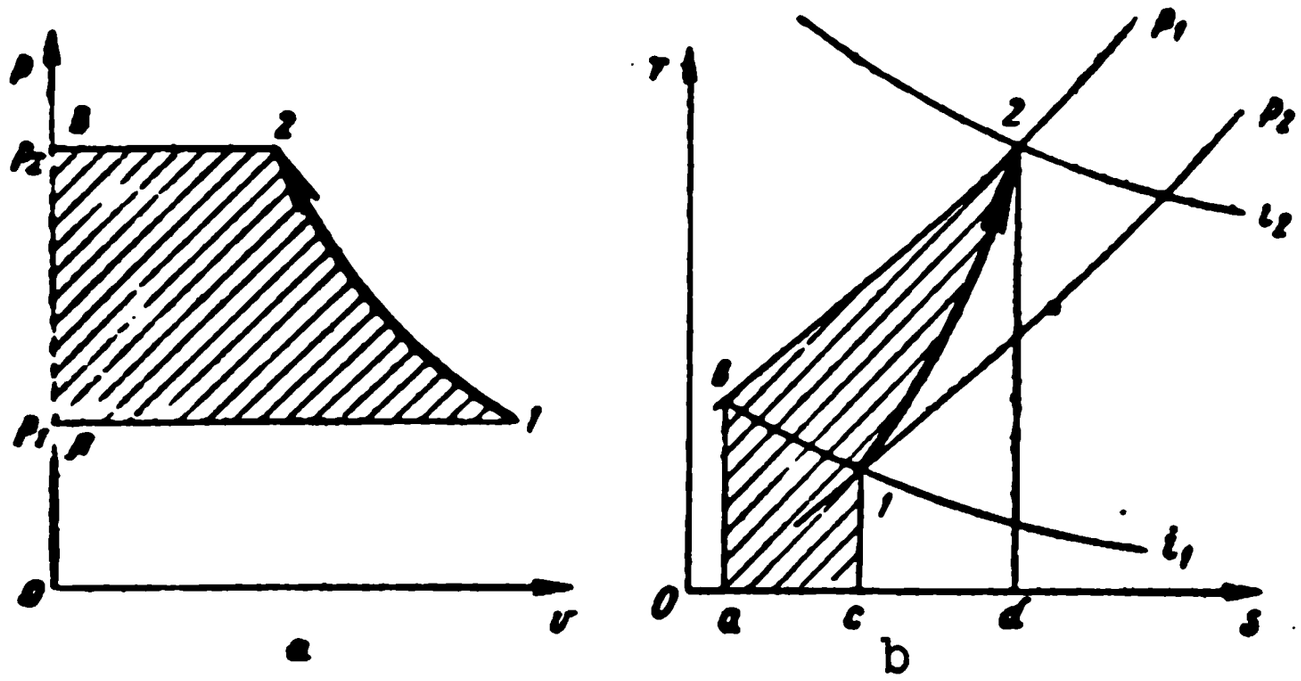

Fig. 36. Graphical representation of the theoretical work of compression.

The theoretical work of compression, i.e., the work that must be expended by the external source of work to obtain 1 kg of compressed gas under the condition that all the processes in the compressor are reversible and the increment of the kinetic energy of the gas is zero (the velocity of the gas at the input to the compressor and at the output are the same), will have on the basis of (1.16) the form

$$l'_{теор} = -\int_{p_1}^{p_2} v dp. \qquad (5.2)$$

It is represented on the pv diagram (Fig. 36a) by the area AB21 bounded by the curve of the compression 1-2 and the ordinate axis. Since q_{1-2} is represented on the Ts diagram (see Fig. 36b) by the area c12d and $i_2 - i_1$, which is equal to q_p, is represented by the area ab2d, the theoretical work of compression $Al'_{теор}$ is represented in

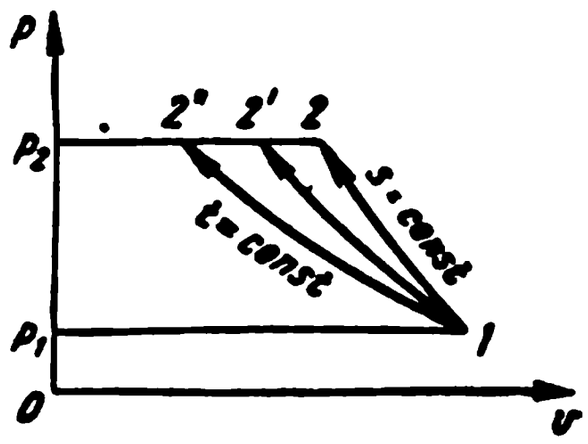

Fig. 37. Adiabatic and polytropic processes of compression (for $n < \kappa$).

accordance with Eq. (5.1) by the area ab2lc (shown shaded in Fig. 36b).

Let line 1-2 on the pv diagram (Fig. 37) represent an adiabatic compression process and let line 1-2' represent a polytropic compression process with $n < \kappa$, i.e., with heat removed from the compressed gas (for an air compressor, for example, $n = 1.2\text{-}1.25$). From the relative location of these lines we see that in adiabatic compression of the gas more work is consumed than in polytropic compression. The least work is consumed in the isothermal compression process 1-2", which is therefore the most suitable.

Heat is removed from the compressed gas usually by water flowing between the double walls of the cylinder.

Let us calculate now the theoretical work of compression as a function of the compression conditions, considering the gas to be ideal.

Cooled compressor with isothermal compression. According to (5.2), the theoretical work of isothermal compression (at temperature t_1) has the form

$$l_{t\,\text{теор}} = -\int_{p_1}^{p_2} \frac{RT_1}{p}\,dp = -RT_1 \ln \frac{p_2}{p_1}. \qquad (5.3)$$

To realize isothermal compression it is necessary to take away from the gas a quantity q of heat, as determined by Eq. (5.1). For an ideal gas

$$q_t = -RT_1 \ln \frac{p_2}{p_1}. \qquad (5.4)$$

Cooled compressor with polytropic compression. The theoretical work of compression has on the basis of (4.32) the form

$$l'_{n\,\text{теор}} = -\frac{n}{n-1} p_1 v_1 \left[\left(\frac{p_2}{p_1}\right)^{\frac{n-1}{n}} - 1\right]. \qquad (5.5)$$

Since $\kappa > n > 1$, it is clear that $-l'_n > -l'_t$. The amount of heat removed is in accord with (4.30)

$$q_n = -\frac{n-\kappa}{(\kappa-1)(n-1)} A p_1 v_1 \left[\left(\frac{p_2}{p_1}\right)^{\frac{n-1}{n}} - 1\right]. \qquad (5.6)$$

Uncooled compressor. The theoretical work of compression corresponding to the reversible adiabatic process in the compressor is

$$l'_{S\,\text{теор}} = -\frac{\kappa}{\kappa-1} p_1 v_1 \left[\left(\frac{p_2}{p_1}\right)^{\frac{\kappa-1}{\kappa}} - 1\right]. \qquad (5.7)$$

Compressor Delivery

The amount of low-pressure gas taken in by a compressor per unit time (or the amount of high-pressure gas delivered by the compressor per unit time) is called the compressor delivery. The compressor delivery is expressed in units of gas volume, for example normal cubic meters per hour or per minute.

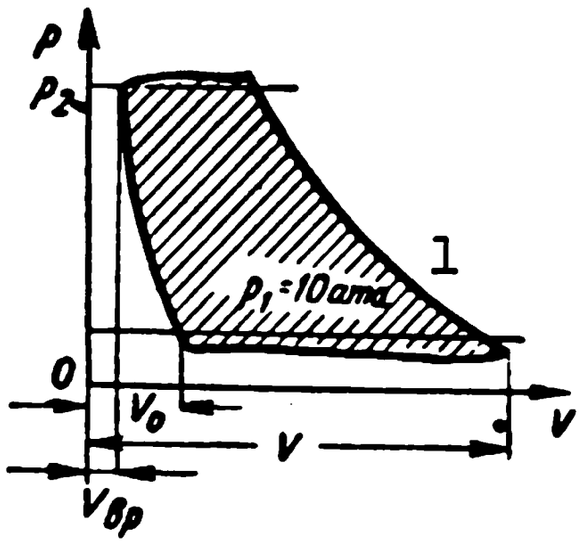

Fig. 38. Actual indicator diagram of compressor. 1) atm abs.

In the cylinder of a piston compressor there is a so-called dead space, which contains the valve unit, not covered upon motion of the piston and amounting usually to 4-10% of the total volume V of the cylinder. In Fig. 38, which shows the actual indicator diagram of a compressor, the volume of the dead space is designated V_{vr}. At the end of the displacement of the gas from the cylinder there remains in the dead space of the compressor some compressed gas. During the reverse motion of the piston, the remaining gas expands and the intake of a new batch of gas into the cylinder begins only after the pressure of the residual gas in the cylinder has dropped from a value p_2 to a value p_1. At the start of the intake, the volume of the residual gas in the cylinder amounts to

$$V_0 = V_{sp} \frac{T_1}{T_s} \frac{p_s}{p_1}.$$

so that the cylinder receives upon further motion of the piston a volume of fresh gas equal to $V - V_0$. In this case the delivery of the compressor will be

$$П = \frac{V - V_0}{\tau} \frac{273}{T_1} \frac{p_1}{p_0} \text{ norm. } m^3. \qquad (5.8)$$

Here p_0 is the atmospheric pressure and τ is the duration of the compression cycle.

The presence of dead space decreases the amount of gas drawn into the cylinder and thereby reduces the compressor delivery. The reduction in the compressor delivery due to the presence of the dead space is characterized by a volumetric efficiency of the compressor λ equal to the ratio of the volume of gas actually drawn into the cylinder to the working volume V_{rab} of the cylinder (difference $V - V_{vr}$)

$$\lambda = \frac{V - V_0}{V_{раб}}. \qquad (5.9)$$

The volumetric efficiency decreases with increasing dead space; usually $\lambda = 0.7\text{-}0.9$.

The amount of gas fed into the compressor, and consequently also the delivery of the compressor, decreases even more as a result of an increase in the specific volume of the gas due to its becoming heated by the hot walls of the cylinder and also due to mixing with the residual gas. Assume, for example, that the gas temperature entering the cylinder is increased as a result of heating from an initial value t_1 to t'_1. Then the amount of gas actually drawn into the cylinder is

$$(V - V_0) \frac{273}{T'_1} \frac{p_1}{p_0} \text{ norm. } m^3,$$

i.e., T'_1/T_1 times smaller than prior to heating of the gas. The overall decrease in the delivery due to the presence of dead space and due to the heating of the gas is characterized by the <u>filling coefficient</u> η_V:

$$\eta_v = \lambda \frac{T_1}{T_2}. \qquad (5.10)$$

With increasing compression pressure p_2 the volume V_0 increases and the volumetric efficiency and consequently the delivery of the compressor decrease; in the limit at a pressure p_{2pr}, at which $V_0 = V$, the volumetric efficiency and the delivery vanish. Calculation shows that in polytropic compression, when $p_{2pr} = p_1(V/V_{vr})^n$ the pressure p_{2pr} for n = 1 or 2 and for V_{vr}/V_{rab} = 0.01-0.03 amounts to 255 and 70 atm abs, respectively, it decreases strongly with increasing V_{vr}.*

The decrease in the delivery of a piston compressor with increasing compression pressure does not make it possible to employ single-stage compressors to obtain high-pressure gas. Usually single-stage piston compressors are used to compress gas to pressures not higher than 10 atm. The ratio of the final pressure p_2 to the initial pressure p_1 is called the pressure ratio in the compressor and will be designated henceforth by β.

Multistage Compressor

The multistage compressor is an aggregate of several single-stage compressors operating in tandem, with intermediate cooling of the compressed gas. The increase in the number of compression stages decreases the pressure ratio in each of the stages and by the same token increases the volumetric efficiency of the compressor and its delivery. Figure 39 shows the principal diagram of a two-stage compressor, while Fig. 40 shows its theoretical pv and Ts diagrams.

As a result of effective cooling in the intermediate heat exchanger, the temperature of the gas entering the second stage is the same as that entering the first. This means that the process of compression approaches in maximal fashion an isothermal process. Because of this the work of compression is decreased compared with the compres-

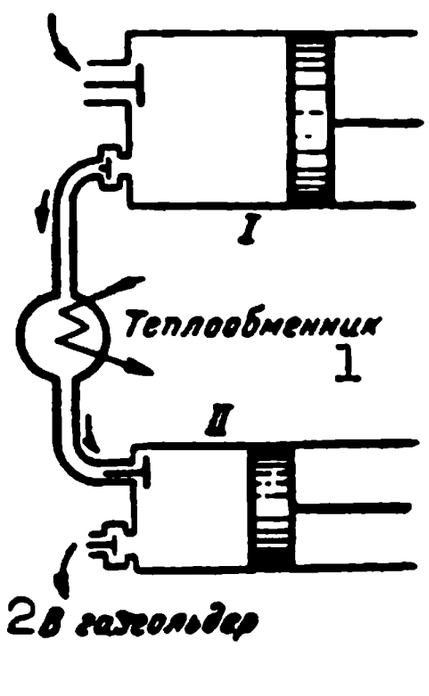

Fig. 39. Principal diagram of two-stage compressor. 1) Heat exchanger; 2) to gas tank.

sion work in a single-stage compressor by the amount of the area shown shaded in the diagram.

In the design of a multistage compressor, it is very important to attain an optimal distribution of the total pressure drop among the stages such that the work consumed in driving the compressor is minimal. If we assume that the polytropic exponent is the same in all stages and has a value $\underline{n}$ and the cooling of the gas after each stage is to the initial temperature t_1, then the theoretical per-unit compression work, which is equal to the total sum of the compression work in the individual stages, can be expressed on the basis of Eq. (5.5) in the following manner:

$$l' = -\frac{n}{n-1} RT_1 \left[\left(\frac{p_{2I}}{p_1}\right)^{\frac{n-1}{n}} + \left(\frac{p_{2II}}{p_{1II}}\right)^{\frac{n-1}{n}} + \ldots + \left(\frac{p_2}{p_{1m}}\right)^{\frac{n-1}{n}} - m \right], \quad (5.11)$$

where $\underline{m}$ is the number of stages.

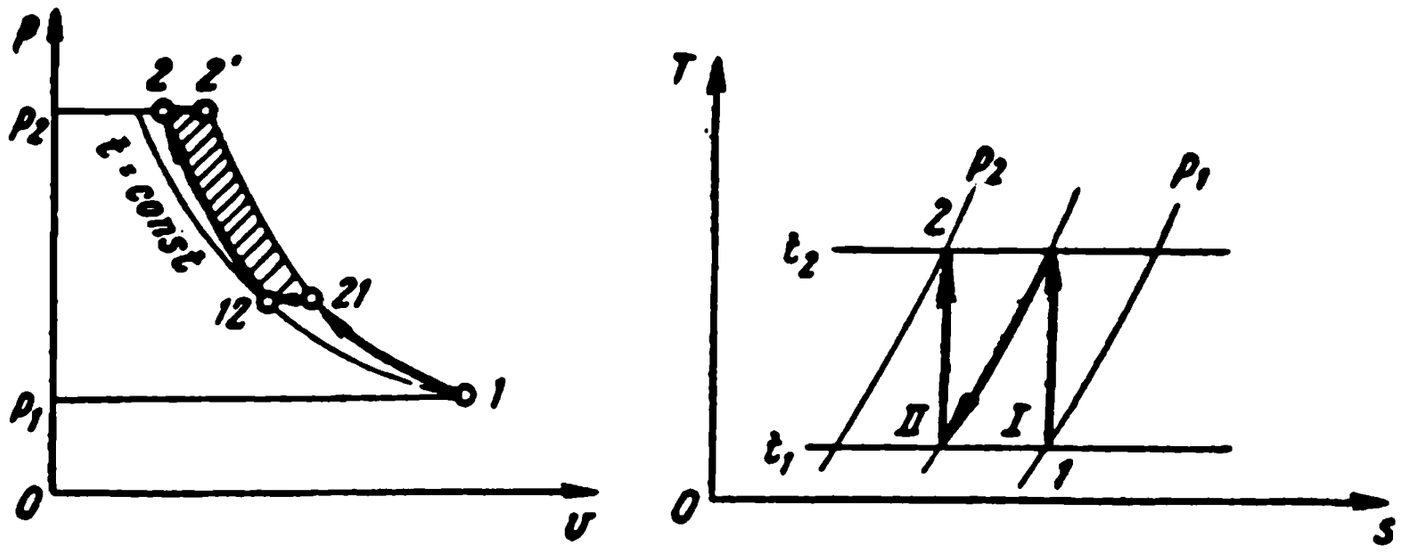

Fig. 40. pv and TS diagrams of two-stage compressor.

It is easy to show that the work of compression l' will be minimal if the ratio of pressures in each of the stages will be the same and equal to

$$\frac{p_{2j}}{p_{1j}} = \sqrt[m]{\frac{p_2}{p_1}}. \quad (5.12)$$

The sought minimal theoretical work consumed in driving the com-

pressor will be

$$l'_{\text{изн}} = -\frac{mn}{n-1} RT_1 \left[\left(\frac{p_2}{p_1} \right)^{\frac{n-1}{mn}} - 1 \right]. \qquad (5.13)$$

The amount of heat carried away in each stage, which is equal also to the amount of heat carried away in the intermediate coolers, will obviously also be the same for all stages. Then the amount of heat carried away in each stage is determined from Expression (5.6):

$$q = -\frac{n-\kappa}{(\kappa-1)(n-1)} ART_1 \left[\left(\frac{p_2}{p_1} \right)^{\frac{n-1}{mn}} - 1 \right]. \qquad (5.14)$$

and the amount of heat q_{khol} carried away in the intermediate heat exchanger under conditions of constant pressure and therefore equal to $-(i_{1j} - i_{2,j-1})$ is given by the equation

$$q_{хол} = -c_p T_1 \left[\left(\frac{p_2}{p_1} \right)^{\frac{n-1}{mn}} - 1 \right]. \qquad (5.15)$$

The working volume of the cylinder of the j-th stage is determined from the delivery of the compressor and from the value of the specific volume of the gas v_{1j} at the start of compression, which as a result of the condition $t_{1i} = t_1$ is equal to $v_1 p_1 / p_{2,j-1}$:

$$v_{1j} = v_1 \left(\frac{p_2}{p_1} \right)^{\frac{1-l}{m}}. \qquad (5.16)$$

Compressor Efficiency

The effective operation of the compressor is characterized by the isothermal or adiabatic efficiency, which is the ratio of the theoretical work (in isothermal or adiabatic compression) to the actual work consumed in driving the compressor (after subtracting the mechanical losses)

$$\eta_{к.\,изот} = \frac{l'_{i.\,теор}}{l'_i}, \qquad (5.17)$$

$$\eta'_{к.\,ад} = \frac{l'_{s.\,теор}}{l'_{ад}}. \qquad (5.18)$$

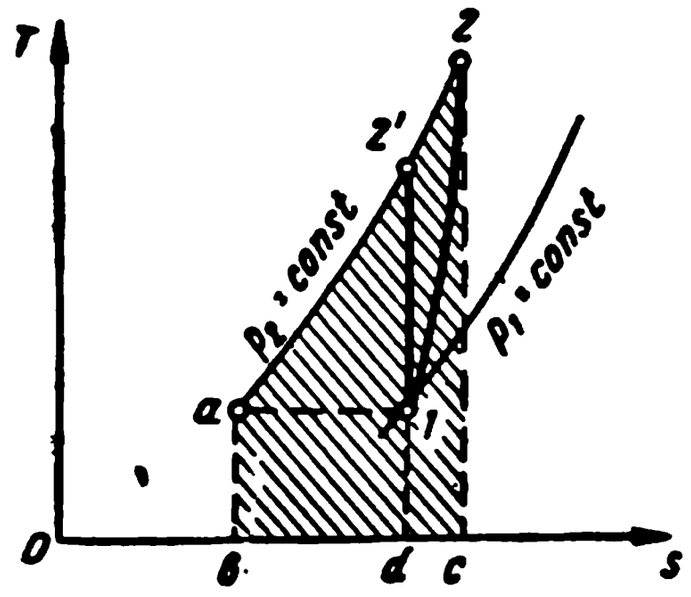

Fig. 41. Irreversible and reversible adiabatic compression processes.

The isothermal efficiency is used to characterize cooled compressors, while the adiabatic one is used for uncooled ones; $\eta_{K,\ izot}$ and $\eta_{K,\ ad}$ are called the relative internal efficiencies of the compressor.

The adiabatic efficiency depends only on the degree of irreversibility of the actual compression, intake, and expulsion of the gas; the value of the isothermal efficiency depends, in addition, on the intensity of heat exchange with the external medium. The more intense the heat exchange, the higher the isothermal efficiency. For a single-stage piston compressor $\eta_{K,izot} = 0.5\text{-}0.8$, $\eta_{K,ad} = 0.85$; for one stage of a centrifugal compressor $\eta_{K,izot} = 0.5\text{-}0.7$, $\eta_{K,ad} = 0.75\text{-}0.80$; for an axial compressor $\eta_{K,ad} = 0.8\text{-}0.85$.

To calculate the adiabatic efficiency of an uncooled compressor, let us plot on the Ts diagram (Fig. 41) the actual, i.e., irreversible, adiabatic process of compression 1-2 and the theoretical (reversible) adiabatic compression 1-2' and let us use Eq. (5.1), according to which

$$l'_{ad} = -\frac{1}{A}(i_2 - i_1).$$

We assume that the compressed gas is ideal and then

$$l'_{ad} = -\frac{1}{A}c_p(T_2 - T_1)$$

and will correspond to the area a2cb on the diagram. The theoretical work of reversible adiabatic compression is

$$l'_s = -\frac{1}{A}c_p(T_{2'} - T_1)$$

and corresponds to the area a2'dc. Consequently, the adiabatic efficiency of the compressor has the following form:

$$\eta_{K.ad} = \frac{T_{2'} - T_1}{T_2 - T_1}.$$

Since the line 1-2' is a reversible adiabat and 1-2 can be regarded as a polytrope with exponent $n > \kappa$, the numerical value of which depends only on the friction force, we get

$$\frac{T_{2'}}{T_2} = \left(\frac{p_2}{p_1}\right)^{\frac{\kappa-1}{\kappa}}; \quad \frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)^{\frac{n-1}{n}}.$$

Using these equations, let us transform the equation for $\eta_{K,ad}$ to the form

$$\eta_{K.ad} = \frac{\left(\frac{p_2}{p_1}\right)^{\frac{\kappa-1}{\kappa}} - 1}{\left(\frac{p_2}{p_1}\right)^{\frac{n-1}{n}} - 1}. \qquad (5.19)$$

§2. PISTON TYPE INTERNAL COMBUSTION ENGINES

Heat engines, and in the more general sense, heat-power installations, are used to convert heat released as a result of thermochemical and other reactions into mechanical work. The heat is usually obtained by burning fuel. It is possible to use for this purpose also nuclear reactions, and particularly the fission of uranium and other fissionable elements.

In the present chapter we consider heat-power installations with thermochemical fuel. Various types of heat-power installations differ from one another in the properties of the working medium employed, the principles used to obtain the mechanical work, and the character of the thermodynamic processes occurring with the working medium. The working medium of internal-combustion engines is a mixture of air and combustible gases or vapors of liquid fuel (in the initial portion of the cycle) and gaseous combustion products (in the remaining parts of the cycle). Internal-combustion engines are of the piston type, of the gas turbine type, and of the jet type.

Cycles of Piston Engines

In internal combustion piston engines, the useful work is obtained by periodic expansion and compression of the working medium which is in the cylinder under the piston, the force from which is transmitted by means of a special mechanism to the consumer of the work.

Various requirements that are imposed on internal combustion engines in accordance with their purpose have led to the development of numerous types of such engines. However, from the thermodynamic point of view (as regards the character of the working cycles employed) these engines can be subdivided into the following three classes:

1) engines which use a cycle in which heat is supplied at a constant volume;

2) engines using a cycle with heat supplied at constant pressure;

3) engines using the so-called mixed cycle, with heat supplied both at constant volume and at constant pressure.

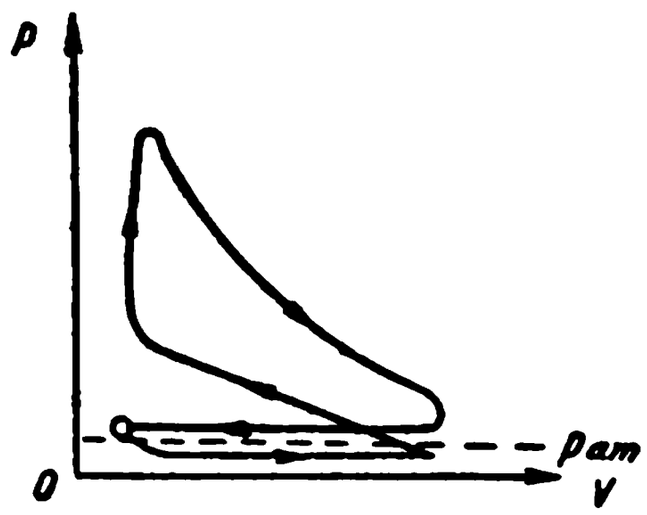

Fig. 42. Construction and indicator diagram of a four-stroke engine.

The operating principle of piston internal combustion engines can be readily explained using as an example a four-stroke engine, the construction and indicator diagram of which are shown in Fig. 42. As the piston moves to the right (first stroke) the working mixture, representing a mixture of air with combustible gas or vapor of liquid fuel (in engines with heat supply at V = = const), or else pure air (in engines with heat supply at p = const and in engines with mixed cycle) is drawn into the engine cylinder through the intake valve 1. As the piston moves to the left with the intake and exhaust valves closed (second stroke), the working mixture is compressed. Af-

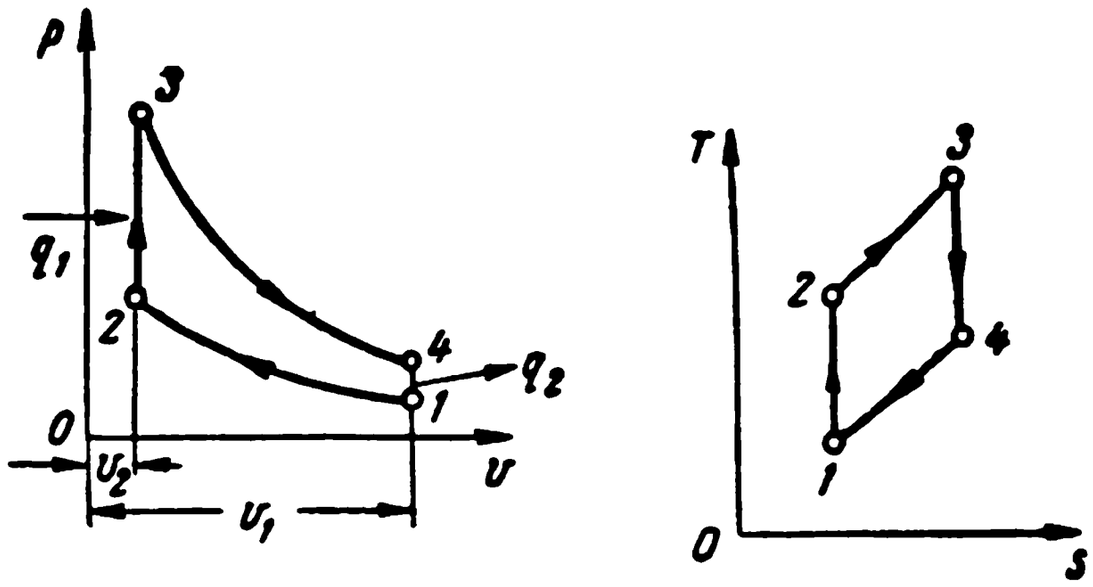

Fig. 43. Theoretical cycle of internal combustion engine with heat supplied at V = const.

ter the piston reaches its extreme left position, the fuel combustion process begins. In engines with heat supplied at V = const, the working mixture is ignited by an electric spark; in mixtures with heat supplied at p = const or operating in accordance with the mix cycle, highly pulverized liquid fuel is injected into the cylinder at the end of the second stroke and, on entering into the air which is strongly heated by the compression (up to 600-800°C), it ignites and burns. After combustion of the fuel (or simultaneously with it), expansion of the combustion products takes place, accompanied by motion of the piston to the right; this third stroke is called the working travel, since it is then that the proper useful work is performed. The fourth and last stroke begins when the piston reaches the extreme right position and the exhaust valve 2 opens and the gaseous combustion products are expelled or exhausted from the cylinder into the atmosphere.

Let us consider the theoretical cycle of an internal combustion engine. The actual compression and expansion of the working medium are replaced arbitrarily in plotting the theoretical cycle by reversible adiabats. The combustion process, as a result of which the working mixture is converted into a new substance (combustion products), is regarded as a reversible supply of heat to an invariant working medium. In addition, it is assumed that in engines where the combustible work-

ing mixture is compressed the combustion takes place instantaneously, i.e., it is assumed that the entire amount of heat is supplied while the piston is stationary and consequently at constant volume. In engines where the air and fuel are supplied separately, the combustion occurs under conditions p = const (in diesel engines, where the pulverization of the liquid fuel is effected by high-pressure air from a special compressor) or at first at V = const and then at p = const (in compressorless engines with pulverization of the liquid fuel under pressure through a mechanical nozzle).

The exhaust of the combustion products from the cylinder into the atmosphere, accompanied by a drop in the pressure and temperature of the working medium, is regarded as an isochoric process.* The theoretical cycle of the internal combustion engine with heat supplied at v = = const is shown in Fig. 43; that with heat supplied at p = const is shown in Fig. 44, while that with heat supplied at v = const and p = = const is shown in Fig. 45. In all these cycles the processes 1-2 (adiabatic compression of the combustible mixture or air), 3-4 (adiabatic expansion of the combustion products), and 4-1 (isochoric exhaust of the combustion products) are common; the difference between cycles lies only in the process 2-3 of heat supply (combustion of the fuel).

The volume ratio $v_1/v_2 = \varepsilon$ at the start and end of the compression is called the <u>degree of compression</u>.

The pressure ratio $p_3/p_2 = \lambda$ is called the <u>degree of additional pressure increase</u> as a result of the combustion of the fuel.

The volume ratio $v_3/v_2 = \rho$ is called the <u>degree of preliminary expansion</u>.

If the working medium in the engine is an ideal gas, the specific heats of which are constant, then the parameter of the working medium

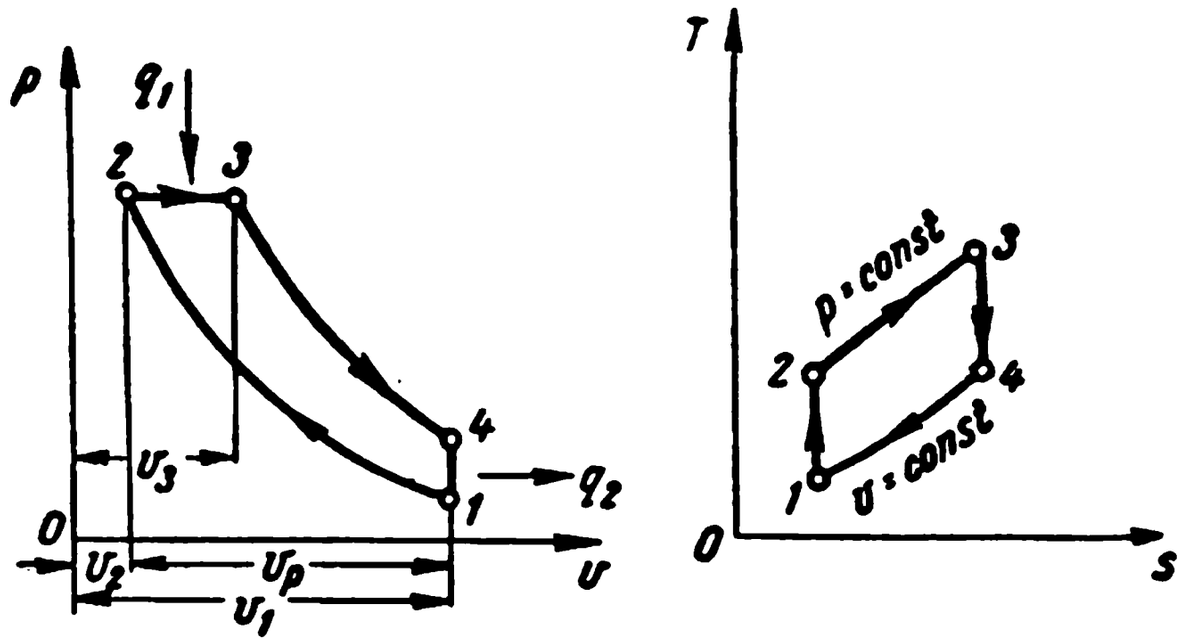

Fig. 44. Theoretical cycle of internal combustion engine with heat supplied at p = const.

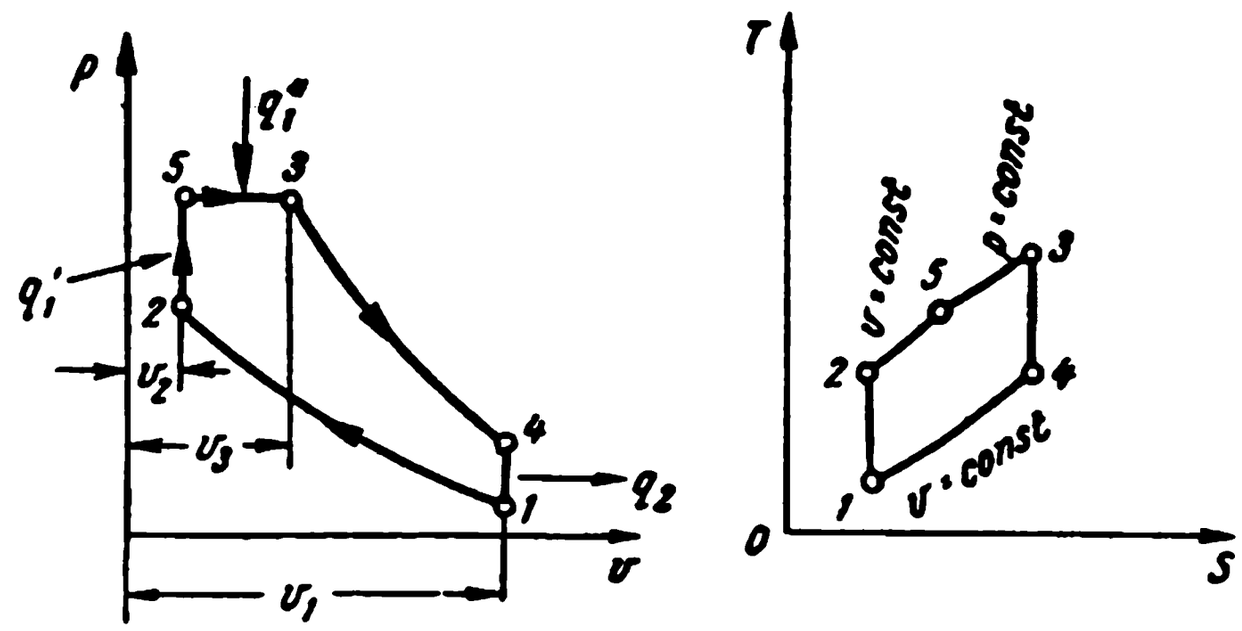

Fig. 45. Theoretical cycle of internal combustion engine with heat supplied partially at V = const and partially at p = = const.

at the terminal points of the cycle (1, 2, 3, 4, and 5) will have the values listed in Table 3.

The pressure p_3 and the temperature T_3 are the highest pressure and temperature in the cycle. The thermal efficiency of a cycle with heat supplied at v = const has, in accordance with the values of q_1 and q_2 listed in Table 3, the form

$$\eta_{t.v} = 1 - \frac{1}{\varepsilon^{\kappa-1}}. \qquad (5.20)$$

It is seen from Eq. (5.20) that the thermal efficiency of a cycle with heat supplied at V = const is the larger, the higher the degree of compression ε. With increasing adiabatic exponent of the working mixture, the thermal efficiency increases when the degree of compres-

TABLE 3

Characteristics of Cycle

	Цикл с подводом тепла при V = const (1)			Цикл с подводом тепла при p = const (2)			Смешанный цикл (3)		
	p	v	T	p	v	T	p	v	T
4 Точка 1	p_1	v_1	T_1	p_1	v_1	T_1	p_1	v_1	T_1
. 2	$\varepsilon^{\kappa} p_1$	$\frac{v_1}{\varepsilon}$	$\varepsilon^{\kappa-1} T_1$	$\varepsilon^{\kappa} p_1$	$\frac{v_1}{\varepsilon}$	$\varepsilon^{\kappa-1} T_1$	$\varepsilon^{\kappa} p_1$	$\frac{v_1}{\varepsilon}$	$\varepsilon^{\kappa-1} T_1$
. 3	$\varepsilon^{\kappa}\lambda p_1$	$\frac{v_1}{\varepsilon}$	$\varepsilon^{\kappa-1}\lambda T_1$	$\varepsilon^{\kappa} p_1$	$\frac{\rho}{\varepsilon} v_1$	$\varepsilon^{\kappa-1}\rho T_1$	$\varepsilon^{\kappa}\lambda p_1$	$\frac{\rho}{\varepsilon} v_1$	$\varepsilon^{\kappa-1}\rho\lambda T_1$
. 4	λp_1	v_1	λT_1	$\rho^{\kappa} p_1$	v_1	$\rho^{\kappa}\lambda T_1$	$\rho^{\kappa}\lambda p_1$	v_1	$\rho^{\kappa}\lambda T_1$
. 5							$\varepsilon^{\kappa}\lambda p_1$	$\frac{v_1}{\varepsilon}$	$\varepsilon^{\kappa-1}\lambda T_1$
5 Степень сжатия ε	8 *5—8* (карбюраторные двигатели) *6—9* (газовые двигатели)			9 *13—18* (величина ε должна удовлетворять условию $\varepsilon^{\kappa-1} > \frac{T_{воспл}}{T_1}$)					
6 Количество подведенного тепла q_1	$u_3 - u_2 = c_V \varepsilon^{\kappa-1} \times (\lambda - 1) T_1$			$i_3 - i_2 = c_p \varepsilon^{\kappa-1} \times (\rho - 1) T_1$			$u_3 - u_2 + l_3 - l_2 = c_V \varepsilon^{\kappa-1} \times [\lambda - 1 + k\lambda(\rho - 1)] T_1$		
7 Количество отведенного тепла q_2	$u_4 - u_1 = c_V (\lambda - 1) T_1$			$u_4 - u_1 = c_V (\rho^{\kappa} - 1) T_1$			$u_4 - u_1 = c_V (\lambda\rho^{\kappa} - 1) T_1$		

1) Cycle with heat supplied at V = const; 2) cycle with heat supplied at p = const; 3) mix cycle; 4) point; 5) degree of compression ε; 6) amount of heat supplied q_1; 7) amount of heat removed q_2; 8) 5-8 (carburetor engines) 6-9 (gas engines); 9) 13-18 (the value of ε should satisfy the condition $\varepsilon^{\kappa-1} > T_{vospl}/T_1$).

sion remains constant.

Replacing $\varepsilon^{\kappa-1}$ in (5.20) by T_2/T_1, we obtain

$$\eta_{t,V} = 1 - \frac{T_1}{T_2}. \qquad (5.21)$$

The ratio of the average temperatures on lines 2-3 and 4-1 of the supply and removal of heat is also T_2/T_1, i.e., $T^{sr}_{podv}/T^{sr}_{otv} = T_2/T_1$. This circumstance explains completely why $\eta_{t,V}$ increases with increasing degree of compression ε and does not depend on the degree of pres-

sure increase λ. With the increasing degree of compression, the temperature ratio T_2/T_1, and consequently also the ratio of the average temperatures of heat supply and removal, increases and consequently the thermal efficiency of the cycle, which is equal to $1 - (T^{sr}_{otv}/T^{sr}_{podv})$, increases. The increase in the degree of pressure increase λ with ε constant does not change the ratio of the average heat-supply and heat-removal temperatures and therefore does not lead to a change in the thermal efficiency of the cycle.

The thermal efficiency of a cycle with heat supplied at p = const has the form

$$\eta_{t,p} = 1 - \frac{\rho^{\kappa} - 1}{k\varepsilon^{\kappa-1}(\rho - 1)}. \qquad (5.22)$$

Formula (5.22) shows that the thermal efficiency of a cycle with heat supplied at p = const increases with increasing degree of compression ε and decreases with increasing degree of preliminary expansion ρ. The increase in thermal efficiency with increasing degree of compression is due to the increase in the average temperature of heat supply in the cycle and to the decrease in the ratio $T^{sr}_{otv}/T^{sr}_{podv}$.

The thermal efficiency of the mix cycle is

$$\eta_{t.cm} = 1 - \frac{\lambda\rho^{\kappa} - 1}{\varepsilon^{\kappa-1}[\lambda - 1 + k\lambda(\rho - 1)]}. \qquad (5.23)$$

The thermal efficiency of the mix cycle, like the thermal efficiency of cycles with isochoric and isobaric heat supply, increases with increasing degree of compression ε, and furthermore depends on λ and ρ.

Comparison of Cycles

In analysis of the efficiency of various types of piston engines, we can confine ourselves to a comparison of the thermal efficiencies of the cycles of these engines, since the effective efficiency changes on going from one engine to another approximately like η_t. It is most advantageous to compare cycles of piston engines for equal maximum

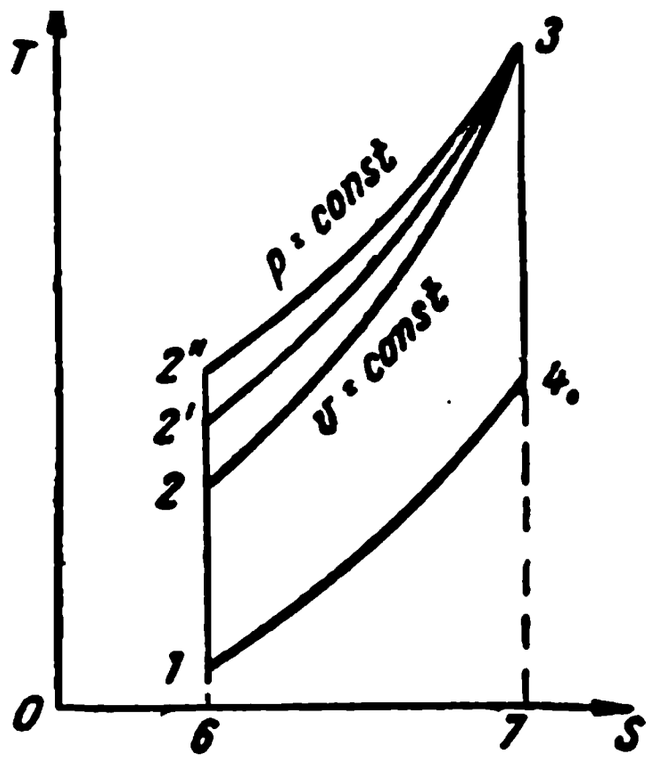

Fig. 46. Comparison of internal-combustion engine cycles with heat supplied at V = const, p = = const, and partially with p = = const and partially with V = const.

pressures and temperatures, for it is precisely these conditions that actually determine the structural features of the engines, their strength, and their operating reliability. For an unambiguous definition of the mix cycle, it is necessary in addition to specify the degree of compression ε_{sm}.

It is obvious that for identical p_{maks} and t_{maks} the value of ε_{sm} will lie between ε_V and ε_p, i.e., $\varepsilon_V < \varepsilon_{sm} < \varepsilon_p$, where ε_V and ε_p are the degrees of compression for cycles with heat supplied at v = const and p = const. Under these conditions the relation between the thermal efficiencies of the cycles under consideration will be as follows:

$$\eta_{t.v} < \eta_{t.cm} < \eta_{t.p},$$

inasmuch as, as can be seen from Fig. 46, the value of q_2, measured by the area 1-4-7-6, is the same for all cycles and the values of q_1, measured by the areas under the corresponding curves of heat supply, are different and satisfy the obvious relation:

$$q_{1.v} < q_{1.cm} < q_{1.p}.$$

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[Footnotes]

144 The increase in pressure in the cylinder is also limited by the flash point of the lubricating oil, which becomes heated when the gas is compressed.

151 In order to expel from the cylinder the gaseous combustion products, whose volume and pressure at the start of the fourth stroke are v_4 and p_4, it is necessary to expand work $l' = v_4 (p_4 - p_1)$. But then on the basis of (2.11) we get $q = u_1 - u_4$, and this is why the exhaust process 4-1 should be regarded as isochoric.

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[List of Transliterated Symbols]

140 теор = teor = teoreticheskiy = theoretical

142 вр = vr = vrednyy = dead

143 раб = rab = rabochiy = working

144 пр = pr = predel'nyy = limiting

146 мин = min = minimal'nyy = minimum

146 хол = khol = kholodil'nik = cooler

146 изот = izot = izotermicheskiy = isothermal

146 ад = ad = adiabaticheskiy = adiabatic

153 воспл = vospl = vosplameneniye = ignition

153 ср = sr = sredniy = average

153 подв = podv = podvod = supply

153 отв = otv = otvod = removal

154 см = sm = smesheniye = mixing

155 макс = maks = maksimal'nyy = maximum

Chapter 6

CONDITIONS OF THERMODYNAMIC EQUILIBRIUM

Each thermodynamic system can be either in equilibrium or in the nonequilibrium state. Among the equilibrium states one distinguishes (as in mechanics) between states of stable and unstable equilibrium.

§1. GENERAL CONDITIONS FOR THE EQUILIBRIUM OF A THERMODYNAMIC SYSTEM

Stable equilibrium of a thermodynamic system is characterized by the fact that when the causes of deviations of the system from the equilibrium state are removed, the system returns by itself to the initial equilibrium state. In this case, within a time during which the thermodynamic equilibrium is established (this time is called the <u>relaxation</u> time), various nonequilibrium and consequently irreversible processes will occur in the system, consisting of the damping of the mechanical motions, equalization of densities or temperatures, etc. In order to take a system out of the state of stable equilibrium, it is necessary to perform (expend) certain work on the system.

An unstable equilibrium differs from a stable one in that once the system is brought out of the equilibrium state, it will not return to the initial state, but will go into a new stable equilibrium state.

We begin our study of the conditions of equilibrium of different thermodynamic systems with an analysis of an isolated system. In such a system the internal energy U and its total volume V have a constant value. Once taken out of the stable-equilibrium state, the system will return after some time to this state and, owing to the irreversibility of the relaxation processes, the entropy of the system increases on

approaching the equilibrium state until it reaches a maximum. This leads to the following condition for thermodynamic equilibrium of an isolated system: <u>the entropy of an isolated system in the state of stable equilibrium has a maximum value</u>:

$$dS = 0;\ d^2S < 0. \tag{6.1}$$

Actually, if an isolated system is in a state with maximum entropy, no deviations of the system from this state can occur, for then the entropy of the system should assume values smaller than the maximum, i.e., decrease, something which is impossible by virtue of the inequality $\Delta S \geq 0$, which expresses the second law of thermodynamics.

In order to establish the conditions for stable equilibrium of a system having specified fixed values of the temperature t and volume V, we recall that in such a system, if it does not perform useful external work, the free energy decreases (in irreversible processes) or remains constant (in reversible processes), but never increases. Therefore <u>the condition for stable equilibrium of a system that has a constant temperature and volume is that the free energy of the system be a minimum</u>

$$dF = 0;\ d^2F > 0. \tag{6.2}$$

In a system having constant temperature t and pressure p and performing no useful external work, the conditions of stable equilibrium can be established analogously on the basis of the deductions obtained in Chapter 3. As is well known (see §6 of Chapter 3), the isobaric potential of such a system decreases for any irreversible process and in the case of a reversible process it maintains a constant value.

Consequently, <u>the sufficient condition for the equilibrium of a system which is at constant pressures and temperature is that the isobaric potential of the system be a minimum</u>:

$$d\Phi = 0;\ \ d^2\Phi > 0. \tag{6.3}$$

The conditions (6.1)-(6.3) for stable equilibrium have a most general

value; they are applicable to all thermodynamic systems.

The state of a system that is in stable equilibrium is called stable. A stable state of a system corresponds to the absolute maximum or minimum of the corresponding characteristic function S, F, or Φ.

The state of unstable equilibrium of a system is called a labile state. Labile states are not encountered in practice, since owing to the action of various perturbing factors the system cannot stay in this state for any considerable time and goes over into a stable state.

Equilibrium states of a system that are stable with respect to the nearest state, but are not stable with respect to some more remote state, are called metastable. Metastable states are observed when the corresponding characteristic function of the system has several extremum points. Each of the metastable states corresponds to a relative extremum, i.e., not to the largest maximum and not to the smallest minimum of the characteristic function. A system which is in a metastable state will go over into a stable state after the lapse of a certain time and in the presence of the necessary conditions.

§2. EQUILIBRIUM OF A HOMOGENEOUS SYSTEM

As we already know from the foregoing, in the state of stable equilibrium all the parts of a homogeneous system should have identical temperature and identical pressure:

$$p = \text{idem}; \quad t = \text{idem}. \tag{6.4}$$

A thermodynamic analysis of the equilibrium states of a homogeneous system shows that in any state of stable equilibrium the following inequalities should be satisfied:

$$\left(\frac{\partial p}{\partial V}\right)_t < 0; \quad C_V > 0. \tag{6.5}$$

From the first inequality of (6.5) it follows that an increase in the volume at constant temperature is always accompanied by a decrease in

pressure. This condition is obvious, for in the opposite case, i.e., if $(\partial p/\partial V)_t > 0$, the state of any body would be absolutely unstable, for the smallest decrease in the volume of the body, for example upon accidental increase in the external pressure, would lead not to an increase in the pressure of the body and thereby to counteraction of the external action, as is the case in stable equilibrium, but to a decrease in the body's own pressure, as a result of which it would become compressed to a volume $V = 0$ by the prevailing pressure of the surrounding medium.

It is appropriate to note that in all real bodies there exists among the homogeneous states that state (generally unique) of stable equilibrium at which the derivative $(\partial p/\partial V)_t$ vanishes. This state is called the critical state or the critical point. In order for the critical state to be stable, the signs in the changes of the pressure Δp and the volume ΔV at $t = \text{const}$ should be opposite in this state. For this purpose it is necessary, as can be seen from the expansion of Δp in powers of ΔV,

$$\Delta p = \left(\frac{\partial p}{\partial V}\right)_t \Delta V + \frac{1}{2}\left(\frac{\partial^2 p}{\partial V^2}\right)_t \Delta V^2 + \frac{1}{6}\left(\frac{\partial^3 p}{\partial V^3}\right)_t \Delta V^3 + \ldots,$$

that simultaneously with $(\partial p/\partial V)_t$ the derivative $(\partial^2 p/\partial V^2)_t$ also vanishes at the critical point (or, in addition, $(\partial^3 p/\partial V^3)_t$ and $(\partial^4 p/\partial V^4)_t$), and the next derivative should have a negative sign. The second inequality (6.5), which stipulates that in the state of stable equilibrium the specific heat of the body C_V be positive, is also perfectly understandable. Let us assume that $C_V < 0$. Then absorption of a certain amount of heat by the body at $V = \text{const}$ (for example, as a result of fluctuations in the medium surrounding the body) would cause a decrease in the temperature of the body as compared with the surrounding medium, as a result of which heat flux would arise from the medium

to the body and this would lead to further decrease of the temperature of the body and consequently also to an increase in the influx of heat, etc. We see therefore that when $C_V < 0$ stable equilibrium of the body (not only homogeneous, but also inhomogeneous) is impossible. From analogous considerations it follows also that the quantity C_p should be positive in the state of stable equilibrium.

§3. PHASE EQUILIBRIUM

From Conditions (6.5) for stable equilibrium of a homogeneous system it follows that not all values of these independent parameters, for example $\underline{t}$ and V, correspond to physically possible states of a homogeneous system. Experiment has shown that there exists a whole region of the parameters $\underline{t}$ and V where the first condition of (6.5) is not satisfied. At these values of the independent parameters the existence of a homogeneous system is impossible, and the latter breaks up into two contiguous different homogeneous parts or phases.

General Conditions for Phase Equilibrium

Any substance can be in different phases, which represent different aggregate states of the matter (gaseous, liquid, and crystalline) and also allotropic modifications of the crystalline state. Each phase is a homogeneous system with like physical properties in all its parts. A characteristic feature of the phase is the presence of boundaries which separate the given phase from other phases in contact with it. The spatial delineation which is possessed by the phases makes it possible to separate them mechanically.

A substance can go over from one phase into another; this transition is called a phase transition or a phase transformation. The transition of a substance from a condensed (solid or liquid) phase into the gaseous phase is called evaporation or vapor formation (and in the case of a solid body, in addition, sublimation), while the reverse

transition is called condensation; the transition from the solid phase into a liquid one is called melting, while the reverse transition from the liquid phase into the solid one is called hardening or crystallization. Phase transitions are accompanied by absorption or release of heat.

The equilibrium coexistence of several contiguous different phases of substance is called phase equilibrium. In order to find the conditions of phase equilibrium, let us consider first the equilibrium state of a system consisting of two phases of one and the same substance.

For phase equilibrium it is necessary first of all that, as in the case of a homogeneous body, the conditions of mechanical and thermal equilibrium be satisfied. This means that the pressure and the temperature of the two phases should be the same*:

$$p^{(1)} = p^{(2)}; \quad t^{(1)} = t^{(2)}.$$

Unlike a homogeneous body, these conditions are not sufficient for the equilibrium coexistence of two phases, each of which can go over into the other. For equilibrium it is necessary in addition that there be no predominant growth of one phase at the expense of the other, i.e., that the stabilities of the phases in the equilibrium state be equal. This third phase-equilibrium condition is obtained from an examination of the equilibrium transition of one phase into another. Let $dG^{(1)}$ kg of the first phase go over into the other. Since this transition occurs during phase equilibrium and consequently, under equal temperature and pressure for both phases, which we assume to be constant, it represents a reversible isothermally isobaric process. At constant p and t the isobaric potential of the system in the equilibrium state should in accordance with Formula (6.3) have a minimum $d\Phi = 0$. But in the case of the two-phase system under consideration we have

$$\Phi = \varphi^{(1)}(p, t)\, G^{(1)} + \varphi^{(2)}(p, t)\, G^{(2)},$$

therefore the equilibrium condition has the form

$$\varphi^{(1)}(p, t)\, dG^{(1)} + \varphi^{(2)}(p, t)\, dG^{(2)} = 0$$

or, recognizing that $G^{(1)} + G^{(2)} = \text{const}$,

$$[\varphi^{(1)}(p, t) - \varphi^{(2)}(p, t)]\, dG^{(1)} = 0.$$

Since $dG^{(1)} \neq 0$, we have

$$\varphi^{(1)}(p, t) = \varphi^{(2)}(p, t). \qquad (6.6)$$

The equation obtained is indeed the sought third condition for phase equilibrium. Consequently, the condition for the equilibrium of a two-phase system is that the temperatures, pressures, and chemical potentials of both phases be equal

$$p^{(1)} = p^{(2)}; \quad t^{(1)} = t^{(2)}; \quad \varphi^{(1)} = \varphi^{(2)}. \qquad (6.7)$$

The mass of the phases does not enter into the equilibrium condition, i.e., a change in the mass of the phases does not upset the equilibrium, provided the phase densities do not change. Solving Eq. (6.6) with respect to p, we obtain the dependence of the pressure of the substance in phase equilibrium on the temperature:

$$p = p_s(t). \qquad (6.8)$$

Two phases can be in equilibrium with each other not for arbitrary p and t, but under fully defined values of these parameters, wherein specification of one of these parameters determines the value of the other. From the equilibrium conditions it follows also that the specific volumes $v^{(1)}$ and $v^{(2)}$ of phases that coexist in equilibrium are functions of the temperature t only. The form of these functions can be obtained from the equation of state of each of the phases by substituting in them the values of p from Eq. (6.8).

Phase Stability

The equality of the chemical potentials $\varphi^{(1)}$ and $\varphi^{(2)}$ expresses identical stability of both phases in the state of phase equilibrium.

Therefore the conditions (6.6) are frequently called the condition for chemical equilibrium.

In order to understand how and by virtue of what factors does a phase transition occur, let us imagine that each phase of the substance can exist in arbitrary states (at a given temperature t it can have a pressure both larger and smaller than p_s) and let us plot graphically for some temperature t the dependence of the chemical potential of each of the phases on the pressure p (Fig. 47). The equality of the chemical potentials $\varphi^{(1)}$ and $\varphi^{(2)}$ of the coexisting phases signifies that at a given temperature t the curves $\varphi^{(1)}(p, t = \text{const})$ and $\varphi^{(2)}(p, t = \text{const})$ cross at the point whose abscissa is equal to the value of the pressure during the phase equilibrium, p_s. With change in temperature, the point where the curves $\varphi^{(1)}$ and $\varphi^{(2)}$ cross shifts accordingly. It is seen from Fig. 47 that at a pressure p that is greater than the equilibrium pressure p_s, the first phase has a smaller chemical potential compared with the second phase, and that at pressures smaller than p_s, to the contrary, the first phase has a larger chemical potential. Thus, when $p > p_s$ we have $\varphi^{(1)} < \varphi^{(2)}$ and when $p < p_s$ we have $\varphi^{(1)} > \varphi^{(2)}$.

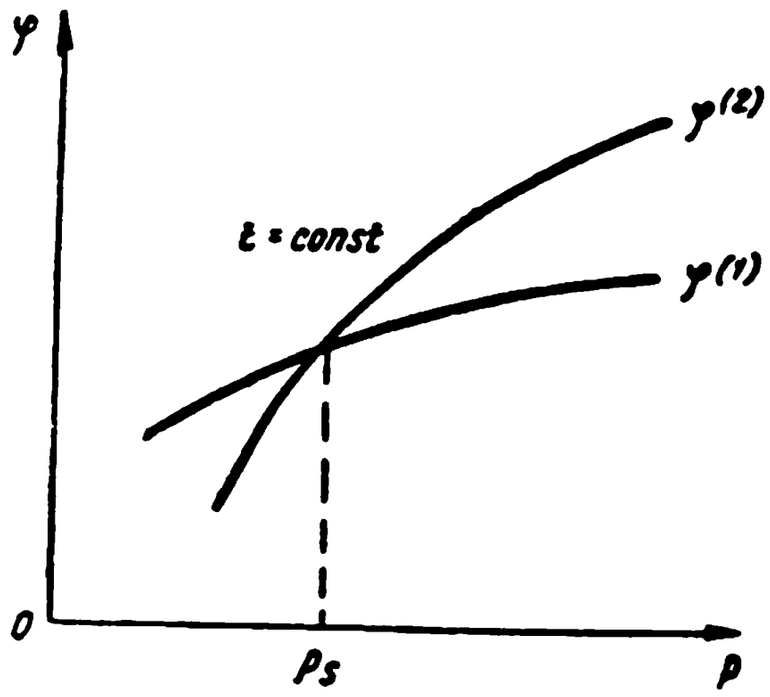

Fig. 47. Character of the dependence of the chemical potential of the phase on the pressure.

The phase which has under given conditions a higher chemical potential will tend to go over into the phase whose chemical potential is lower. In fact, at a constant value of the temperature and pressure, the isobaric potential of any thermodynamic system decreases, $d\Phi < 0$, as the system tends toward equilibrium, which yields for a system consisting of two phases

$$\varphi^{(1)} dG^{(1)} + \varphi^{(2)} dG^{(2)} < 0,$$

and since the total amount of substance $G^{(1)} + G^{(2)} = G$ does not change, we have $dG^{(2)} = -dG^{(1)}$ and consequently

$$[\varphi^{(1)} - \varphi^{(2)}] dG^{(1)} < 0.$$

It follows from this inequality that when $\varphi^{(1)} > \varphi^{(2)}$ we have $dG^{(1)} < 0$ and when $\varphi^{(1)} < \varphi^{(2)}$ we have $dG^{(1)} > 0$, i.e., the substance always tends to go over from a phase where the chemical potential is larger into a phase with the lower chemical potential. In other words, the phase that under given conditions (for given p and t) has the lowest value of the chemical potential is the most stable.

In the region $p > p_s$, the lowest chemical potential is possessed by the first phase, and in the region $p < p_s$ by the second phase. Consequently, when $p > p_s$ the first phase is more stable; therefore the body in this region will be in the form of the first phase (or at any rate will tend to go over into it). When $p < p_s$ the second phase is more stable and the body will be in the form of the second phase. When $p = p_s$ the chemical potentials of both phases are the same: none of the phases has advantages over the other with respect to stability, so that the body will exist in the form of mixture of the two phases in equilibrium.

pt Phase Diagram

If we plot in the pt plane the curve $p = p_s(t)$, then the points of this curve will correspond to those states for which phase equilibrium exists (Fig. 48). Therefore the curve $p_s(t)$ itself has been called the phase-equilibrium curve and the entire pt diagram has been called the phase diagram. The phase diagram is plotted, in addition to the pt plane, also in the pv, Ts planes, etc., and also in thermodynamic space p, t, v.

The points lying on both sides of the phase-equilibrium curve on

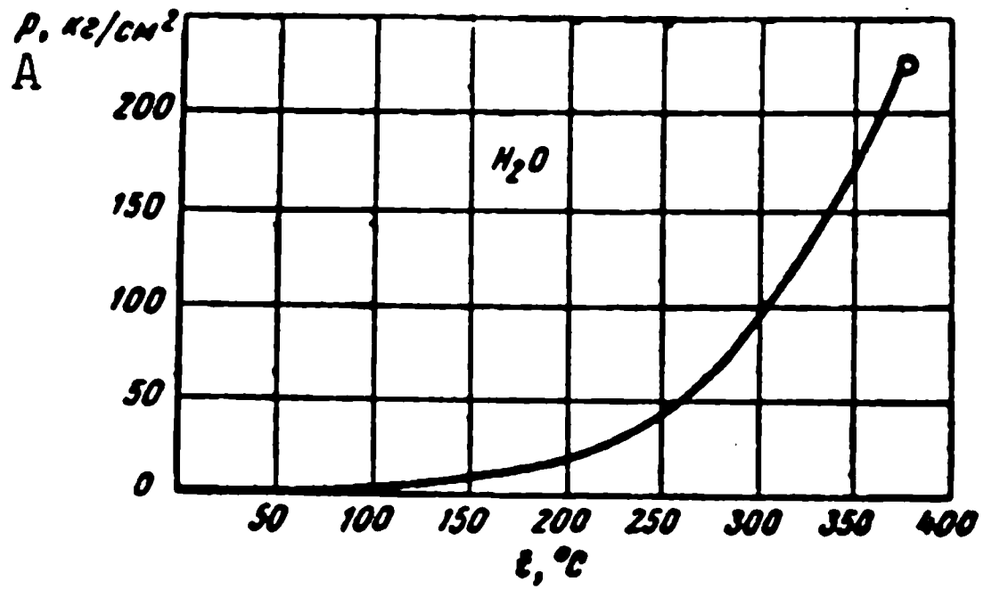

Fig. 48. Curve of equilibrium states of two phases. A) p, kg/cm^2.

the pt diagram correspond to homogeneous states of the body, i.e., to separate phases; the upper points (those above the curve or to the left of it) correspond to a denser state of the matter and the lower ones (to the right of the phase-equilibrium curve) to a less dense state of the matter. For example, in the case of equilibrium between the liquid and gaseous states, the region of liquid state is above the curve of the phase equilibrium, while the region of gaseous state is below this curve; the phase-equilibrium curve represents in this case the curve of saturated vapor tension.

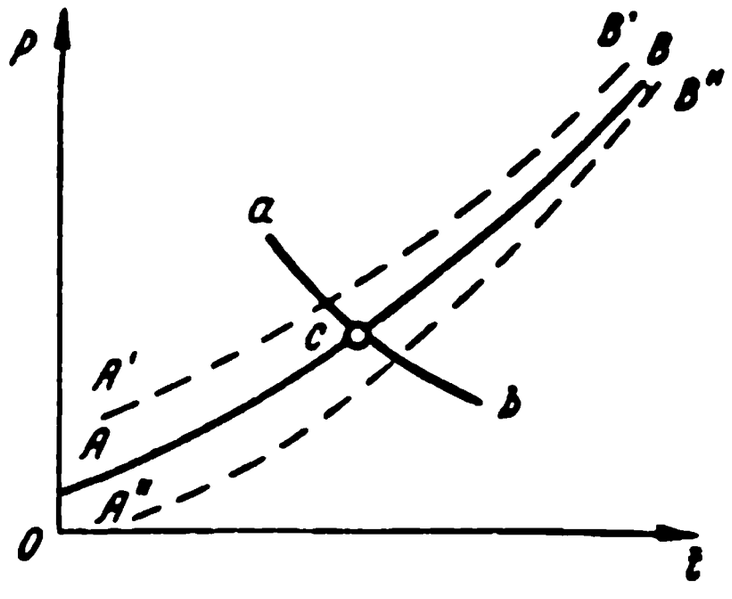

Fig. 49. Boundaries of metastable states of a substance.

When the state of the body varies along the line ab, which crosses the phase-equilibrium curve AB (Fig. 49) at the point of intersection C, stratification of the phases takes place with subsequent transformation of one phase into another.

In the case of a sufficiently cautious variation of the state of the body, the stratification of the phases on going through the phase-equilibrium curve may sometimes not occur, so that the body will remain homogeneous (in the form of a single phase) in that region, too, where it should have broken up into two phases. Therefore the liquid state of a substance can be encountered

below the phase-equilibrium curve and the gaseous state above this curve. However, the value of the chemical potential of the liquid state below the curve of phase equilibrium will (Fig. 47) be larger than the value of the chemical potential of the gaseous state, while above the phase-equilibrium curve the situation will be reversed: the chemical potential of the liquid state will be smaller than the chemical potential of the gaseous state. Therefore the liquid state of matter below the phase-equilibrium curve (this state is called a <u>superheated</u> liquid) will be less stable compared with the gaseous state and sooner or later will go over into the latter. Equally, the gaseous state above the phase-equilibrium curve, called the <u>supercooled</u> or <u>supersaturated</u> vapor, will be less stable compared with the liquid state. A superheated liquid and a supersaturated vapor are metastable states of the substance. The region of metastable states is relatively small. The boundary of this region in Fig. 49 is shown conventionally in the form of two dashed curves A'B' and A"B".

The curve of phase equilibrium $p_s(t)$ between the liquid and gaseous phases always makes a sharp angle with the Ot axis, i.e., the pressure of the saturated vapor increases with temperature. The curve of phase equilibrium of a liquid and its vapor has a finite extent and terminates at the point K (Fig. 50). At temperature values higher than at the point K, the phase transition of the liquid into vapor or vice versa does not occur. This end point of the liquid-vapor phase-equilibrium curve is called the <u>critical point</u> and the temperature and pressure corresponding to it are called the <u>critical temperature</u> t_k and the <u>critical pressure</u> p_k. Consequently, at a temperature above critical no liquefaction of gas (conversion into liquid) occurs.

The existence of a final (critical) point on the phase-equilibrium curve of the liquid and solid phases shows that at high temperatures

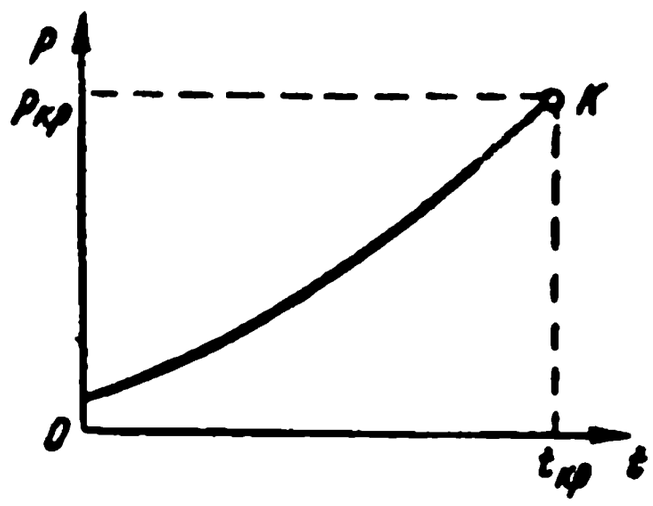

Fig. 50. Curve of phase equilibrium for liquid and vapor.

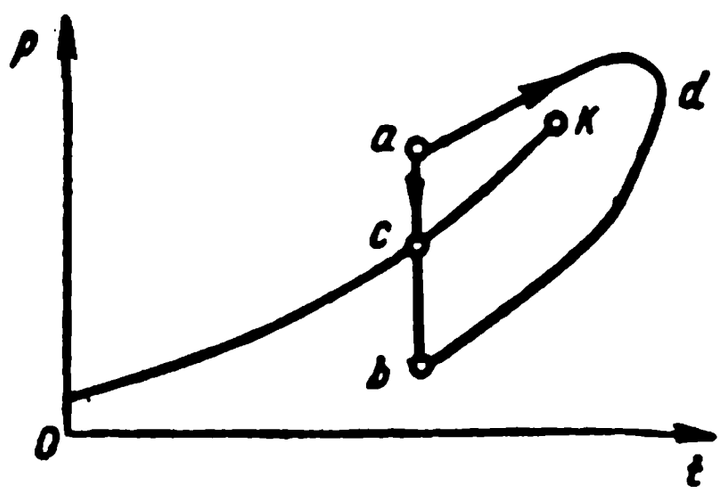

Fig. 51. Continuous transition from liquid to gas.

(large t_k) there is no principal difference between the liquid and gaseous states of the substance at high temperatures; the liquid and gas differ from each other under these conditions only quantitatively but not qualitatively.

It is possible to go from the liquid to the gas, for example from the point a to the point b and back, in addition to following the path ab (Fig. 51), which crosses the phase-equilibrium curve and which is accompanied by a phase transition at the point c, also along some line adb which envelops the critical point K. Since the line adb does not cross the phase-equilibrium curve anywhere, there will be no phase transformation at any of the points of this line. This means that the substance remains homogeneous all the time and nonetheless its initial and final states a and b correspond to different phases: the initial to liquid and the final to gas. To indicate where the transition from the liquid to the gas took place is impossible in this case (it is impossible to demarcate between the liquid and gaseous states at high temperatures).

The region of the liquid state is arbitrarily called the region above the phase-equilibrium curve between the ordinate axis and the critical isotherm; all the remaining region of the single-phase state is regarded as the gas region. The absence of a principal difference between the liquid and the gas in the critical and supercritical regions is due to the identical character of interaction between the molecules in liquids and in gases at high temperatures, as a result of

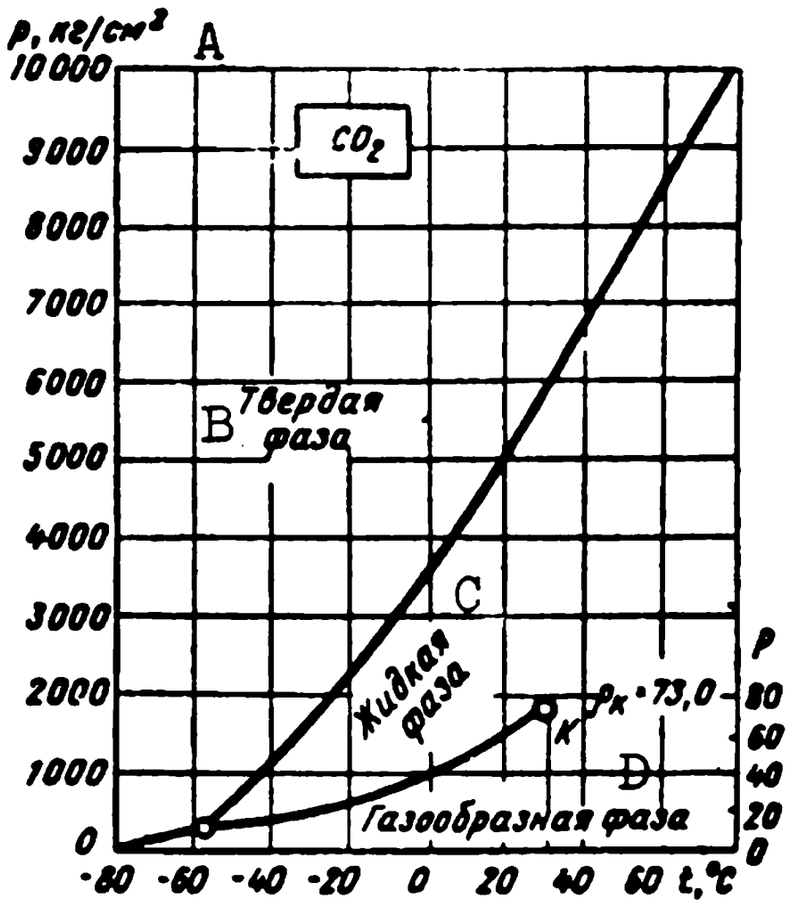

Fig. 52. Phase-equilibrium curve of crystalline, liquid, and gaseous phases of CO_2. A) kg/cm^2; B) solid phase; C) liquid phase; D) gaseous phase.

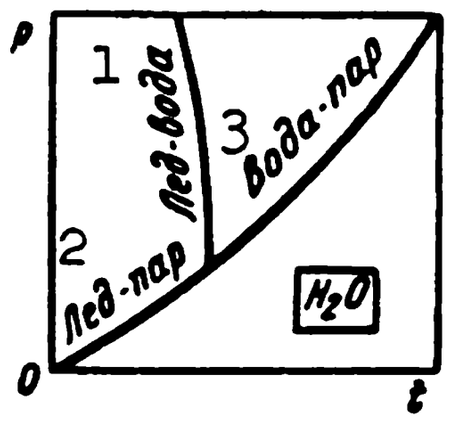

Fig. 53. Phase-equilibrium curve of crystalline, liquid, and gaseous phases of H_2O. 1) Ice-water; 2) ice-vapor; 3) water-vapor.

which the liquid and gas can be regarded as two isotropic phases of the substance which differ from one another only quantitatively.

The difference between the liquid and crystalline or gaseous and crystalline states of matter has a different character. The crystalline and liquid or gaseous states differ from each other qualitatively, since the crystalline state is an anisotropic phase of matter, while the liquid (or gaseous) state is an isotropic phase. Therefore a continuous transition from the solid state into the liquid is hardly possible, as a result of which the phase-equilibrium curve between the crystalline and the liquid phases does not have an end and apparently there exists no critical point in the phase transformation from the crystalline phase to the liquid. At the same time, it must be noted that at a temperature close to the crystallization point there are similar features in the properties of the crystalline and liquid bodies. At low temperatures the liquid is closer to the solid state and its properties than to the gaseous state. This is confirmed by the presence of a certain molecular structure in liquids near the crystallization point.

The phase-equilibrium curve $p_s(t)$ of the crystalline and liquid

phases can make both an acute and an obtuse angle with the positive direction of the Ot axis. In a phase equilibrium between the crystalline and gaseous phases, this angle will always be acute (Figs. 52 and 53). The phase-equilibrium curves of each two out of three phases cross; as a result of this the entire pt plane is broken up into regions corresponding to the crystalline, liquid, and gaseous phases. At the point of intersection of the three curves of the phase equilibrium the three different phases of the substance will be in contact and in equilibrium. Points in which three phases of substance are in equilibrium are called triple points. The values of the pressure and temperature at the triple point can be analytically determined from the equilibrium conditions of the three phases, conditions which have the same form as Conditions (6.7).

The third of these conditions (with account of equality of the pressures and temperatures of all three coexisting phases) yields two equations with two unknowns

$$\varphi^{(1)}(p, t) = \varphi^{(2)}(p, t) = \varphi^{(3)}(p, t),$$

the solution of which yields the sought values of the pressure and temperature at the triple point.

More than three phases of the substance cannot exist in equilibrium. Indeed, in the case of equilibrium of four phases we would have for the two variables p and t, which determine the position of the equilibrium point, the three equations

$$\varphi^{(1)}(p, t)=\varphi^{(2)}(p, t)=\varphi^{(3)}(p, t)=\varphi^{(4)}(p, t),$$

which are incompatible and which do not lead to definite values of p and t. It follows therefore that the points of equilibrium of four phases do not exist. The result obtained is a consequence of the so-called Gibbs phase rule, according to which the number of coexisting phases can exceed the number of independent components by not more

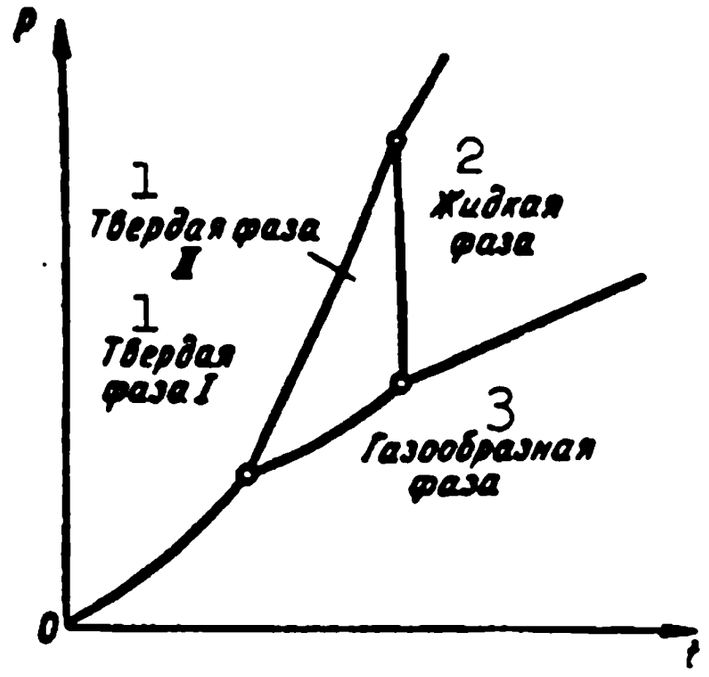

Fig. 54. Triple point of equilibrium of two crystalline modifications and the liquid phase. 1) Solid phase; 2) liquid phase; 3) gaseous phase.

than two.

When a solid crystalline body is heated at constant pressure, a transition takes place into the liquid or directly into the gaseous state, depending on the pressure. If the pressure is smaller than the pressure at the triple point or much larger than the pressure of the critical point, the substance will go from the crystalline state directly into the gaseous state, bypassing the liquid state. If the pressure is larger than the pressure at the triple point and smaller than the pressure at the critical point, then the substance will go from the crystalline state first into the liquid state and then upon further heating into the gaseous state.

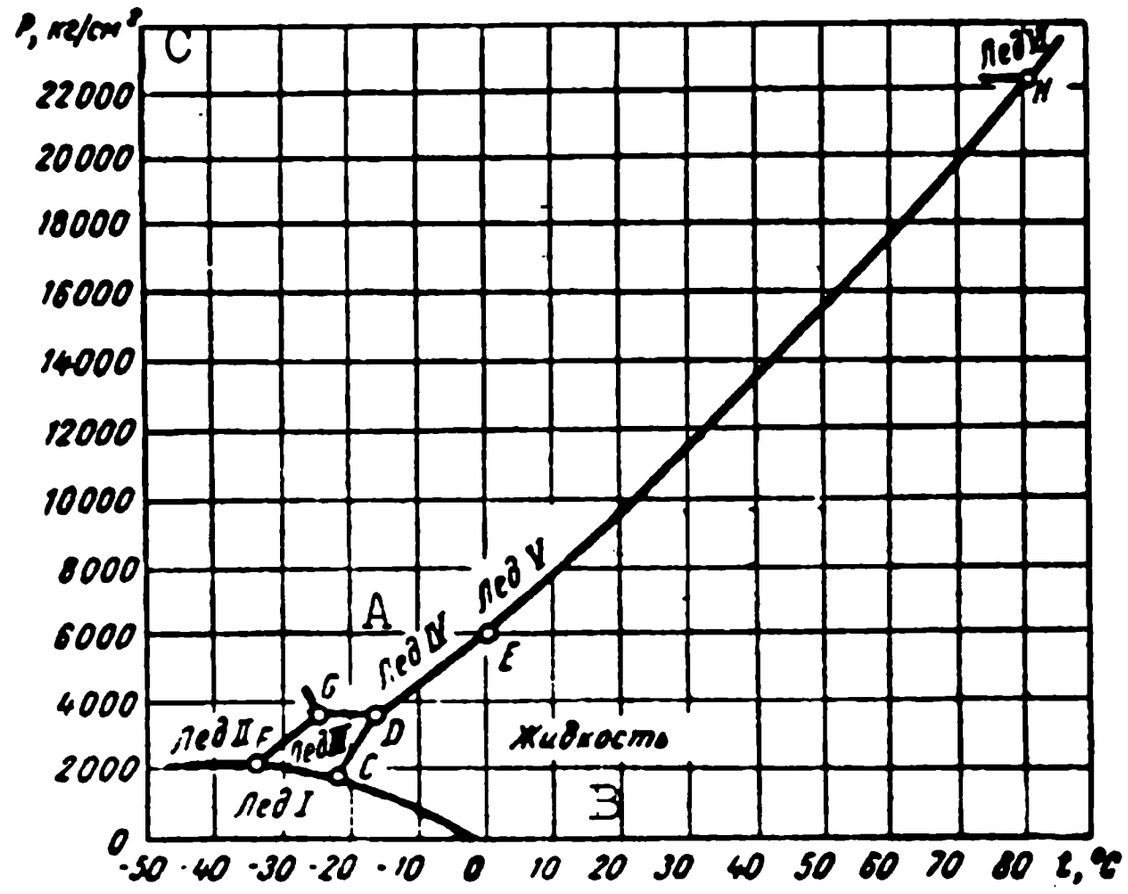

Fig. 55. Phase diagram for water.
A) Ice; B) liquid; C) kg/cm^2.

The phase-equilibrium curve between the crystalline and gaseous phases goes through the origin O, i.e., as $T \rightarrow 0$ the vapor pressure above the crystal is equal to zero. At absolute zero of temperature

all the substances are in the crystalline state, independently of the pressure. An exception is helium, which at pressures smaller than 25 atm remains liquid down to absolute zero. Many substances have several crystalline or allotropic modifications (phases). In such substances the phase diagram will have not one but several triple points. At some of these points two crystalline modifications and a liquid (or gaseous) phase, or else three crystalline phases, will be in equilibrium (Fig. 54). The equilibrium of the gaseous, liquid, and one of the crystalline phases will occur at only one point, which is the main triple point. The phase diagram for water, for which six crystalline modifications are known, is shown in Fig. 55.

pv Phase Diagram

In addition to the phase diagram in the pt plane, considerable interest is attached to the phase diagram in the pv plane. The latter differs in the fact that the region of equilibrium of the two phases, which on the pt diagram is represented by a line, occupies a certain area in the pv plane (Fig. 56).

Area I on this diagram represents the region of the gaseous state, area III represents the liquid state (this region is bounded by a portion of the critical isotherm KE), while area V is that of the crystalline state of the substance. The two-phase states of the substance are represented by areas II, IV, and VI, respectively. In any of these regions, the substance is in the form of a mixture of two coexisting phases. Region II represents the region of equilibrium existence of the liquid and gaseous phases, region IV is the region of equilibrium coexistence of the liquid and solid phases, and region VI is that of equilibrium coexistence of the crystalline and gaseous phases. Each of these regions is bounded by some line, which is called the boundary line.

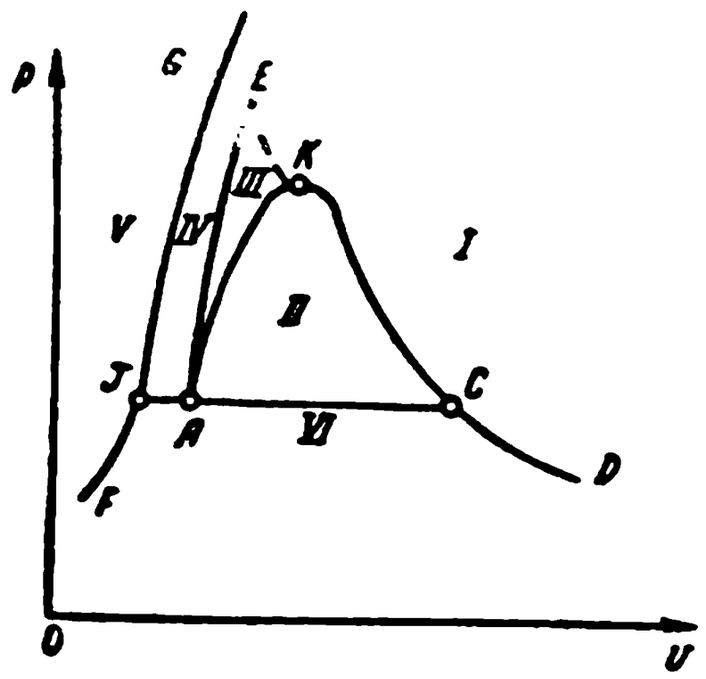

Fig. 56. Region of equilibrium of two phases.

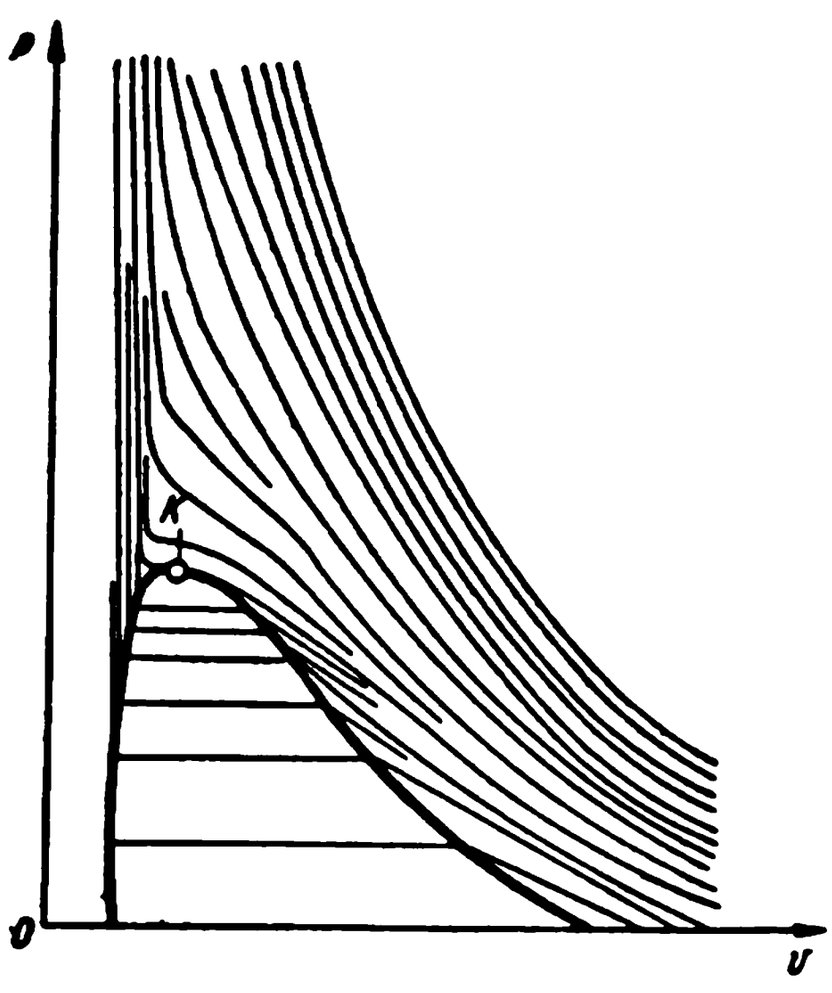

Fig. 57. General view of the isotherms in the region of the liquid and gaseous states of a substance.

The value of the abscissa of any of the points of the boundary curve is the specific volume v of the corresponding phase on the phase-equilibrium curve. Thus, the curve KCD determines the volume of the saturated vapor v"; curve AK determines the volume v' of the liquid under the pressure of its own saturated vapor (boiling liquid); curve AE is the volume of the liquid phase in equilibrium with the solid phase, and curve FJG is the volume of the crystalline phase. The course of the curves KCD and AK shows that with increasing temperature the volume of the boiling liquid v' increases and the volume of the saturated vapor v" decreases; at the point K, the values of v' and v" become equal. The point K is the critical point of the given substance. The segment JC corresponds to equilibrium of all three phases (triple point on the pt diagram).

The phase transition from the crystalline to the liquid or to the gaseous state and from the liquid state to the gaseous state occurs at constant pressure without a change in temperature and is represented on the pv diagram by a horizontal segment (called the straight-line segment of the isotherm) moving, respectively, from the curve FJC to the curve AE or CD and from the left branch of the boundary curve AKC to the right branch of this curve. Each of these segments

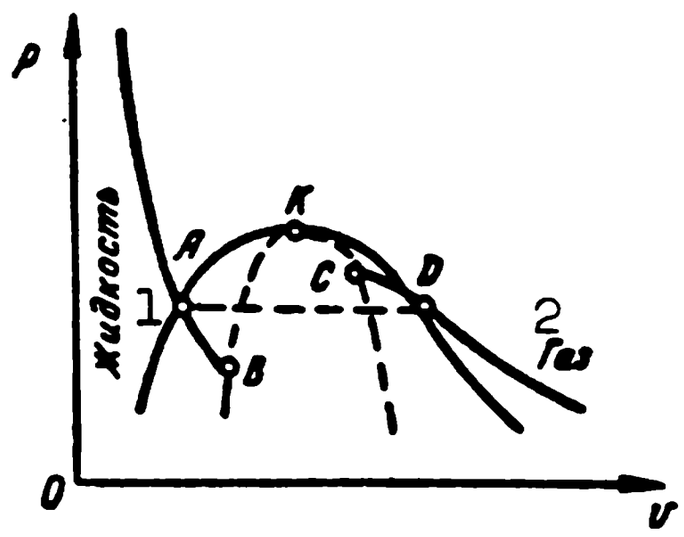

Fig. 58. Character of the isotherms near the metastable states of homogeneous matter (after Van der Waals). 1) Liquid; 2) gas.

has a length equal to the increment in the volume in the phase transition. The over-all view of the isotherm in the region of the liquid and gaseous states of the substance is shown in Fig. 57. In the region of homogeneous (single-phase) states of the substance, each of the isotherms, in accordance with the condition $(\partial p/\partial v)_t < 0$ for the stability of the equilibrium, moves from left to right and downward, while in the region of the two-phase states it moves parallel to the Ov axis.

If we plot on the pv diagram in addition to the stable states also the metastable states of the homogeneous substance, then the isotherms will have the form shown in Fig. 58. At each of the points of the isotherm, the substance is in a homogeneous state and therefore $(\partial p/\partial v)_t < 0$. The limiting points B and C are the boundary states of the homogeneous substance. At these points the substance is, respectively, in the form of a superheated and supersaturated vapor; the derivative $(\partial p/\partial v)_t$ at the boundary points B and C vanishes, as a result of which the point B can be regarded as the point of the minimum of the isotherm, while the point C is its maximum. The straight-line segment AD of the isotherm represents an equilibrium phase transition. With increasing temperature, the isotherms shift upward and the extent of the linear portion decreases. At the critical temperature t_k the length of the straight-line portion of the isotherm vanishes: the points B and C of the critical isotherm merge with the critical point K. But the point B is a minimum and C a maximum of the isotherm, consequently at the point where they coincide (at the critical point) the isotherm should have an inflection. It follows therefore that at the

critical point the derivatives $(\partial p/\partial v)_t$ and $(\partial^2 p/\partial v^2)_t$ should vanish and the next partial derivative $(\partial^3 p/\partial v^3)_t$ should be negative. This result is in full agreement with the results reported in §1 of Chapter 6.

If a continuous line is drawn through the limiting points B and C of all the isotherms, then this line will indicate the boundary of the metastable or (in the more general sense) homogeneous states of the substance. At all points of this boundary line we have $(\partial p/\partial v)_t = 0$, which can be regarded as an equation for this line.

§4. THERMODYNÁMIC DIAGRAMS AND TABLES

A thermodynamic analysis of the processes involved in the change of the state of bodies calls for analysis of the properties of these bodies. The thermodynamic properties of the technically most important working media have by now been sufficiently thoroughly investigated. On the basis of exact experiments for such important substances as water, carbon dioxide, ammonia, and others, special tables have been compiled, containing in systematized form data on the thermodynamic properties of these substances. For a thermodynamic analysis it is necessary to have in addition to tables also thermodynamic diagrams, with the aid of which it is convenient to analyze all possible processes and cycles. The most useful are the Ts, is, and it diagrams.

Ts Diagram

On the Ts diagrams the abscissas represent the values of the specific entropy s of the given substance and the ordinates the temperature T (or t, $^\circ$C). A grid of isobars (lines p(t, s) = const) and of equal-enthalpy lines (lines i(t, s) = const) is plotted on the Ts plane, and sometimes also the isochores (the lines v(t, s) = const). In addition, the boundary curve is drawn, while in the region of the two-phase liquid-vapor state lines of equal steam contents (x = const,

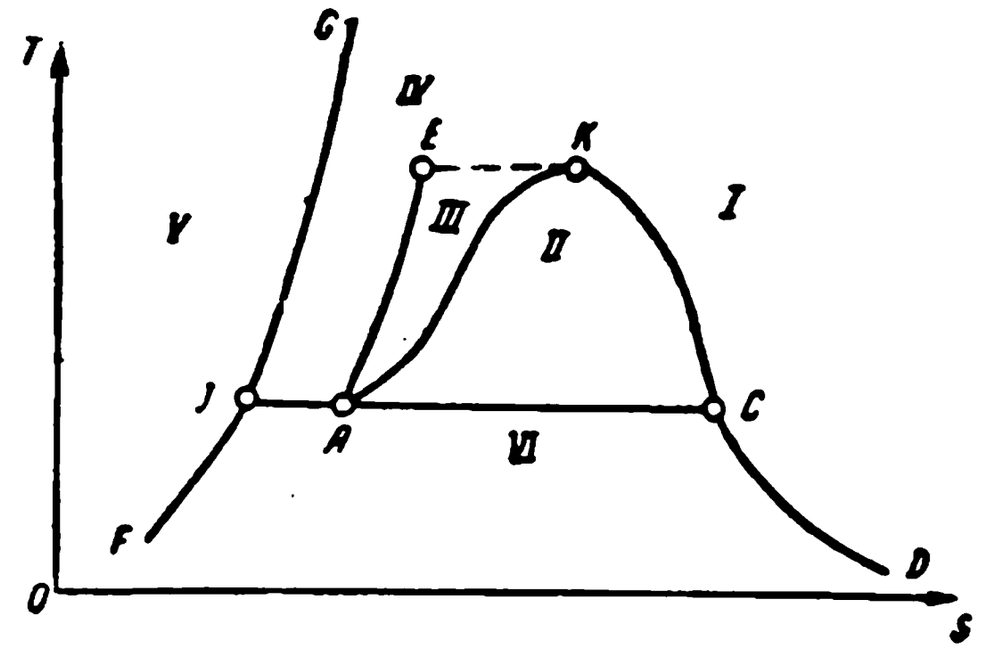

Fig. 59. General view of Ts diagram.

where $x = G''/(G' + G'')$ are drawn.

The boundary curve and also the isobars, the isochores, and the equal-enthalpy lines are drawn on the basis of processing of experimental data, and in some cases on the basis of theoretical considerations.

An over-all view of a Ts diagram is shown in Fig. 59. Here AKC is the boundary curve between the liquid and vapor phases; the point K of this curve is the critical point. The branch AK of the boundary curve, also called the left boundary curve, is the line of the boiling liquid (in equilibrium with the saturated vapor); the branch KD or the right boundary curve is the line of dry saturated vapor. Region I is the region of superheated vapor, region III is the liquid region, and region II corresponds to the equilibrium coexistence of the liquid and vapor phases. Region V represents the region of the solid state of the substance; region IV represents the region of equilibrium coexistence of the solid and liquid phases, while region VI is the region of equilibrium coexistence of the solid and vapor phases. The line JC, which is the demarcation between boundaries IV and VI, II and VI, determines the temperature of the equilibrium coexistence of all three phases, i.e., the triple point of the substance.

In technical thermodynamics it is customary to confine oneself

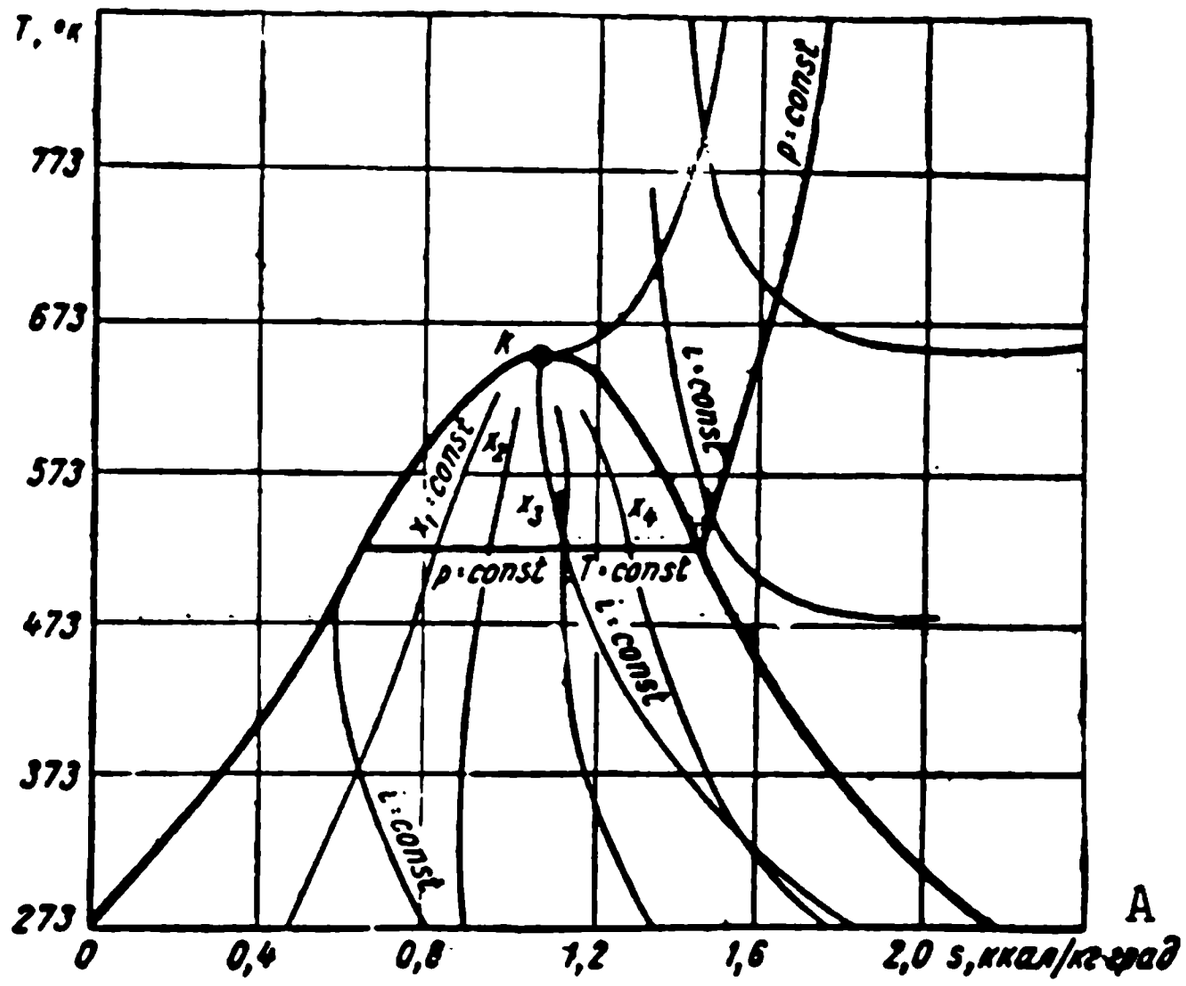

Fig. 60. Ts diagram for water. A) kcal per kg-deg.

only to that part of the diagram which pertains to the gaseous and liquid states of the substance.

The isobars have a slope $(\partial t/\partial s)_p$ equal in accordance with Eq. (3.15) to T/c_p. Since T is essentially a positive quantity and c_p is greater than 0 for all bodies, we also have $(\partial t/\partial s)_p > 0$. Therefore the isobars in the region of the single-phase state of the substance represent rising continuous curves, as is seen from the Ts diagram for water (Fig. 60). The higher the pressure of the body, the higher are the corresponding isobars located; these do not cross each other. The indicated placement of the isobars follows from Relation (3.51) according to which $(ds/dp)_t$ is equal to $-(dv/dt)_p$. For gases $(\partial v/\partial t)_p > 0$, and therefore $(\partial s/\partial p)_t < 0$; on shifting horizontally to the right (when $ds > 0$) we have $dp < 0$, i.e., p decreases. The same is true for a liquid, except for the region in which $(\partial v/\partial t)_p < 0$ (such a region exists, as is well known, for water at $t < 4^\circ C$). In this region $(\partial s/\partial p)_t > 0$, and consequently the arrangement of the isobars is reversed; the higher the pressure, the lower the isobar. The change in the course of location of the isobars* occurs on the boundary of this

region, on the line $(\partial v/\partial t)_p = 0$. In the region of the two-phase state of the substance the isobars coincide with the isotherms and represent horizontal straight lines.

At the critical point $(\partial t/\partial s)_p = 0$ (the critical point K is the point of maximum of the boundary curve).

The slope of the isochore is in accordance with (3.14) equal to T/c_V, and since $T > 0$ and $c_V > 0$ it follows therefore that $(\partial t/\partial s)_V > 0$. Thus the isochore, like the isobar, is a rising continuous curve and the slope of the isochore exceeds the slope of the isobar. The larger the specific volume, the lower is the corresponding isochore situated.

Lines of equal heat content or equal-enthalpy lines run as shown in Fig. 60. In the region of the two-phase state and near it, the slope of the equal-enthalpy line is negative, $(\partial T/\partial s)_i < 0$, and therefore equal-enthalpy lines represent descending curves.

The is Diagram

The abscissas of this diagram (Fig. 61) are the values of the specific entropy s, while the ordinates are the values of the specific enthalpy i; the diagram is marked with a grid of isotherms $t(i, s) =$ $=$ const, isobars $p(i, s) =$ const, and in some cases also isochores $v(i, s) =$ const. In addition, as in the Ts diagram, the boundary curves are drawn and in the region of the two-phase "liquid-saturated vapor" state one draws lines of equal steam contents.

The isobars on the is diagram have the form of rising intersecting curves with slope $(\partial i/\partial s)_p$, equal in accordance with (3.13) to the absolute temperature T. The isobars move smoothly through the boundary curve without jumps in the values of the derivatives. The isobars corresponding to the greater pressure are located above the isobars with the smaller pressure. This can be readily verified by considering the

transition from lower values of i to larger ones along the vertical line s = const. Indeed, when ds = 0 we have di = Avdp, from which it follows, inasmuch as v > 0, that di and dp have identical signs, i.e., when moving upward along the vertical the pressure increases.

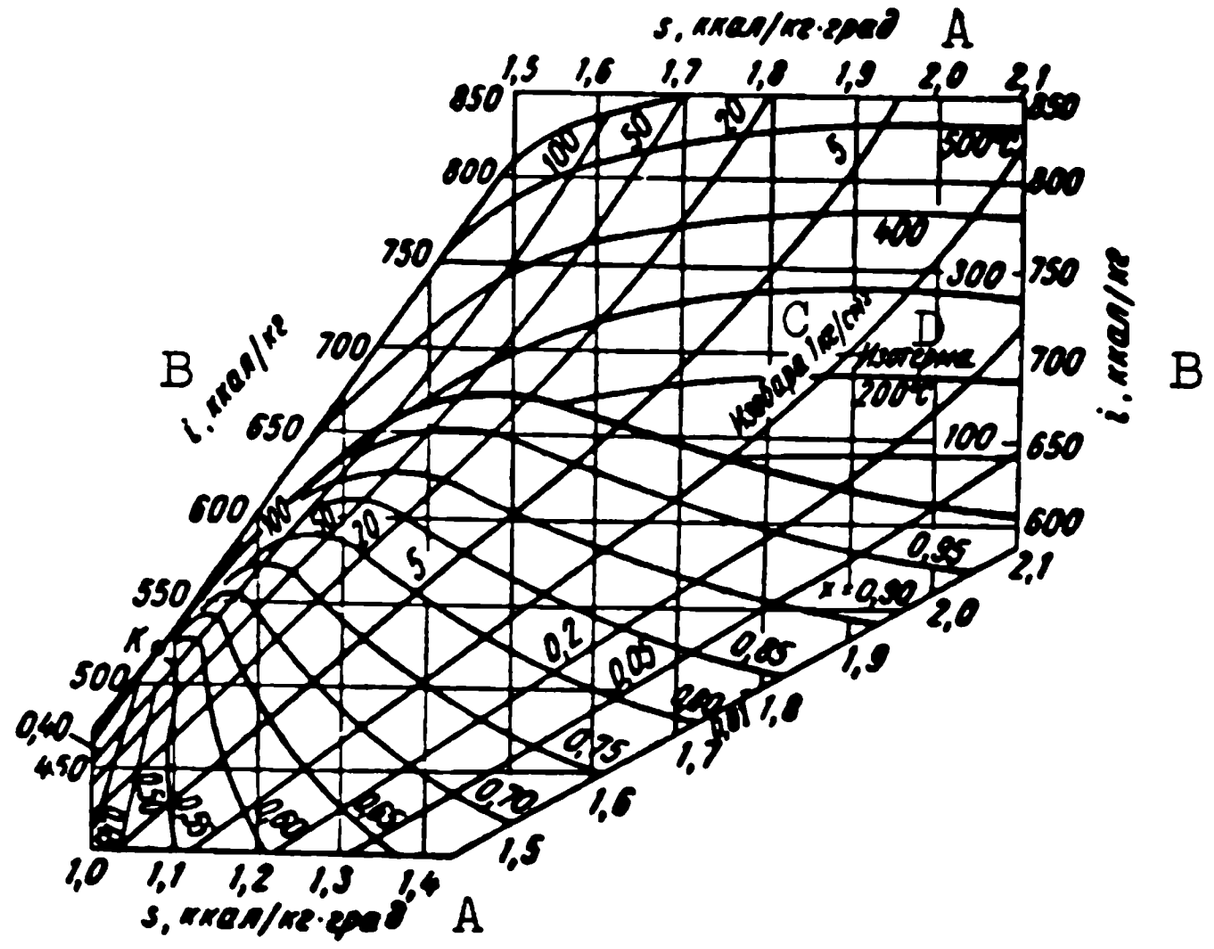

Fig. 61. Scheme of is diagram. A) kcal per kg-deg; B) kcal/kg; C) isobar, 1 kg/cm^2; D) isotherm.

The isotherms on the is diagram in the region of the gaseous state of the matter are curves which have a positive slope near the saturation curve with the convex part upward. The larger the temperature t, the higher the corresponding isotherm. In the region of the two-phase state of the substance the isotherms coincide with the isobars and represent straight lines with positive slope equal to T. The length of the straight-line portion of the isotherm decreases with increasing t and turns into a point at $t = t_k$. Therefore at the critical point the derivative $(\partial i/\partial s)_t$ is equal to T_k, i.e., it has a positive value; it follows from this that the critical point is on the rising part of the boundary curve, as is seen from Fig. 61.

The isochores on the is diagram are represented by curves with positive slope. The larger the specific volume of the substance, the

lower is the corresponding isochore located.

The is diagram is very convenient for an exact determination of the enthalpy difference at the start and at the end of the process. Therefore this diagram is widely used to calculate processes of thermally insulated flow, which are characterized by the fact that the increase in the kinetic energy per kilogram of moving gas is equal, as was shown in §7 of Chapter 2, to the loss in the specific enthalpy. With the aid of the is diagram it is easy to determine the amount of heat q obtained by 1 kg of the working medium in an isobaric process, this value being equal to the difference in the specific enthalpies in the final and initial states.

The it Diagram

This diagram is less widely used than the Ts and is diagrams, but it is used quite frequently in various calculations. The abscissas (Fig. 62) represent the values of the temperature t and the ordinates the values of the specific enthalpy i and, in addition, an isobar grid, the boundary curve, and the line of constant heat contents are also drawn.

Inasmuch as $(\partial i/\partial t)_p = c_p$, in the region of the single-phase (gaseous and liquid) states of the substance all the isobars have a positive slope. In the region of the two-phase state of the substance, where $c_p = \infty$, the isobars pass parallel to the ordinate axis. It follows therefore that at the critical point we have $(\partial i/\partial t)_p = \infty$, i.e., the critical point K is on the boundary curve to the right.

The amount of heat q obtained by 1 kg of the medium in the isobaric process 1-2, which is equal to $i_2 - i_1$, is represented on the it diagram by the length of a perpendicular dropped from point 2 of the given isobar on a straight line parallel to the Ot axis and passing through the point 1. The simplicity of the graphic determination of

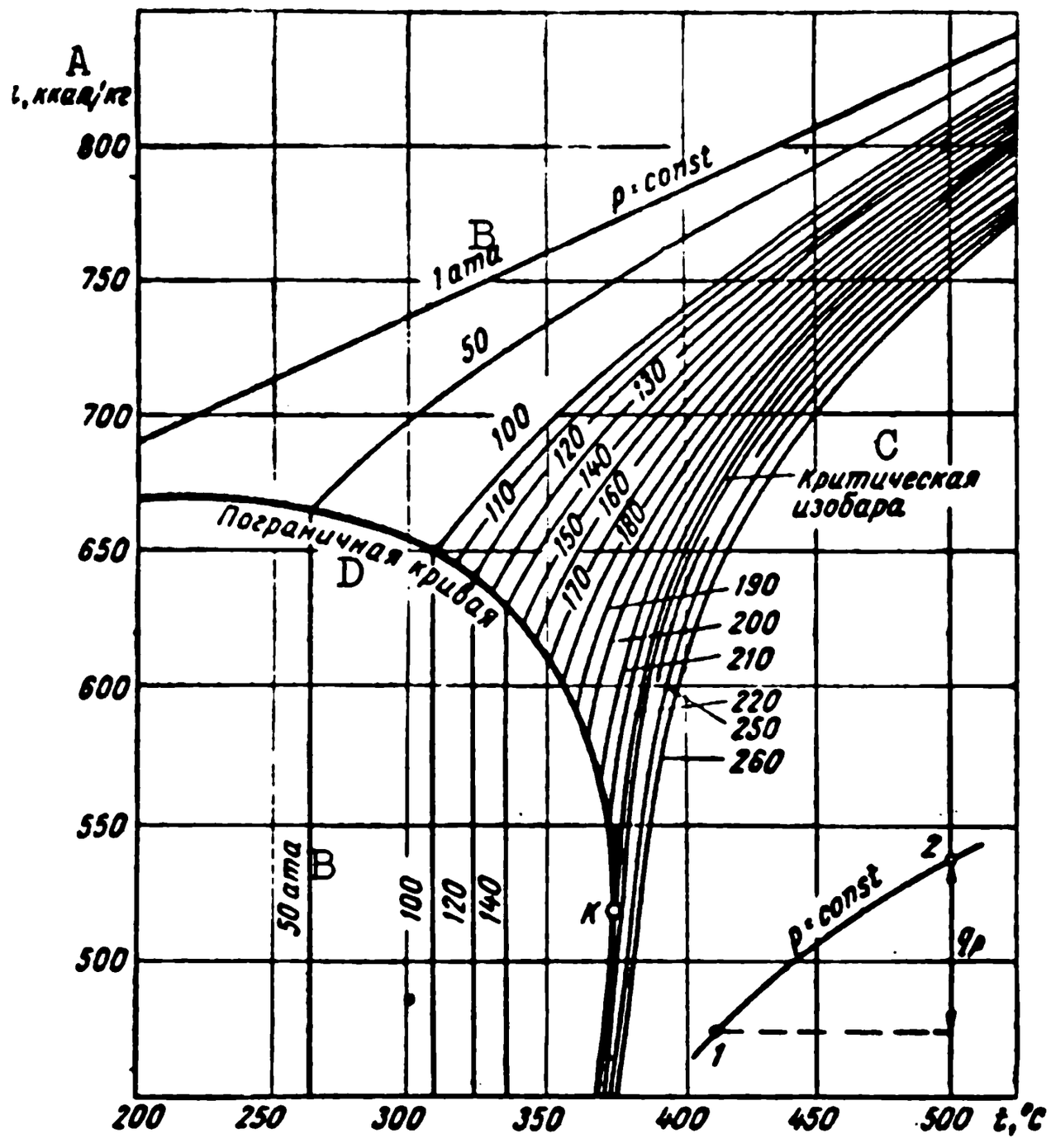

Fig. 62. Scheme of it diagram. A) kcal/kg; B) atm abs; C) critical isobar; D) boundary curve.

the heat of the isobaric process is the main advantage of the it diagram. This diagram is quite convenient for the design of heat exchangers. Assume, for example, a heat exchanger in which the heat carrier giving up the heat is substance 1 and the working medium which receives the heat is substance II. The initial temperatures of both bodies are $t_0^{(1)}$ and $t_0^{(2)}$, and their flows by weight are $G^{(1)}$ and $G^{(2)}$ kg/sec.

If no useful technical work is performed by the flow and the increase in the kinetic energy is equal to zero and there is no leakage of heat, then all the heat given up by the first substance is absorbed by the second. But in this case, in accordance with (2.28) $Q = I_0 - I$ and the following heat-balance equation will therefore hold:

$$G^{(1)}\left(i_0^{(1)} - i^{(1)}\right) = G^{(2)}\left(i^{(2)} - i_0^{(2)}\right).$$

Denoting $G^{(2)}/G^{(1)}$ by g, we have

$$i_0^{(1)} - i^{(1)} = g\left(i^{(2)} - i_0^{(2)}\right).$$

This equation, being the consequence of the second equation of (2.28), remains in force also for viscous liquids when the pressure is not constant, but decreases in the flow direction. With the aid of the heat-balance equation we can determine from the condition that relates the final temperatures of the heat carrier and the working medium (for example, from the equation $t^{(1)} = t^{(2)} + \Delta$, where Δ is a specified value) the parameters of both bodies at the output of the heat exchanger. The final temperature is particularly easy to determine graphically. For this purpose we draw lines showing the change in state of the substance $i = i(t)$ during the flow* for 1 kg of substance I and for g kg of substance II (Fig. 63) and we find on these curves the points C^I and C^{II}, which are situated at a distance Δ in the vertical direction and at equal distances in the horizontal direction relative to the initial points A and B. Since we have in accordance with the construction $i_A^{(1)} - i_C^{(1)} = g(i_C^{(2)} - i_B^{(2)})$, the points C^I and C^{II} represent the state of the substances I and II at the output of the heat exchanger; the ordinate of each of these points represents the sought final temperature t_{kon}^I and t_{kon}^{II}.

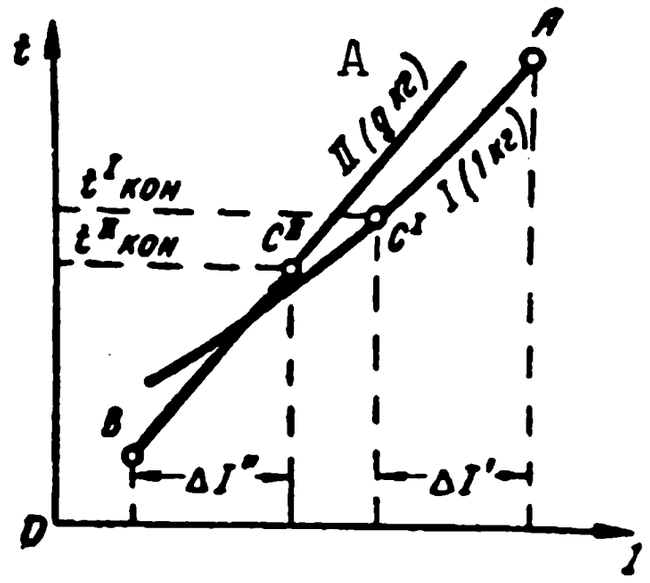

Fig. 63. Diagram for the determination of the final temperature in the case of heat exchange between two streams of liquid. A) kg.

The calculation becomes simpler if in place of I we measure along one of the axes the difference $I - I_0$, i.e., the amount of heat Q (a Qt diagram).

§5. HEAT OF PHASE TRANSITION

The amount of heat absorbed (or released) in an equilibrium isothermal transition of 1 kg of substance from one phase to another is called the <u>specific heat of phase transition (transformation)</u>. Depending on the type of phase transition, a

distinction is made between the heats of evaporation (or vapor formation), condensation, melting, and solidification. An equilibrium isothermal phase transition is in accordance with the conditions (6.7) also isobaric, and therefore the specific heat of the phase transition can be determined from the following two equations:

$$r = i^{(2)} - i^{(1)}, \tag{6.9}$$

$$r = T(s^{(2)} - s^{(1)}). \tag{6.10}$$

The specific heat of phase transition is therefore equal to the difference in the specific enthalpies of the two phases and also to the difference in the specific entropies of the two phases multiplied by the absolute temperature at which the transition occurs.

On the phase-equilibrium curve the pressure p and the volume of each phase $v^{(1)}$, $v^{(2)}$, and also the values of i and s, are functions of the temperature. Therefore the heats r of the phase transition are a function of the temperature. To find the form of this dependence, let us differentiate Eq. (6.10) with respect to T, as a result of which we obtain on the basis of (3.42)

$$\frac{dr}{dT} = \frac{r}{T} + (c_V^{(2)} - c_V^{(1)}) + AT\left[\left(\frac{\partial p}{\partial T}\right)_V^{(2)} \cdot \frac{dv^{(2)}}{dT} - \right.$$
$$\left. - \left(\frac{\partial p}{\partial T}\right)_V^{(1)} \cdot \frac{dv^{(1)}}{dT}\right]. \tag{6.11}$$

Phase transitions accompanied by absorption or release of heat are called first-order phase transitions. They are characterized by a jump-like change in many properties of the substance: the internal energy, the volume, the entropy, the enthalpy, and the compressibility. Some properties, for example the isobaric potential, do not experience jumps in first-order phase transitions. First-order phase transitions include evaporation and condensation, melting and crystallization, and many transitions from one crystalline modification to another.

In addition to first-order phase transitions, there exist phase

transitions for which no heat is released or absorbed. Such phase transitions are called second-order phase transitions. In second-order phase transitions no jump-like changes occur in the internal energy, the volume, the entropy, or the enthalpy, but on the other hand the specific heats and the coefficients of thermal expansion change jumpwise at the transition point. Second-order phase transitions are usually observed in crystals; they consist of changes in the degree of symmetry of the crystal and of transitions to higher symmetry occurring at a definite temperature. In general there are no other phase transitions except the first- and second-order transitions.

§6. THE CLAPEYRON-CLAUSIUS FORMULA

As was shown in §3 of the present chapter, the pressure of two phases in equilibrium is a function of the temperature. To establish this dependence we make use of Condition (6.6) that the chemical potentials of both phases be equal. Let us assume that the conditions of equilibrium have changed somewhat, i.e., the parameters p and T have been subjected to small increments dp and dT. In this new equilibrium position the chemical potentials of both phases will also be equal

$$\varphi^{(1)}(p+dp,\ T+dT)=\varphi^{(2)}(p+dp,\ T+dT).$$

Expanding the functions $\varphi^{(1)}$, $\varphi^{(2)}$ in powers of dp and dT and confining ourselves, owing to the smallness of dp and dT, to the first terms of the expansion, we obtain

$$\varphi^{(1)}(p,\ T)+\left(\frac{\partial\varphi}{\partial T}\right)_p^{(1)}dT+\left(\frac{\partial\varphi}{\partial p}\right)_T^{(1)}dp=$$

$$=\varphi^{(2)}(p,\ T)+\left(\frac{\partial\varphi}{\partial T}\right)_p^{2}dT+\left(\frac{\partial\varphi}{\partial p}\right)_T^{(2)}dp.$$

Since

$$\left(\frac{\partial\varphi}{\partial T}\right)_p=-s,\quad\left(\frac{\partial\varphi}{\partial p}\right)_T=Av,$$

and

$$\varphi^{(1)}(p,\ T)=\varphi^{(2)}(p,\ T),$$

we have

$$-s^{(1)}\,dT + Av^{(1)}\,dp = -s^{(2)}\,dT + Av^{(2)}\,dp.$$

But

$$s^{(2)} - s^{(1)} = \frac{r}{T};$$

therefore

$$\frac{r}{T}\,dT = A\,(v^{(2)} - v^{(1)})\,dp$$

or

$$\frac{dp}{dT} = \frac{r}{AT\,(v^{(2)} - v^{(1)})}. \qquad (6.15)$$

Equation (6.15) is called the Clapeyron-Clausius formula. The quantity dp/dT entering into this formula is the derivative of the pressure with respect to the temperature, taken along the phase-equilibrium curve. The Clapeyron-Clausius formula determines the change in pressure along the phase-equilibrium curve or, what is the same, the dependence of the equilibrium pressure of both phases on the temperature.

Since on going over from the crystalline or from the liquid state into the gaseous state the volume increases and the heat of evaporation r is a positive quantity, we have $dp/dT > 0$. This means that the pressure of the saturated vapor above a solid body or above a liquid increases with increasing temperature. Upon equilibrium of the solid and liquid phases the derivative dp/dT on the equilibrium curve can be both positive and negative. Inasmuch as r is always larger than zero, the sign of the derivative dp/dT will depend on the sign of the difference in the volumes $v_{zh} - v_t$ of the liquid and solid phases. If $v_{zh} > v_t$, we also have $dp/dT > 0$; when $v_{zh} < v_t$, we have $dp/dT < 0$.

For most substances $v_{zh} > v_t$ and $dp/dT > 0$, i.e., the solidification temperature rises with increasing pressure. However, for some substances there are exceptions. Thus, the specific volume of liquid

water is smaller than the specific volume of ice and $dp/dT < 0$, meaning that an increase in pressure causes a decrease in the melting temperature of ice. This anomaly of water has, as is well known, a tremendous practical significance. In addition to water, a similar anomaly is observed in bismuth and gallium.

The volume difference of the liquid and solid phases $v_{zh} - v_t$ is quite insignificant for all substances, so that the derivative dp/dT usually has a large value and the curve of phase equilibrium between the solid and the liquid is almost perpendicular to the temperature axis.

§7. PHASE TRANSITIONS FOR UNEQUAL PHASE PRESSURES

In §3 of Chapter 6 we considered the conditions of equilibrium of two contiguous phases, with respect to which it was assumed that the boundary between phases is free of the action of any external forces. In this case only the pressure forces of the phases themselves act on the separation boundary, as a result of which the equilibrium conditions lead to the requirement that the pressures, temperatures, and chemical potentials of both phases be equal.

Let us assume now that on the separation boundary there act in addition to the pressure also other forces. As before, for the equilibrium of the system it is necessary to satisfy the following: 1) mechanical equilibrium, namely that the sum of all forces acting on the surface of the phase separation be equal to zero; 2) thermal equilibrium of the system (equality of the temperatures of both phases); 3) chemical equilibrium (equality of the chemical potentials of both phases). Consequently, the phase-equilibrium conditions have in this case the form

$$p^{(1)} - p^{(2)} + p = 0; \quad t^{(1)} = t^{(2)}; \quad \varphi^{(1)}(p^{(1)}, t) = \varphi^{(2)}(p^{(2)}, t), \qquad (6.16)$$

where $p^{(1)}$ and $p^{(2)}$ are the pressures of the first and second phases,

while $\underline{p}$ is the pressure on the separation boundary between the phases due to the presence of the external forces.

Let us assume that the phase-equilibrium conditions have changed and the parameters $p^{(1)}$, $p^{(2)}$, T have become equal to $p^{(1)} + dp^{(1)}$, $p^{(2)} + dp^{(2)}$, T + dT. With the aid of the third condition in (6.16) we can establish the connection between the variations of the parameters upon shift in equilibrium. For this purpose we expand

$$\varphi^{(1)}(p^{(1)} + dp^{(1)},\ T + dT) \quad \text{and} \quad \varphi^{(2)}(p^{(2)} + dp^{(2)},\ T + dT)$$

in powers of dT and of $dp^{(1)}$ and $dp^{(2)}$, respectively, and confine ourselves to the first terms of the expansion. As a result we get

$$\frac{r}{AT}\,dT = v^{(2)}\,dp^{(2)} - v^{(1)}\,dp^{(1)}, \qquad (6.17)$$

where $\underline{r}$, $v^{(1)}$, $v^{(2)}$ are the values of these quantities in the initial state. Equation (6.17) is a generalization of the Clapeyron-Clausius equation for the case of unequal pressures of phases coexisting in equilibrium. This equation is used to investigate the phase transitions, for example melting, occurring upon adding to the external pressure of one of the phases and also to study the boiling and condensation processes under the action of capillary forces.

Capillary forces (surface-tension forces) occur, as is well known, in a thin surface layer on the phase-separation boundary (this layer is also called the capillary layer) as a result of mutual attraction of the molecules. The surface-tension forces strive to contract the capillary layer and reduce its surface. Owing to the surface tension on any curved separation boundary between the phases, additional normal forces arise, the magnitude of which per unit surface is called the <u>capillary</u> or <u>surface pressure</u>. The capillary pressure is equal to

$$p_\sigma = \sigma\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right). \qquad (6.18)$$

where ρ_1 and ρ_2 are the principal radii of curvature of the surface at

the given point and σ is the surface-tension coefficient. Formula (6.18) is called the Laplace formula.

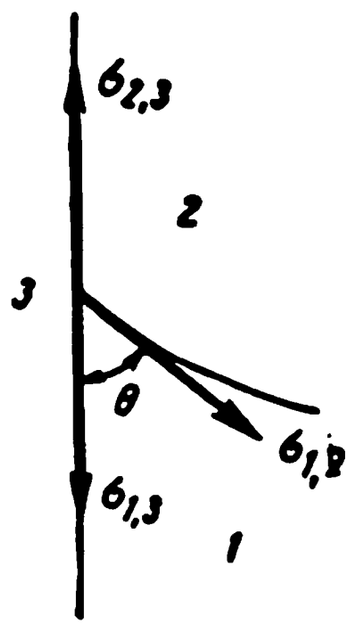

Fig. 64. Region of one- and two-phase states of a system.

In the case of a spherical phase-separation surface

$$p_\sigma = \frac{2\sigma}{Q}. \qquad (6.19)$$

The coefficient of surface tension σ characterizes the values of the surface-tension forces on the separation boundary between two phases in equilibrium and is therefore a function of only one variable, for example the temperature.

At the critical point, where the liquid and gaseous phases do not differ from each other, the separation boundary between the two phases vanishes and the coefficient of surface tension vanishes. The radius of curvature is considered positive in the case of a concave surface of the denser phase and negative for a convex surface.

The condition of mechanical equilibrium on a curved phase-separation surface in the presence of surface-tension forces will have the form

$$p^{(2)} - p^{(1)} = \sigma\left(\frac{1}{Q_1} + \frac{1}{Q_2}\right). \qquad (6.20)$$

i.e., the pressure difference between the less dense and more dense phases should be equal to the capillary pressure.

The presence of surface tension leads to a curving of the surface of the liquid in the case of contact between three bodies: solid, liquid, and gas.

Let the region 2 on Fig. 64 be occupied by the gas, region 1 by the liquid, and region 3 by the solid. All three bodies bound on a certain line perpendicular to the plane of the figure. To a unit length

of this line there are applied three surface-tension forces, due to the interaction between the first and second, second and third, and first and third bodies and equal, respectively, to $\sigma_{1,2}$, $\sigma_{2,3}$, $\sigma_{1,3}$. Each of these forces is directed inside the separation boundary between the corresponding two bodies. In the equilibrium state the resultant of all three forces of surface tension should not have a component along the surface of the solid body:

$$\sigma_{2,3} = \sigma_{1,3} + \sigma_{1,2}\cos\theta. \qquad (6.21)$$

The angle θ between the surface of the liquid and the surface of the solid is called the border angle. It follows from (6.21) that if $\sigma_{2,3} > \sigma_{1,3}$, i.e., if the surface tension between the gas and the solid is larger than between the solid and the liquid, then the border angle θ is acute; in this case the liquid partially wets the solid surface. When $\sigma_{2,3} < \sigma_{1,3}$ the border angle θ is obtuse and the liquid does not wet the solid surface. In the case of complete wetting of the solid by the liquid, when $\theta = 0$, the liquid flows without limit over the surface of the solid. In this case equilibrium between the three contiguous bodies, expressed by (6.21), is never established. Likewise, no equilibrium is established for liquids which absolutely do not wet the solid, i.e., for $\theta = \pi$. In this case a thin film of adsorbed gas is formed on the surface of the solid.

The state of a thermodynamic system in the presence of surface tension will be determined by the temperature t and volume V of the system and by the area Ω of the phase-separation surface. Accordingly, the internal energy U, the free energy F, and the entropy S of the system can be represented in the form of a sum of two terms, the first of which depends on volume effects and represents the known expressions for U(t, V), F(t, V), and S(t, V), while the second expresses the dependence of U, F, and S, on the surface phenomena, i.e., on the

coefficient of surface tension σ and on the area Ω of the phase-separation surface:

$$U = U(T, V) + A\left(\sigma - T\frac{d\sigma}{dT}\right)\Omega;$$

$$F = F(T, V) + A\sigma\Omega;$$

$$S = S(T, V) - A\frac{d\sigma}{dT}\Omega.$$

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[Footnotes]

162 We assume first that there are no capillary forces.

177 The indicated variation in the course of the isobars gives the impression that they intersect. Actually the isobars do not intersect.

182 For flow without friction these will be isobars; for flow with friction the form of the line $i = i(t)$ is determined by the aggregate of the flow equation (see §1, Chapter 8). If the liquid is incompressible, this beingsatisfied with sufficient accuracy away from the critical point, then $di = c_p dt$ and therefore the equation of the line $i = i(t)$ has the following form: $i - i_0 = \int_{T_0}^{T} c_p dT$.

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[List of Transliterated Symbols]

167 кр = kr = kriticheskiy = critical

182 кон = kon = konechnyy = final

185 ж = zh = zhidkost' = liquid

185 т = t = tverdoye telo = solid

Chapter 7

VAPORS OF LIQUIDS

§1. EVAPORATION AND CONDENSATION

Physical Processes Occurring in Evaporation

An equilibrium phase transition from the liquid state to the gaseous state for a given pressure – from the point 1 on the left branch of the boundary curve to point 2 on the right branch (Fig. 65) – occurs at a fully defined temperature [see Condition (6.7)] and is represented by the straight-line portion of the isotherm AB. From the molecular point of view, the transition from the liquid to the vapor consists of evaporation of the liquid, i.e., of an escape of a certain fraction of the molecules, which are in thermal motion, to the outside of the confines of the liquid phase.

Let us assume that a liquid is contained in a closed vessel (for example in a cylinder under a piston) and occupies part of the vessel. The molecules escaping from the liquid fill the free space above the surface of the liquid and form saturated vapor.* Part of these molecules returns to the liquid as a result of thermal motion. Between the transition of the molecules from the liquid to the vapor and the reverse transition of the molecules from the vapor to the liquid a dynamic equilibrium is established, as a result of which the density of the molecules above the liquid and consequently also the saturated vapor pressure assume for a given temperature a perfectly defined value. With increasing temperature the equilibrium shifts, causing a corresponding change in the density and saturated-vapor pressure.

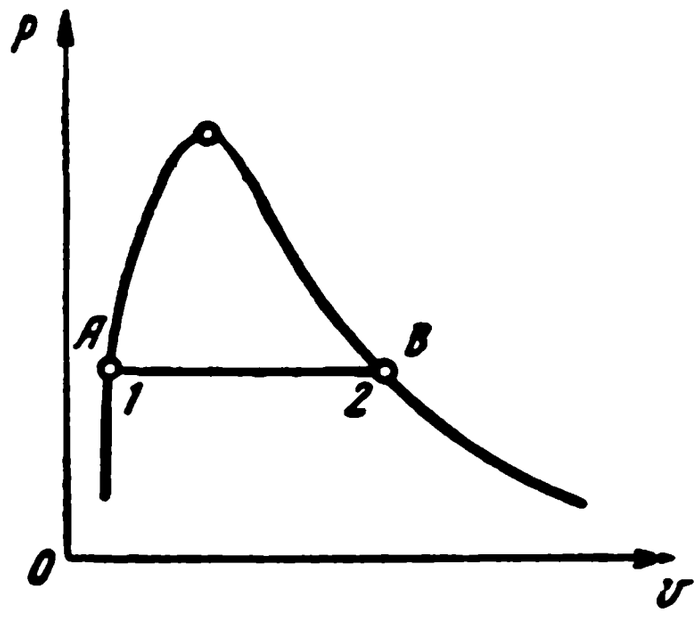

Fig. 65. Isobaric transition from liquid to gaseous state.

The evaporation of the liquid occurs not only from the free surface, but also inside the vapor bubbles which are formed in the liquid itself under certain conditions. If formation of vapor bubbles is possible, they arise in the liquid in large amounts and, since their summary surface exceeds by many times the free surface of the liquid, evaporation inside the vapor bubbles acquires a predominant value. Vapor bubbles are produced on the walls of the vessel, where there are projections (or pits) of roughness. Once produced, the vapor bubble becomes the center of evaporation of the liquid. The dimensions of the vapor bubble increase with increasing evaporation from it, as a result of which an increase takes place in the lifting force, which is proportional to the bubble and under the influence of which the bubble breaks away from the wall after reaching a given dimension and overcomes the forces of hydrodynamic resistance of the surrounding liquid, floating to the top of the liquid where it collapses. The bubble that has floated to the top is replaced by a newly formed vapor bubble. By motion of vapor bubbles from the lower layers of the liquid to its surface the vapor formed inside the liquid is continuously transported to the space above the liquid.

Boiling of Liquids

The process of intense evaporation of a liquid, occurring throughout its volume with formation of a large number of vapor bubbles, is called boiling of a liquid. In the case of equilibrium boiling, the temperature of the liquid and the pressure on it must satisfy Condition (6.7). It is necessary at the same time to supply heat to the

liquid from the outside.

The boiling temperature for a liquid at a given external pressure is the temperature at which the pressure of the saturated liquid vapor (above its plane surface) is equal to the external pressure. For example, when water boils in an open vessel the boiling temperature should be 100°C, since it is precisely at this temperature that the pressure of the saturated steam is equal to the atmospheric pressure.

Usually all liquids contain air and other gases in solution. Such liquids boil as soon as the temperature is raised to the boiling temperature. In those cases when the liquid does not contain dissolved gases, the boiling of the liquid starts at a somewhat higher temperature than that corresponding to the equality of the saturated-vapor pressure and the atmospheric pressure, i.e., the liquid is superheated at the start of the boiling. The phenomenon of superheating of a boiling liquid can be explained as follows. Inside the vapor bubble produced in the liquid, the saturated-vapor pressure will be less, owing to the action of surface-tension forces on the surface of the bubble, than the saturated-vapor pressure on the plane surface at the same temperature.

Therefore, when the temperature of the liquid reaches the boiling temperature the saturated-vapor pressure on the surface of the liquid in the vessel (owing to the relatively large dimensions of the vessel, this surface does not differ in practice from a plane surface) becomes equal to the external pressure, the pressure of the saturated vapor inside the steam bubbles in the liquid, which is always less than the pressure of the saturated vapor above the plane surface, will be lower than atmospheric. As a result of this, the steam bubble, if it is somehow produced inside the liquid, will be crushed by the larger external pressure. For a vapor bubble to be produced in the liquid it is

necessary that the saturated-vapor pressure p''_ρ inside the bubble become equal to the total pressure on the bubble. The latter consists of the following: 1) the external pressure p_0 on the liquid; 2) the capillary pressure due to the action of the surface-tension forces on the boundary of the vapor bubble and the liquid and equal in the case of a spherical bubble to $p_\sigma = 2\sigma/\rho$; 3) the hydrostatic pressure γh, which we neglect from now on owing to the smallness of the depth $\underline{h}$

$$p''_\rho = p_0 + \frac{2\sigma}{\rho}. \qquad (7.1)$$

Inasmuch as the saturated-vapor pressure is an increasing function of the temperature, in order for the saturated-vapor pressure in the bubble to reach the necessary value the temperature of the liquid at the heated walls of the vessel, where the steam bubbles are produced, should increase compared with the boiling temperature t_s (for a given external pressure) by a certain positive amount Δt for which Condition (7.1) is satisfied. The positive quantity Δt, equal to the difference between $\underline{t}$, where boiling begins at a given external pressure, and t_s, where the saturated-vapor pressure equals the latter, is called the <u>degree of liquid superheat</u> during boiling. The superheat of the liquid is the necessary condition for boiling; without superheat, no vapor bubbles can be produced in a pure liquid.

In the presence of dissolved air or other gases in the liquid, the evaporation occurs in air bubbles, as a result of which the action of the surface-tension forces is offset and does not influence the boiling, and in particular does not lead to a noticeable superheating of the liquid.

<u>Condensation of Vapor</u>

The compression of saturated vapor, carried out without removal of heat, can be accompanied, depending on the initial conditions, both

by condensation of the vapor (its conversion into liquid) and by the transition of the vapor from the saturated state into the unsaturated one. When heat is removed, compression of a saturated vapor always leads to condensation of the vapor. The equilibrium condensation of the saturated vapor for a given constant external pressure will occur, in accordance with the phase-equilibrium conditions (6.7), at constant temperature. Graphically this process is represented by segment 2-1 of the isotherm on Fig. 65. The amount of heat carried away is equal in this case to the heat of evaporation of the liquid under the same conditions.

Condensation is the reverse of evaporation of the liquid and consists of formation, over the entire volume occupied by the saturated vapor, of liquid drops which increase in size with increasing compression of the vapor. The formation of the liquid drops is particularly intense if the vapor contains dust particles or electrically charged particles (particularly ions), which can serve as condensation centers. In this case, which is most typical for ordinary conditions, the slightest compression of the saturated vapor causes it to condense. If there are no extraneous condensation centers, then the compression of the saturated vapor to a definite limit will not lead to the formation of liquid drops (the condensation is retarded). The reason for it lies in the following. Equilibrium motion of the vapor above a small drop is always larger than above a plane surface of the liquid, and as a result the vapor, which is saturated with respect to a plane surface, is not saturated with respect to a small liquid drop. Therefore the small liquid drop, if it were to be formed in the vessel where there exists saturated vapor (relative to the plane surface), would not be in equilibrium with the vapor and would be evaporated. In order for the drop not to evaporate, it is necessary to increase the vapor pressure

(for example, by compression) to equal the vapor pressure saturating the space above the drop of given size. At that instant, equilibrium is established between the drop and the vapor and further compression of the vapor will give rise to condensation of the vapor on the drop and cause the latter to grow.

Thus, condensation of saturated vapor in the absence of extraneous condensation centers begins only after drops of definite size have been produced in the vapor. The drops having these critical dimensions (radius ρ_{kr}) and in equilibrium with the vapor are called <u>nucleation drops</u>.

<u>Saturated Vapor Pressure Above a Curved Liquid Surface</u>

The quantitative connection between the saturated-vapor pressure and the curvature of the surface separating the liquid and gaseous phases can be determined with the aid of the general equation (6.17). Assume that for a temperature <u>t</u> and for a plane separation surface between phases the pressure of the saturated vapor is p_s. If the phase-separation boundary is curved somewhat at the same temperature, for example it becomes spherical with radius of curvature ρ, then the pressures of both phases change by $\Delta p' = p'_\rho - p_s$ and $\Delta p'' = p''_\rho - p_s$, respectively, and from Eq. (6.17) we have

$$v'(p'_\rho - p_s) = v''(p''_\rho - p_s).$$

Inasmuch as $p''_\rho = p'_\rho + 2\sigma/\rho$, we get from this

$$\left.\begin{aligned} p''_\rho &= p_s - \frac{2\sigma}{\rho}\frac{v'}{v''-v'}, \\ p'_\rho &= p_s - \frac{2\sigma}{\rho}\frac{v''}{v''-v'}. \end{aligned}\right\} \qquad (7.2)$$

At sufficiently low temperatures the volume of the liquid in the denominator of the right half of (7.2) can be neglected and v" can be set equal to RT/p_s, as a result of which we obtain

$$p''_\rho = p_s\left(1 - \frac{2\sigma}{\rho}\cdot\frac{v'}{RT}\right).$$

Since the expression in the parentheses represents the first two terms of the expansion of $e^{-(2\sigma/\rho)\cdot(v'/RT)}$ in powers of $(2\sigma/\rho)\cdot(v'/RT)$, we get

$$p_\rho' = p_s e^{-\frac{2\sigma v'}{RT}\cdot\frac{1}{\rho}}. \qquad (7.3)$$

From (7.3) it is clear that the equilibrium vapor pressure above the liquid drop when $\rho < 0$ will be larger than the saturated-vapor pressure above a plane surface having the same temperature. Inside the vapor bubble of spherical form located in the liquid, the saturated-vapor pressure for $\rho > 0$ will be smaller than the saturated-vapor pressure above the plane surface. From (6.17) we can determine further the variation of the temperature of the phase transition with variation of the radius of curvature of the separation surface at a constant pressure of one of the phases.

Let us assume that an initially flat separation surface has been curved somewhat, so that its radius of curvature became equal to ρ and the pressure on the liquid phase did not change. Since by assumption the pressure p' of the liquid phase is the same before and after the curving of the surface, the temperature, which had the same value for the two coexisting phases, should change; an equal change should occur also in the pressure of the second phase. The temperature of the phases which are in equilibrium with the separation surface curved will be $T + \Delta T$, where T is the temperature of the same phases in the case of equilibrium on the flat separation surface, while the pressure of the second phase is $p_\rho = p_s + (2\sigma/\rho)$. Substituting the values $\Delta p' = 0$ and $\Delta p'' = 2\sigma/\rho$ in (6.17) we obtain

$$\Delta T = \frac{2A\sigma v'' T}{r}. \qquad (7.4)$$

In this formula v", r, and σ are the saturated-vapor volume, the heat of evaporation, and the surface-tension coefficient at temperature T

(prior to the change in the curvature of the surface).

When the curvature of the phase-separation surface changes under conditions where the pressure in the gaseous phase is constant, the formula for the change in the equilibrium phase temperature has the following form:

$$\Delta T = \frac{2A\sigma v' T}{r}.$$

Formula (7.4) can be used to determine the degree of superheat of a liquid containing no dissolved extraneous gases and boiling under conditions of constant, say atmospheric pressure on the liquid. It is necessary to bear in mind here that the vapor bubbles produced on the walls of the vessel have a radius of approximately the same magnitude as the projections (pits) in the roughness of the walls.

The equations (7.2), which express the conditions for the equilibrium between the liquid and gaseous phases separated by a spherical surface, are particularly convenient for the analysis of the formation in a given phase of nuclei of a new phase (inasmuch as the nuclei have the form of very small liquid drops or very small vapor bubbles). The radius of curvature which enters in (7.2) is the critical radius of the nucleus of a new phase. If we denote the critical radius of the nucleus of the liquid phase by ρ'_{kr} and that of the gaseous phase by ρ''_{kr}, then we get from (7.2)

$$\varrho'_{kp} = -\frac{2\sigma v'}{(v'' - v')\left(p''_{\varrho} - p_s\right)},$$

$$\varrho''_{kp} = -\frac{2\sigma v''}{(v'' - v')\left(p'_{\varrho} - p_s\right)}.$$

Consequently, if in a gaseous phase having a pressure p''_{ρ} a liquid drop has been produced with radius ρ'_{kr} or larger, then such a drop will be stable and after a certain time it should either increase in size or at least keep its size unchanged. On the other hand, if the

initial dimension of the drop is smaller than ρ'_{kr}, then such a drop cannot exist for a long time and will soon decrease and vanish completely. With respect to the drop with radius larger than ρ'_{kr}, vapor with pressure p''_{ρ} will be unstable and, therefore, if we place such a drop in the vapor, the latter will start to condense on it until it goes over into the liquid phase.

The formation of nucleation drops, as is well known, is greatly facilitated if there are present in the vapor extraneous condensation centers. Without them the nucleation drops can form only by fluctuations in the density of the vapor phase (by accumulation of a certain, generally considerable number of molecules). The small probability of such fluctuations causes a delay in the condensation up to a certain limit and, as a consequence, the formation of supersaturated vapor with pressure larger than p_s.

In addition to dropwise condensation, so-called capillary condensation can occur in narrow capillaries, slots, and pores of the solid body, under the condition that the liquid phase wets the surface of this body. Since the saturation pressure above the concave meniscus of the liquid, formed in the capillaries, will in accordance with Condition (7.3) be smaller than above the flat surface, the vapor condensation in capillaries will begin at pressures lower than the saturated-vapor pressure (relative to the flat surface).

In equal fashion, in the liquid phase, a vapor bubble with radius smaller than ρ''_{kr} will be unstable, whereas with respect to a vapor bubble with radius larger than ρ''_{kr} the liquid itself will be unstable. From the expressions for ρ_{kr} it follows that the larger the pressure of the main phase is compared with the equilibrium pressure on the flat separation surface, i.e., the more supersaturated the vapor is or the more heated the liquid, the smaller the critical dimension of the

nucleus and the faster the transition from the initial to the final phase can take place. The critical radius is directly proportional to the coefficient of surface tension ($\rho_{kr} = 0$ when $\sigma = 0$). This dependence explains why minute solid particles (dust particles) can serve as condensation centers provided the surface tension on the boundary between these particles and the liquid is smaller than on the boundary between the liquid and its vapor.

Parameters of a Two-Phase System

Let us denote the relative content by weight of the vapor phase in a two-phase system, i.e., in an equilibrium mixture of liquid and saturated vapor (such a mixture is also called wet steam or vapor), by x:

$$x = \frac{G''}{G' + G''}.$$

where G' is the weight of the liquid phase and G" is the weight of the vapor phase. The quantity x is called the degree of dryness of the wet vapor.

The specific volume of the mixture is

$$v = (1 - x)\, v' + x v''. \tag{7.5}$$

Analogously, because of the additivity of the thermodynamic functions, we have

$$u = (1 - x)\, u' + x u''; \tag{7.6}$$

$$i = (1 - x)\, i' + x i''; \tag{7.7}$$

$$s = (1 - x)\, s' + x s''. \tag{7.8}$$

Here v", u", s", i" and v', u', s', i' are the specific volume, the internal energy, the entropy, and the enthalpy of the saturated vapor and of the liquid in equilibrium with it, respectively.

It follows from (7.6)-(7.8) that the internal energy, the entropy, and the enthalpy of a two-phase system depend on x and t or p (are functions of two independent parameters). Generally the state of equi-

librium of a two-phase system is determined by two parameters, which can be any pair of parameters $\underline{p}$, $\underline{t}$, $\underline{v}$, or $\underline{x}$, except $\underline{p}$ and $\underline{t}$. It follows from this, in particular, that all the relationships between the partial derivatives of the thermodynamic quantities presented in §7 of Chapter 3 for the independent variables $\underline{t}$ and $\underline{v}$ (but not $\underline{p}$ and $\underline{t}$) hold also for two-phase states.

Since the pressure of the saturated vapor is a function of the temperature $p = p_s(t)$, any partial derivative of the pressure with respect to the temperature, for example $(\partial p/\partial t)_V$, $(\partial p/\partial t)_I$, $(\partial p/\partial t)_x$, etc., will be equal to dp_s/dt in a two-phase system.

The specific heat c_V of a two-phase system at constant volume, which is equal to the derivative $(\partial u/\partial t)_V$, has at all points of the two-phase region a finite positive value. The specific heat c_p of a two-phase system at constant pressure, which in accordance with Eq. (3.47) is equal to

$$c_p = c_V - AT\frac{\left(\frac{\partial p}{\partial t}\right)_v^2}{\left(\frac{\partial p}{\partial v}\right)_t}.$$

has at all points of the two-phase region an infinitely large value, inasmuch as $(\partial p/\partial v)_t = 0$.

Properties of Matter at the Critical Point

At the critical point, unlike all other points of the boundary curve, the properties of both phases (liquid and gaseous) are identical. This means that the critical state is one and the same limiting physical state for both the homogeneous substance or for the substance that has broken up into two phases. It follows from this that on going through the critical point from the region of the homogeneous state into the two-phase region or back, the properties of the substance should vary continuously and, in particular, the entropy and the vol-

ume and certain derivatives of the thermodynamic functions (but not all of them) should not experience discontinuity at the critical point, in contradiction to what takes place at other points of the boundary curve. Therefore, for example, at the critical point the value of the partial derivative of p with respect to t along the portion of the critical isochore lying in the homogeneous region should be equal to the value of this derivative on the lower portion, which lies under the boundary curve (in the two-phase region), but since in the two-phase region $(\partial p/\partial t)_V = dp_s/dt$, we get for $t = t_{kr}$

$$\left(\frac{\partial p}{\partial t_к}\right)_v = \frac{dp_s}{dt_к} \qquad (7.9)$$

Since the derivative dp_s/dt_k is positive for all substances, the partial derivative $(\partial p/\partial t_k)_V$ has a positive value at the critical point. The equality of the derivatives $(\partial p/\partial t_k)_V$ and dp_s/dt_k signifies that the saturated-vapor tension curve $p = p_s(t)$ and the critical isochore $v(p, t) = v_k$ have a common tangent at the critical point (Fig. 66).

The boundary curve in the vicinities of the critical point is symmetrical relative to the vertical axis passing through the critical point and has the form of a parabola. Therefore near the critical point we have

$$v' - v_к = v_к - v'. \qquad (7.10)$$

In order to find the dependence of v' and v" on t, let us use the expansion of p in powers of $(v - v_k)$ and $(t_k - t)$ which, taking into account the fact that $(\partial p/\partial v_k)_t = 0$ and $(\partial^2 p/\partial v_k^2)_t = 0$, has the form

$$p = p_к - \left(\frac{\partial p}{\partial t_к}\right)_v (t_к - t) - \frac{\partial^2 p}{\partial t_к \partial v_к}(t_к - t)(v - v_к) +$$

$$+ \frac{1}{6}\left(\frac{\partial^3 p}{\partial v_к^3}\right)_t (v - v_к)^3 + \ldots.$$

For a given temperature and for $v = v'$ or $v = v''$, p has one and

the same value; recognizing further that in the near-critical region Eq. (7.10) holds true, we obtain from the expression for p

$$\left.\begin{aligned} v'' - v_\kappa &= \nu \sqrt{t_\kappa - t}, \\ v_\kappa - v' &= \nu \sqrt{t_\kappa - t}, \end{aligned}\right\} \qquad (7.11)$$

where ν is a certain constant coefficient. From Eqs. (7.11) we see that at the critical point

$$\frac{dv''}{dt_\kappa} = -\frac{dv'}{dt_\kappa} \to -\infty. \qquad (7.12)$$

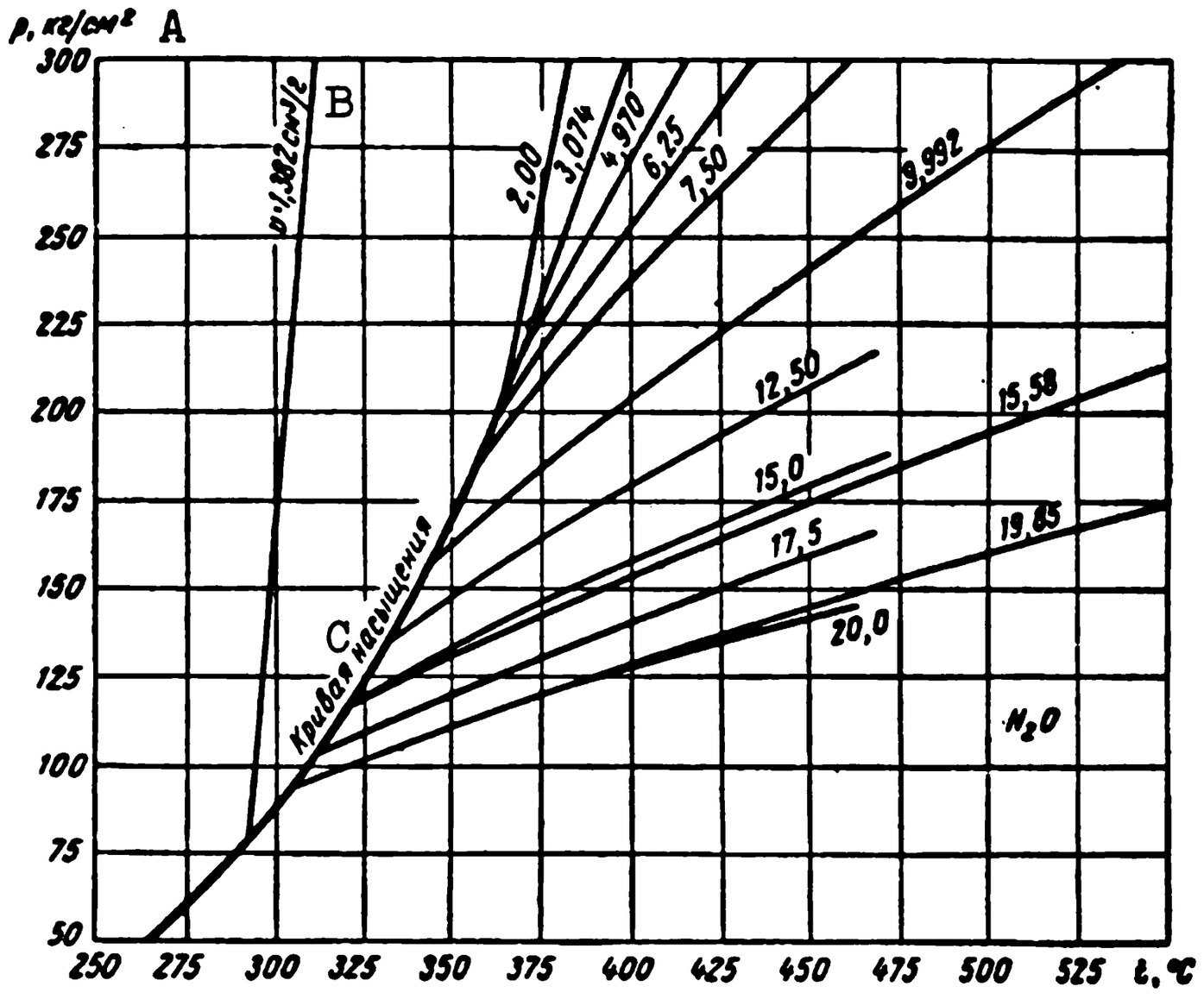

Fig. 66. Course of the saturated-vapor tension curve and of the critical isochore near the critical point. A) kg/cm^2; B) cm^3/g; C) saturation curve.

The specific heat c_V has at the critical point a finite value, while the specific heat c_p assumes at the critical point, as at all other points of a two-phase region, an infinite value.

An experimental study of the properties of water vapor and other substances in the high-temperature and pressure region, undertaken in recent years, has disclosed some new properties of real gases. Thus, for example, it has been established that in the supercritical region

the maxima of the curves $c_p(p, t) = \text{const}$ (each of these curves can be called the line of constant specific heat) lie almost on a line starting from the critical point. This is clearly seen from Fig. 67. Inasmuch as along the curve $c_p(p, t) = \text{const}$ we have

$$\left(\frac{\partial c_p}{\partial p}\right)_t \frac{dp}{dt} + \left(\frac{\partial c_p}{\partial t}\right)_p = 0, \qquad (7.13)$$

and at the point of the maximum of this curve $dp/dt = 0$, the derivative $(\partial c_p/\partial t)_p$ also vanishes at this point. Thus, at all points of the line passing through the maxima of the constant-specific heat curves $c_p(p, t) = \text{const}$, including at the critical point, we have $(\partial c_p/\partial t)_p = 0$.

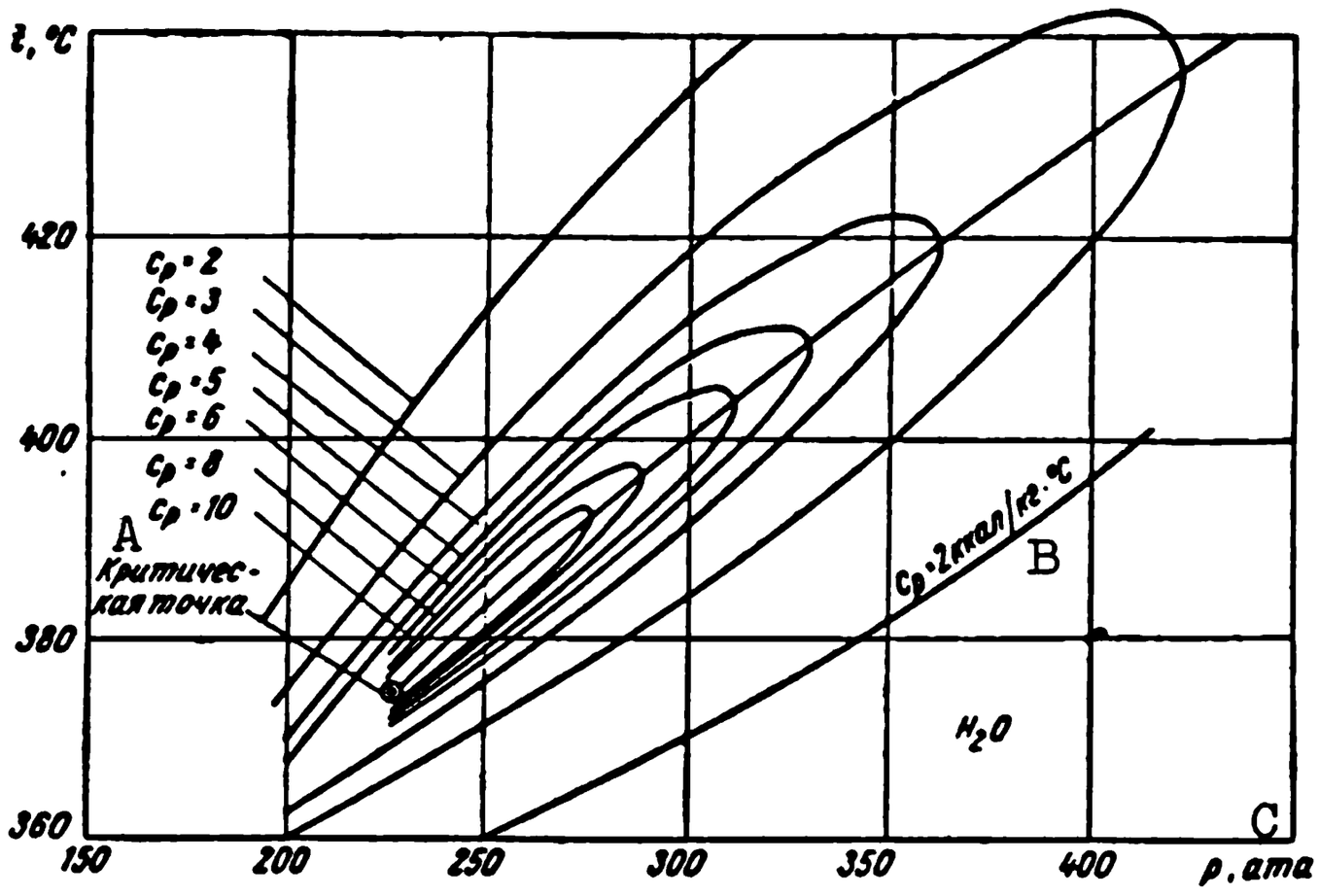

Fig. 67. The lines $C_p = \text{const}$. A) Critical point; B) kcal/kg-$^\circ$C; C) atm abs.

§2. SATURATED VAPOR

Heat of Evaporation, Density and Pressure of Saturated Vapor

The heat of evaporation r of a liquid represents essentially a positive function of the temperature (see §5 of Chapter 6). The form of this dependence is shown in Fig. 68. At the critical point, where all the differences between the liquid and the gas disappear, the heat of evaporation vanishes. With decreasing temperature, r increases. The dependence of r on t at temperatures that are not very low compared

with t_k is given with sufficient accuracy by the following formula

$$r = T[\varkappa_1 + \varkappa_2(t_k - t)]\sqrt{t_k - t}. \qquad (7.14)$$

It is seen from this formula that in the region of the critical point the heat of vapor formation is proportional to the square root of the temperature difference $t_k - t$.

The volume v" of the saturated vapor and the volume v' of the liquid under the pressure of its own saturated vapor satisfy the following empirical relationship:

$$\frac{1}{v'} + \frac{1}{v''} = \frac{2}{v_k} + \lambda(t_k - t), \qquad (7.15)$$

where the coefficient λ has a positive value close to unity for all substances. Equation (7.15) expresses the so-called rule of straight-line diameter. This name becomes understandable if we take into consideration the fact that in the coordinates t, 1/v the diameter of the curve $1/v_s(t)$ is a straight line (Fig. 69).

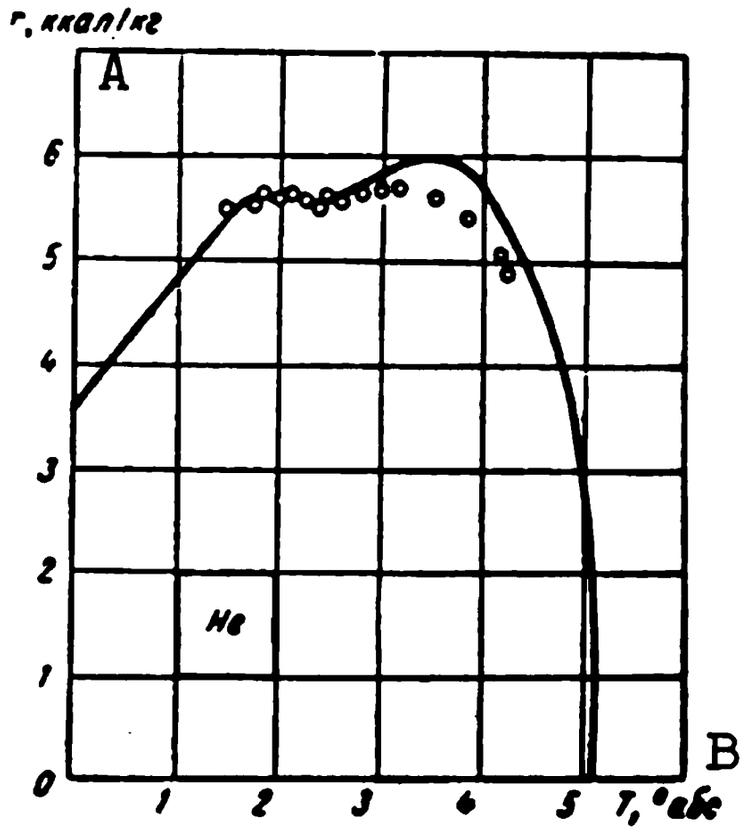

Fig. 68. Dependence of the heat of evaporation on the temperature. A) kcal/kg; B) abs.

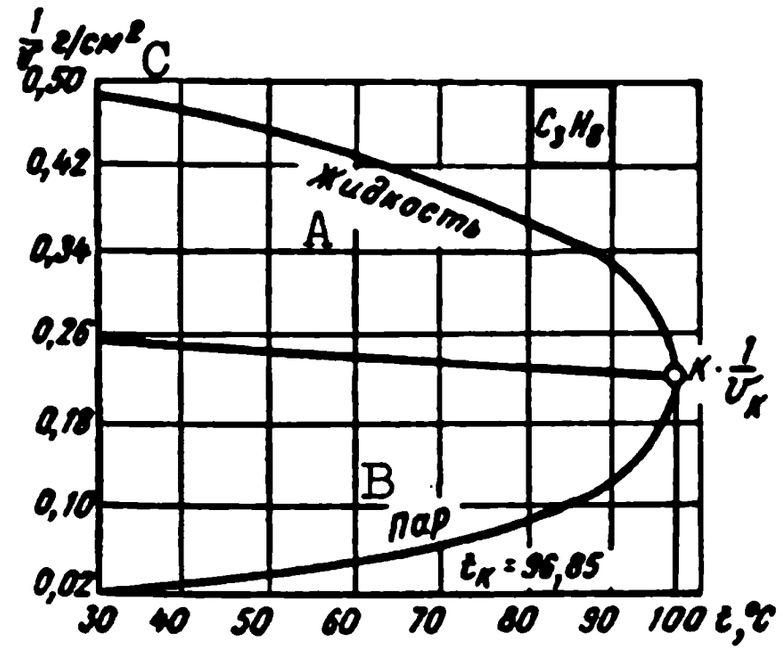

Fig. 69. Rule of straight-line diameter. A) Liquid; B) vapor; C) g/cm^2.

It is interesting to note that a dependence analogous to the empirical relation (7.15) is observed for the values of the viscosity coefficients and heat conduction coefficients on the boundary curve. At temperatures considerably lower than critical, the volume v' of a liquid under the pressure of its saturated vapor varies so little with the temperature that it can be regarded as a constant quantity. Therefore for small values of the ra-

tio T/T_k, the left boundary curve on the pv diagram is almost vertical. On the other hand, the smaller the temperature, the stronger the temperature variation of the volume v" of the saturated vapor. The dependence of the saturated-vapor pressure on the temperature is determined by the Clapeyron-Clausius formula (6.15)

$$\frac{dp}{dT} = \frac{r}{AT(v' - v')}.$$

from which it follows that since $r > 0$, $v'' > v'$, the saturated-vapor pressure increases with increasing temperature (Fig. 70).

If r, v', and v" are known functions of the temperature, then by integrating the Clapeyron-Clausius formula we can establish the dependence of p on T in explicit form. At sufficiently low temperatures, when the volume of the liquid v' can be neglected compared with the volume of the saturated vapor v" and the volume v" can on the basis of the Clapeyron-Mendeleyev formula be set approximately equal to RT/p, we have

$$\frac{dp}{dT} = \frac{rp}{ART^2}.$$

Assuming further that r is a linear function of the temperature, we obtain the following approximate equation for p:

$$\ln p = \frac{B}{T} + C \ln T + D, \qquad (7.16)$$

where B, C, and D are constants.

Specific Heat of Saturated Vapor

In §6 of Chapter 2 and §3 of Chapter 3 we gave a general definition of the specific heat of a process characterized by some constant parameter X:

$$c_X = T\left(\frac{\partial s}{\partial T}\right)_X.$$

Choosing as the parameter X the degree of dryness of the vapor x, we find the following expression for the specific heat of saturated vapor

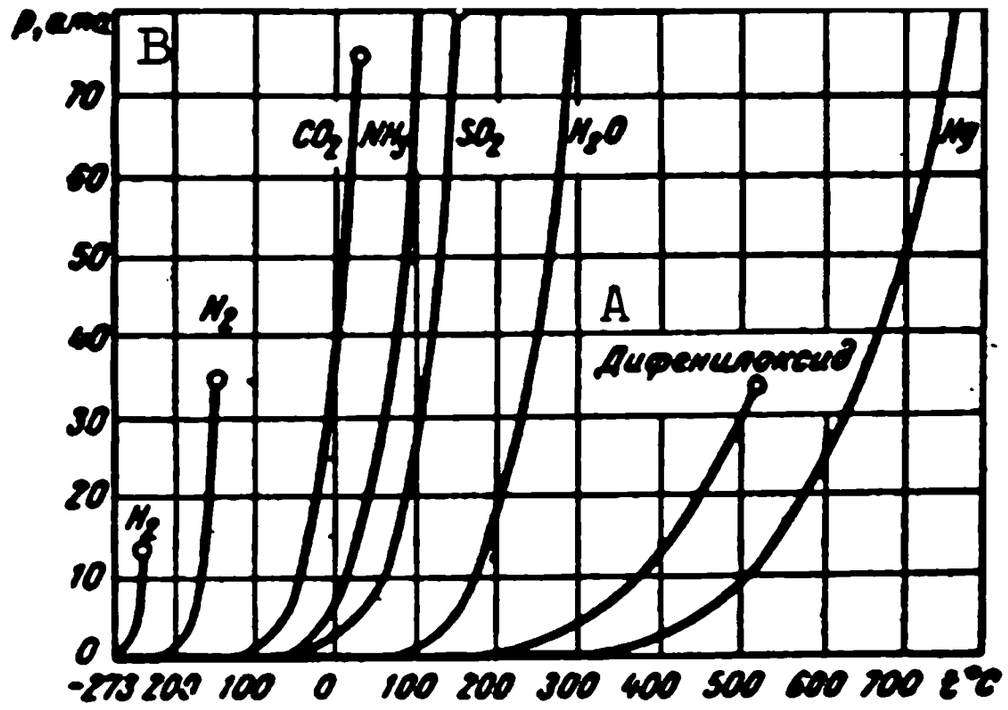

Fig. 70. Dependence of saturated-vapor pressure on the temperature. A) Diphenyl oxide; B) atm abs.

(specific heat along the right-hand boundary curve):

$$c' = T\left(\frac{\partial s}{\partial T}\right)_{x=1}. \qquad (7.17)$$

and the specific heat of a liquid in equilibrium with its own vapor (specific heat along the left boundary curve),

$$c' = T\left(\frac{\partial s}{\partial T}\right)_{x=0}. \qquad (7.18)$$

The specific heat of saturated vapor is the amount of heat which must be consumed in order to heat 1 kg of saturated vapor by $1^{\circ}C$ in such a way that the steam remains saturated during the entire heating process.

In Eqs. (7.17) and (7.18) $(\partial s/\partial T)_{x=1}$ and $(\partial s/\partial T)_{x=0}$ are the derivatives of the entropy $\underline{s}$ with respect to T along the right and left boundary curves, respectively. Since $\underline{s}$ on the boundary curve depends only on the temperature, these derivatives will be equal to ds"/dT and ∂s'/∂T. Subtracting (7.18) from (7.17) we obtain

$$c'' - c' = T\frac{d}{dt}(s'' - s')$$

or, since s" − s' = r/T,

$$c'' - c' = \frac{dr}{dT} - \frac{r}{T}. \qquad (7.19)$$

At low pressures or, more accurately, at temperatures that are sufficiently small compared with the critical temperature, the specific heat c' is approximately equal to the specific heat c'_p at the same point. To verify this, let us use the obvious relation

$$\left(\frac{\partial T}{\partial s}\right)_x = \left(\frac{\partial T}{\partial s}\right)_p + \left(\frac{\partial T}{\partial p}\right)_s \cdot \left(\frac{\partial p}{\partial s}\right)_x.$$

from which, taking into account the fact that on the basis of Eq. (3.57) we have

$$\left(\frac{\partial T}{\partial p}\right)_s = \frac{AT}{c_p}\left(\frac{\partial v}{\partial T}\right)_p$$

and in the liquid state the coefficient of thermal expansion $1/v(\partial v/\partial T)_p$ is close to zero, while the derivative $(\partial p/\partial s)_{x=0} = dp/ds'$ is finite, we obtain

$$\left(\frac{\partial T}{\partial s}\right)_{x=0} \approx \left(\frac{\partial T}{\partial s}\right)_p$$

or

$$c' \approx c'_p.$$

Figure 71 shows the actual course of the isobars for relatively low pressures. The isobar shown on this figure is strongly distorted as compared with the left boundary curve. In fact, the isobars drawn from the points of the lower portion of the left boundary curve, i.e., at relatively low pressures, pass in accordance with the condition $(\partial T/\partial s)_{x=0} \approx (\partial T/\partial s)_p$ slightly above this curve and in practice merge with it.

At low pressures the derivative dr/dT is approximately equal to $c''_p - c'_p$ and therefore Eq. (7.19) assumes the form

$$c'' = c''_p - \frac{r}{T}. \qquad (7.20)$$

Let us determine now the sign of the specific heats c" and c'. For this purpose we substitute into Eqs. (7.17) and (7.18) the value of the derivative $(\partial s/\partial T)_v$:

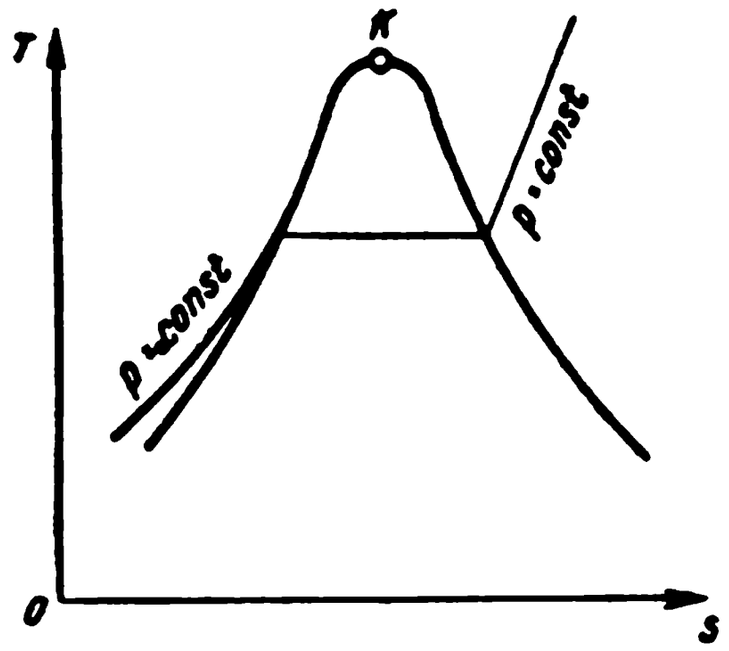

Fig. 71. Actual course of the isobars at low pressures.

$$\left(\frac{\partial s}{\partial T}\right)_x = \left(\frac{\partial s}{\partial T}\right)_v + \left(\frac{\partial s}{\partial v}\right)_T \left(\frac{\partial v}{\partial T}\right)_x.$$

Recognizing that

$$\left(\frac{\partial s}{\partial T}\right)_v = \frac{c_v}{T};\quad \left(\frac{\partial v}{\partial T}\right)_{x=1} = \frac{dv''}{dT};\quad \left(\frac{\partial v}{\partial T}\right)_{x=0} = \frac{\partial v'}{\partial T},$$

and that in accordance with Eq. (3.41)

$$\left(\frac{\partial s}{\partial v}\right)_T = A\left(\frac{\partial p}{\partial T}\right)_v.$$

we obtain

$$c'' = c_v'' + AT\left(\frac{\partial p}{\partial T}\right)_v \cdot \frac{dv''}{dT}. \qquad (7.21)$$

$$c' = c_v' + AT\left(\frac{\partial p}{\partial T}\right)_v \cdot \frac{dv'}{dT}. \qquad (7.22)$$

Let us find first the value of c" at the critical point. It follows from (7.21) that at the critical point, where c_V and $(\partial p/\partial T_k)_V = dp/dT_k$ have finite positive values and the derivative $dv''/dT_{kr} \to -\infty$, the specific heat c" of the saturated vapor tends to negative infinity:

$$c_k'' \to -\infty. \qquad (7.23)$$

Inasmuch as c" is a continuous function of the temperature, it follows therefore that also near the critical point, i.e., at $t \leq t_k$, the specific heat of the saturated vapor of any substance will have a negative value.*

The negative sign of the specific heat of saturated vapor denotes that when the vapor is heated the heat will not be absorbed but released. The reason for it becomes clear if we take into account the following circumstance. In order for the vapor to become saturated as it heats up, its volume must decrease (the vapor must be subjected to compression during heating), which calls for the expenditure of external work. If the work to compress the vapor carried out by the external source of work during heating the vapor is larger than the increase of the internal energy of the vapor, then the heat drawn from

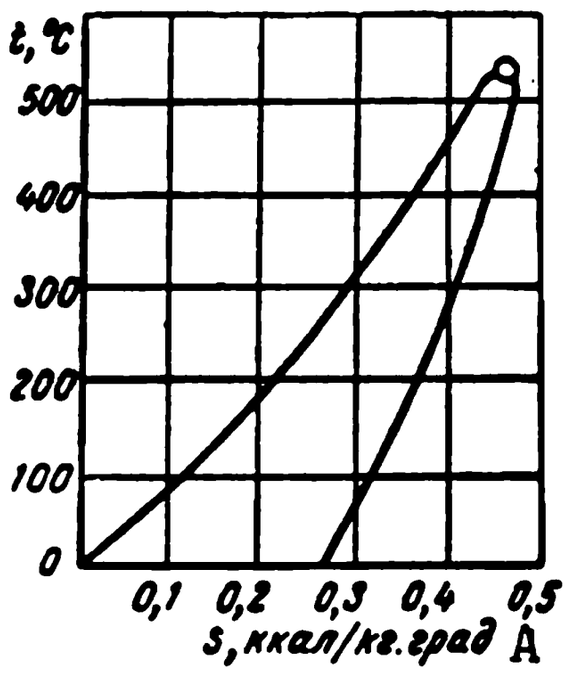

Fig. 72. The Ts diagram of diphenyl oxide. A) kcal per kg-deg.

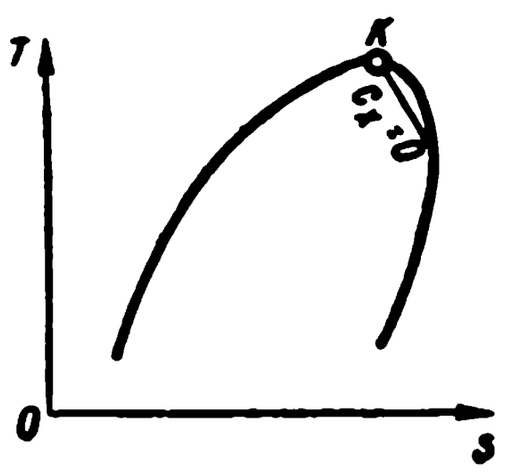

Fig. 73. The Ts diagram for hydrocarbons.

to the contrary, some part of the compression work will in itself be released in the form of heat. Since heating of a liquid involves an increase in its volume and, consequently, the performance of work, the specific heat of a liquid which is in equilibrium with its own vapor is always positive. It is further seen from (7.22) that at the critical point c' becomes equal to positive infinity. The sum of the specific heats of the saturated vapor and of the liquid in equilibrium with it, $c''_k + c'_k$, has at the critical point, as can be seen from Eqs. (7.21) and (7.22) and from Condition (7.10), a finite positive value equal to $2c_{Vk}$. As the temperature decreases from $t = t_k$, the specific heat of the saturated vapor, c", remaining negative, increases in absolute value. At a certain temperature the value of c" may reach zero, after which it can become positive. Such a variation of the specific heat c" is observed, for example, in diphenyl oxide (Fig. 72) and a few hydrocarbons (Fig. 73). In water and several other liquids c" has a negative value up to the extreme state of equilibrium between the liquid and vapor phases (determined by the temperature t_{tr} of the triple point).

Entropy, Internal Energy, and Enthalpy of Saturated Vapor

Knowing the specific heats c" and c' of the saturated vapor and of the liquid in equilibrium with it, we can obtain readily general expressions for their entropy, internal energy, and enthalpy. Since the increase in the entropy ds' along the left boundary curve is equal to c'dT/T, the value of the entropy of the liquid phase at the point

of the left boundary curve, corresponding to the temperature $\underline{t}$, is

$$s' = \int_{T_0}^{T} \frac{c'}{T} dT + s'_0 \qquad (7.24)$$

where s'_0 is the entropy on the boundary curve at temperature t_0, which is chosen to be the initial temperature.

The entropy s" of saturated vapor at temperature $\underline{t}$ is accordingly

$$s'' = \int_{T_0}^{T} \frac{c'}{T} dT + \frac{r}{T} + s'_0. \qquad (7.25)$$

At temperatures that are relatively low compared with t_{kr}, when $c' \approx$ $\approx c'_p$ with c'_p of the liquid phase regarded as constant, it follows from (7.24) and (7.25) that

$$s' = c'_p \ln \frac{T}{T_0} + s'_0, \qquad (7.26)$$

$$s'' = c'_p \ln \frac{T}{T_0} + \frac{r}{T} + s'_0. \qquad (7.27)$$

These equations can be used for an approximate calculation of s' and s".

The specific internal energy u' and the enthalpy i' of the liquid have the following form:

$$u' = \int_{T_0}^{T} c' dT - A \int_{T_0}^{T} p \frac{dv'}{dT} dT + u'_0; \qquad (7.28)$$

$$i' = \int_{T_0}^{T} c' dT + A \int_{T_0}^{T} v' \frac{dp}{dT} dT + i'_0. \qquad (7.29)$$

The values of u" and i" can be obtained from Eqs. (7.28) and (7.29) on the basis of the equations

$$u'' = i'' - Apv''; \quad i'' = i' + r.$$

At relatively low temperatures compared with t_k, when the volume v' of the liquid and the derivative dv'/dT can be neglected and c' can be assumed equal to c'_p, the equations for the internal energy and enthalpy of the liquid and of the saturated vapor in equilibrium assume the following simple form

$$u' = c'_p (T - T_0) + u'_0. \qquad (7.30)$$

$$u'' = c'_p(T - T_0) + r - Ap\upsilon'' + u'_0; \qquad (7.31)$$

$$i' = c'_p(T - T_0) + i'_0; \qquad (7.32)$$

$$i'' = c'_p(T - T_0) + r + i'_0. \qquad (7.33)$$

§3. WET STEAM. MAIN PROCESSES

We shall henceforth define wet steam as a mixture of saturated steam and liquid in equilibrium. Frequently wet steam refers to a mixture of saturated steam and liquid in which the liquid is in the form of minute drops suspended in the steam.

Isothermal Process (t = const)

If the temperature remains constant as the state of wet steam changes, then the pressure will also be constant. It will be equal to the pressure of the saturated steam p_s at the given temperature. This means that the isothermal process is at the same time an isobaric process. In coordinates ts and pv, the isothermically isobaric process of variation of the state of wet steam is represented by a horizontal straight line. Inasmuch as $(\partial i/\partial s)_p = T$, this process is represented in is coordinates by a straight line whose slope is equal to the absolute temperature T (Fig. 74).

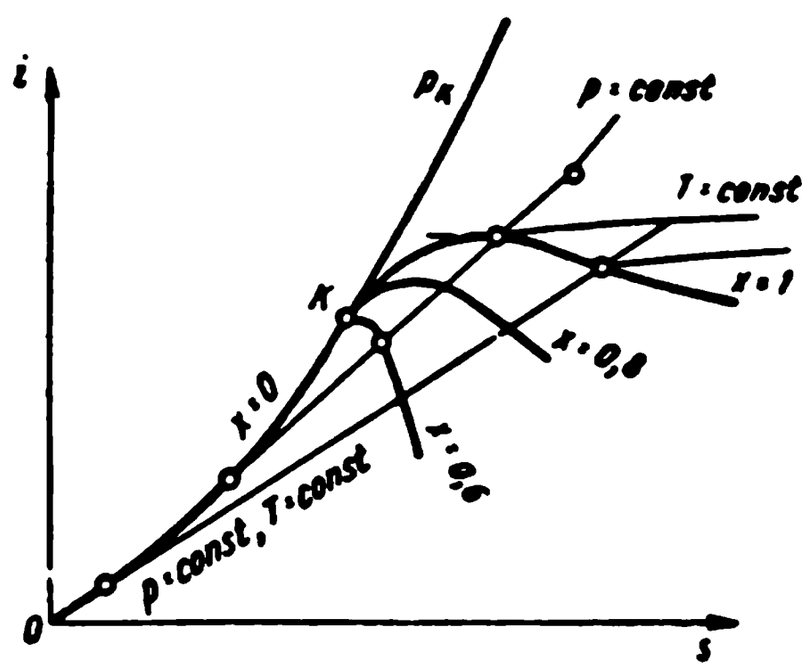

Fig. 74. Isobaric process of variation of the state of wet steam in is coordinates.

The Process x = const

This process is characterized by the constancy of the weight ratio of the liquid and gaseous phases. Graphically the process x = const is represented by a line which is called the line of constant degree of dryness. In order to plot this line we make use of the additivity properties of the volume v, the entropy s, and the enthalpy i. From Eqs. (7.5), (7.7), and (7.8) it follows that

$$x = \frac{v - v'}{v'' - v'} = \frac{s - s'}{s'' - s'} = \frac{i - i'}{i'' - i'}.$$

Since the equilibrium phase-transition process is represented in the coordinates pv, ts, and is by a segment of a straight line (the straight-line portion of the isotherm) of length v" – v'; s" – s'; i" – – i', it follows from these equations that the curve x = const divides the phase-transition line into two segments such that the length of the first (or left-hand) segment, equal to v – v', s – s', i – i', amounts to one x-th of the entire straight line of phase transition, while the length of the second (right-hand) segment v" – v, s" – s, i" – i is equal to (1 – x)-th part of the phase-transition line. The following vary along the line of constant degree of dryness: the temperature, the pressure, and the volume of the wet steam. The change in pressure along the line of constant dryness can be determined from the Clapeyron-Clausius formula

$$\left(\frac{\partial p}{\partial T}\right)_x = \frac{xr}{AT(v - v')}.$$

The change in entropy of wet steam along the line x = const is determined by the equation

$$\left(\frac{\partial s}{\partial T}\right)_x = \frac{c_x}{T}. \qquad (7.34)$$

where c_x is the specific heat on the line x = const (the specific heat of the wet steam).

Using Eq. (7.7) for the entropy of wet steam, we obtain

$$c_x = (1 - x)c' + xc''. \qquad (7.35)$$

The per-unit specific heat of wet steam c_x is numerically equal to the amount of heat necessary to heat 1 kg of wet steam by $1^{\circ}C$ while the degree of dryness of the steam is kept constant. With the aid of Eq. (7.19) we can rewrite (7.35) in the form

$$c_x = c' + x\left(\frac{dr}{dT} - \frac{r}{T}\right). \qquad (7.36)$$

Depending on the temperature $\underline{t}$ and on the degree of dryness of the steam $\underline{x}$, the quantity c_x can be either positive or negative. Since the specific heat c' of a liquid in equilibrium with saturated vapor is positive, at small degrees of dryness c_x will also have a positive sign.

At temperatures sufficiently remote from the critical temperature t_k, the value of c' is approximately equal to that of c'_p and therefore

$$c_x \approx c'_p + x\left(\frac{dr}{dT} - \frac{r}{T}\right). \qquad (7.37)$$

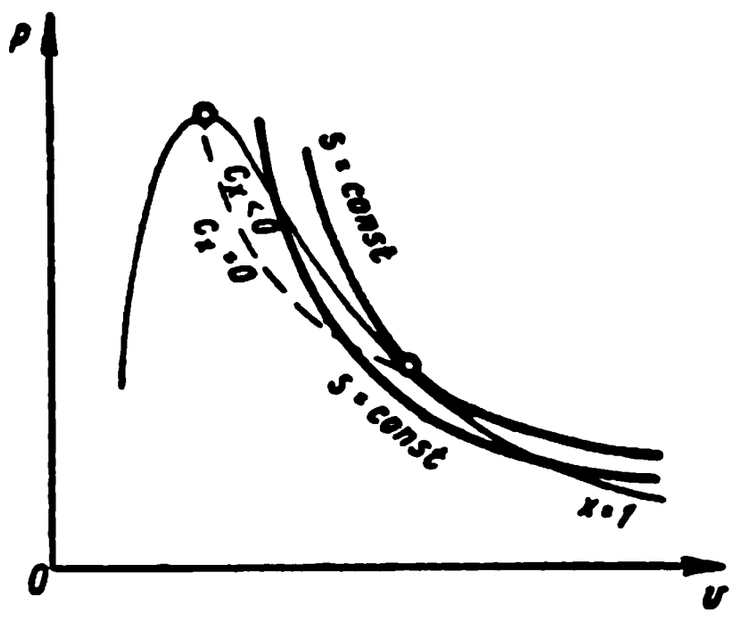

Fig. 75. Course of the line of zero specific heat of wet steam.

The specific heat of wet steam c_x is variable along the line x = const and as already indicated it can assume both positive and negative values. If we join the points of all the lines x = const in which $c_x = 0$, we obtain the line of zero specific heat of wet steam (Fig. 75). This line starts at the critical point and in general should cross the right-hand boundary curve. Above the line of zero specific heat we have $c_x < 0$, while below we have $c_x > 0$. At the point where the lines x = = const and $c_x = 0$ cross, the entropy regarded as a function of $\underline{t}$ (along the line x = const) reaches a maximum.

Isochoric Process [v = const or $(1 - x)v' + xv'' = \text{const}$]

Since in an isochoric process dv = 0 and v' and v'' are functions of the temperature $\underline{t}$ only, we get from (7.5)

$$\frac{dv'}{dt} + x\frac{d}{dt}(v'' - v') + (v'' - v')\left(\frac{dx}{\partial t}\right)_v = 0.$$

The derivative dv''/dt is negative at all temperatures and dv'/dt is positive (in some substances, for example water, the derivative dv'/dt away from the critical point may become negative). Therefore the derivative $(\partial x/\partial t)_V$ can have either a negative or a positive sign.

In other words, in an isochoric process the degree of dryness can either decrease or increase with decreasing temperature. Thus, for example, if the initial state of the substance was near the left-hand boundary curve, then the decrease in temperature at v = const will be first accompanied by an increase in x and then by a decrease. If initially x ≈ 1, then the decrease in temperature for v = const leads to a decrease in x.

Isentropic Process [s = const or (1 − x)s' + xs" = const]

It is seen on Fig. 76 that in a reversible transition from state a to a nearly lying state b the amount of heat dq absorbed by 1 kg of wet steam is $c_x dT + r dx$, i.e.,

$$Tds = c_x dT + r dx. \tag{7.38}$$

In an isentropic process ds = 0, and consequently

$$c_x dT + r dx = 0$$

or

$$\left(\frac{\partial x}{\partial T}\right)_s = -\frac{c_x}{r}. \tag{7.39}$$

This expression is the fundamental equation of the isentropic process. With the aid of Eq. (7.36) it can be reduced to the form

$$\frac{c'}{T} dT + d\left(\frac{xr}{T}\right) = 0.$$

For large pressures, when $c' \approx c'_p$ and is constant, this equation can be integrated. As a result we obtain

$$c'_p \ln \frac{T}{T_0} + \frac{xr}{T} - \frac{x_0 r_0}{T_0} = 0. \tag{7.40}$$

It follows from (7.39) that when $c_x > 0$ we have $(\partial x/\partial T)_S < 0$ and when $c_x < 0$ we have $(\partial x/\partial T)_S > 0$. Consequently, adiabatic cooling of wet steam in the region where the specific heat of wet steam c_x is positive will be accompanied by evaporation, while in the region where c_x is negative it is accompanied by condensation.

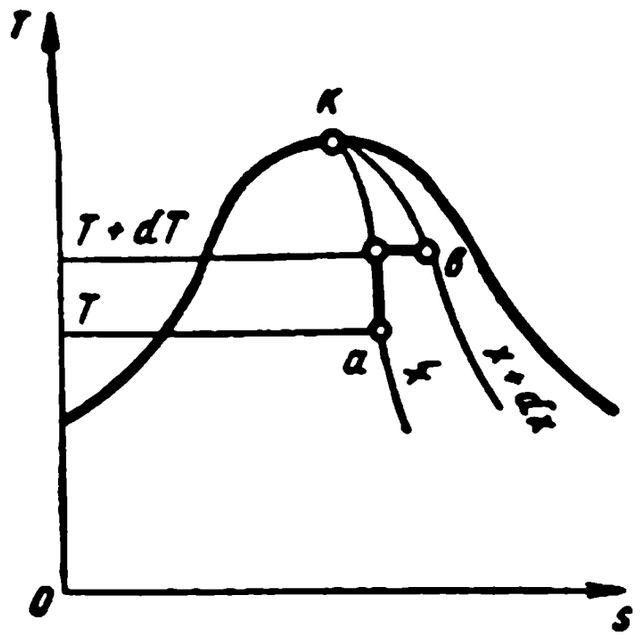

Fig. 76. Elementary isentropic-isothermal transition.

The change in pressure in an isentropic process is determined by the Clapeyron-Clausius formula:

$$\left(\frac{\partial p}{\partial T}\right)_s = \frac{r}{AT(v'' - v')}.$$

Since the right half of this equation is always positive, the differentials dp and dT should have identical signs, i.e., a decrease in pressure is always accompanied by a cooling of the vapor, and an increase in pressure by heating.

To ascertain how the volume of wet steam changes in the isentropic process, we use Eq. (3.48)

$$\left(\frac{\partial T}{\partial v}\right)_s = -\frac{AT}{c_v}\left(\frac{\partial p}{\partial T}\right)_v.$$

For wet steam, as for all other substances, $c_V > 0$ and the derivative $(\partial p/\partial T)_V = dp_s/dT$ is positive everywhere. Therefore $(\partial T/\partial v)_S < 0$. This means that the differentials dT and dv have opposite sign in an isentropic process of variation of the state of wet steam. Thus, isentropic expansion of wet steam leads always to a decrease in the pressure and temperature of the wet steam, while isentropic compression of wet steam causes an increase in its pressure and temperature.

On the basis of Eq. (7.39) we conclude that isentropic expansion of wet steam is accompanied by evaporation of the liquid phase if the specific heat c_x of the wet steam is positive and by condensation of the vapor if c_x is negative. In particular, near the critical point, where the specific heat of the saturated vapor c" is negative, its adiabatic expansion leads to the condensation of the vapor,* while adiabatic compression brings the saturated steam into a superheated state. Adiabatic expansion of the liquid, owing to the positive sign of its specific heat c', is always accompanied by evaporation. It fol-

lows therefore that in that part of the region of two-phase states where $c_x < 0$, the isentropic curves are steeper than the lines of constant degree of dryness $x = \text{const}$ and, in particular, steeper than the right boundary curve (see Fig. 76), so that the isentrope drawn from the region of superheated steam will cross the right-hand boundary curve. Where $c_x > 0$, the slope of the isentropes is smaller than for the line $x = \text{const}$ and for the right-hand boundary curve, so that the isentrope drawn from the region of superheated steam does not cross the right-hand boundary curve and with increasing volume it moves farther away from it. At the point of the right-hand boundary curve where the specific heat c" of the saturated vapor vanishes, the isentrope is tangent to the boundary curve. The isentrope drawn from the point of the liquid state of the substance will approach with increasing volume the left-hand boundary curve, cross it, and then move away from it into the region of the two-phase state of the substance.

Isentropic variation of the state of wet steam can be described also by the equation

$$\left(\frac{\partial p}{\partial v}\right)_s = -k\frac{p}{v}, \tag{7.41}$$

which is an analog of the equation of the isentropic process for an ideal gas. The coefficient k is called the local adiabatic exponent of wet steam. This positive quantity depends on the state of the wet steam: on the temperature t (or pressure p) and on the degree of dryness x of the steam.

At relatively low temperature, when a saturated vapor can be regarded as an ideal gas, it follows from (7.38) and from the Clapeyron-Clausius and Clapeyron-Mendeleyev formulas that

$$k = \frac{1}{1 - \frac{Apv}{r}\left[2 - \frac{T}{xr}\left(c'_p + x\frac{dr}{dT}\right)\right]}. \tag{7.42}$$

Equal-Enthalpy Process [i = const or $(1 - x)i' + xi'' = \text{const}$]

From the thermodynamic identity and Eq. (7.38) it follows that

$$di = c_x dT + r dx + A v dp. \qquad (7.43)$$

Substituting into the obtained equation the value of dp from the Clapeyron-Clausius formula and the value of c_x from (7.36) and taking account of the fact that di = 0 for the equal-enthalpy process, we obtain

$$\left[c' + x\left(\frac{dr}{dT} - \frac{r}{T}\right) + \frac{xvr}{T(v - v')}\right] dT + r\,dx = 0. \qquad (7.44)$$

This is the sought equation for the equal-enthalpy process of wet steam.

At low pressures and not too small values of x, when $c' \approx c'_p$ and constant, while $v - v' \approx v$, Eq. (7.44) can be made integrable

$$c'_p dT + d(xr) = 0,$$

from which it follows that

$$c'_p (T - T_0) + xr - x_0 r_0 = 0. \qquad (7.45)$$

It is seen from (7.44) that near the left boundary curve, i.e., at $x \approx 0$ and $c_x > 0$, we have $(\partial x/\partial T)_I < 0$. At low pressures near the right-hand boundary curve, i.e., for x = 1, we have $(\partial x/\partial T)_I < 0$ and at relatively high pressures, when the specific heat $c'' < 0$ and is appreciable in absolute magnitude, we have $(\partial x/\partial T)_I > 0$. This means that along the line i = const the derivative $\partial x/\partial T$ reverses its sign.

§4. SUPERHEATED STEAM

Superheated or unsaturated steam at a pressure p has a temperature t which is larger than the saturation temperature t_s at the same pressure. The temperature difference $t - t_s$ is the degree of superheat of the steam.

The specific enthalpy, internal energy, and entropy of superheated steam are, respectively, equal to:

$$i = i''(p) + \int_{T_s'', p}^{T, p} c_p dT; \qquad (7.46)$$

$$u = u''(p) + \int_{T_{s''}, p}^{T, p} c_p dT - Ap(v - v''); \qquad (7.47)$$

$$s = s''(p) + \int_{T_{s''}, p}^{T, p} \frac{c_p}{T} dT. \qquad (7.48)$$

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[Footnotes]

191 Saturated vapor in the form of a thin vapor layer exists also in an open vessel above the surface of the liquid.

209 In connection with this result, we must point out the error in the statement frequently encountered in the literature that there supposedly exist substances for which the value of c" is everywhere positive.

216 This property of the steam, in accordance with the statements made in §1 of Chapter 7, explains why supersaturated steam can form when saturated steam is adiabatically expanded.

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[List of Transliterated Symbols]

196 кр = kr = kriticheskiy = critical

210 тр = tr = troynoy = triple

Chapter 8

FLOW OF GASES AND VAPORS

§1. FUNDAMENTAL EQUATIONS OF FLOW

Continuous flow of gas is regarded in thermodynamics as an equilibrium process. Each of the macroscopic parts of the moving gas is considered to be in thermodynamic equilibrium and to have fully defined values of the parameters ($\underline{p}$ and $\underline{t}$, $\underline{i}$ and $\underline{s}$, etc.), varying continuously with time and on going from one point of the stream to another. The foregoing pertains also to flow with friction and flow with heat supply. Thus, the flow of gas is considered in all cases to be an externally irreversible process. In some cases the flow of gas is intermittent in character. In this case, discontinuities occur at individual points or in regions of the stream, or else jumps take place in the values of the flow velocity and the thermodynamic parameters of the gas.

Let us consider <u>stationary</u> or steady-state continuous flow of gas in a tube. In the most general case, when the flow is accompanied by the supply of heat to the moving gas and the latter performs useful work on some external object (this work is called also "technical work"), the value of which per kilogram of gas will be denoted by l'_m, the flow equation (see §7 of Chapter 2) has the form

$$i + A\frac{w^2}{2g} + Ah = i_1 + A\frac{w_1^2}{2g} + Ah_1 + q - Al'_m$$

or in differential form

$$di + \frac{A}{g} = w\,dw + A\,dh = dq - A\,dl'_m. \qquad (8.1)$$

Here q is the amount of heat obtained by 1 kg of gas from the external and internal sources of heat along the path from the initial point 1 to the given point (the heat of friction q_{tr} released in the gas as a result of the action of friction forces is not contained in the quantity q); h is the geometric height of the center of the tube.

The change of the entropy of the gas in a real flow process has the following form

$$ds = \frac{dq + dq_{тр}}{T}.$$

and in accordance with the thermodynamic identity

$$Tds = di - Avdp. \qquad (8.2)$$

The increment in the kinetic energy $(w^2/2g) - (w_1^2/2g)$ represents the available work l' which can be performed by 1 kg of moving gas on an external work object; together with $l'_m + (h - h_1)$ it constitutes the total useful external work, which in accordance with (1.16) is equal to $-\int vdp'$

$$\frac{w^2}{2g} - \frac{w_1^2}{2g} + l'_m + h - h_1 = -\int_{p_1'}^{p'} v\,dp'. \qquad (8.3)$$

where the integral $\int vdp'$ is taken along the real curve of the process.

Since $-\int_{p_1'}^{p'} vdp'$ differs from $-\int_{p_1}^{p} vdp$ by the work done by the friction forces q_{tr}, this equation can be rewritten as

$$\frac{w^2}{2g} - \frac{w_1^2}{2g} + l'_m + h - h_1 = -\int_{p_1}^{p} v\,dp - \frac{1}{A} q_{тр}.$$

In the derivation of Eqs. (8.1) and (8.3) it was assumed that the flow velocity w, the pressure p, the temperature t, and other gas parameters are constant over the section of the tube (have the same value at all points of a plane perpendicular to the axis of the tube). This is not perfectly exact. The flow velocity changes along the section of the tube from a maximum value on the tube axis to zero at the

walls. Therefore all the equations presented above are rigorously correct only for an individual current tube. For a tube of finite cross section these equations should be regarded as approximate and the larger the Reynolds number Re = $w\gamma d/hg$ in the stream, the more accurately they are satisfied. At very large Reynolds numbers it is possible to neglect the viscosity of the gas and to assume that there are no friction forces at all. The values of the temperature $\underline{t}$ and of the pressure $\underline{p}$ of the gas in each section of the tube are connected with the value of the specific volume $\underline{v}$ by the equation of state of the given gas:

$$p = p(t, v). \tag{8.4}$$

In addition to the equations presented, gas flow satisfies still another equation of continuity (law of conservation of the amount of matter), which in the case of stationary flow through the tube has the form

$$\frac{w\Omega}{v} = \frac{w_1\Omega_1}{v_1}, \tag{8.5}$$

where Ω is the cross section of the tube.

Thus, stationary flow of gas through a tube of any cross section is described by four equations: (8.1), (8.2), (8.4), and (8.5), which contain the four unknowns $\underline{p}$, $\underline{t}$, $\underline{v}$, and $\underline{w}$.

If in thermally insulated flow of gas there are no friction forces or they are so small that their effect can be neglected, then ds = 0 and the flow is isentropic. Let us assume further that the moving gas is ideal [Eq. (8.4) is in this case the Clapeyron-Mendeleyev equation]. From the condition ds = 0 it follows that the state of the moving gas varies along the reversible adiabat

$$pv^{\varkappa} = p_1v_1^{\varkappa}.$$

Therefore in place of Eqs. (8.1) and (8.5) we can choose for one-dimen-

sional isentropic flow of an ideal gas the equations (for $h = h_1$)

$$i + A\frac{w^2}{2g} = i_1 + A\frac{w_1^2}{2g} - Al'_m;$$

$$pv_x = p_1 v_{1x};$$

$$pv = RT;$$

$$\frac{w\Omega}{v} = \frac{w_1\Omega_1}{v_1}.$$

For an ideal gas with constant specific heat we have $i = c_pT + i_0$, $c_p = [k/(k-1)]AR$, and therefore Eq. (8.1) for $l'_m = 0$, $q = 0$, and $h = h_1$ can be rewritten in the form

$$\frac{k}{k-1}RT + \frac{w^2}{2g} = \frac{k}{k-1}RT_1 + \frac{w_1^2}{2g}.$$

Let us assume that a stream of ideal gas with a flow velocity $\underline{w}$ is completely stopped – the stream velocity has become equal to zero.

If we denote the absolute temperature of the stopped gas stream by T_{tor} (T_{tor} is called the stagnation temperature), then

$$T_{тор} = T + \frac{k-1}{kR}\frac{w^2}{2g}.$$

Velocity of Sound

In a continuous matter, be it a gas, liquid, or solid, any change in the state of any small part of the medium (local perturbation) due to the action of some external forces is transmitted to the surrounding parts of the medium and propagates from the point of occurrence with a certain velocity, which depends on the elastic properties and the density of the medium.

A perturbation is called small if the changes it produces in the parameters of the medium (pressure, density, temperature, etc.) are sufficiently small compared with the values of these parameters in the unperturbed medium. Examples of small perturbations are the sound waves propagating in the medium, which consist of alternating compressions and rarefactions which occur periodically at each point of the

tion of small perturbations is identical with the velocity of propagation of sound waves and is called the velocity of sound.

TABLE 4
Velocity of Sound in Gases (at 15°C)

1 Газ	κ	a, м/сек 2
3 Гелий	1,66	1000
4 Водород . . .	1,41	1320
5 Кислород . .	1,40	323
6 Азот	1,40	347
7 Воздух . . .	1,40	342
8 Углекислота	1,31	273

1) Gas; 2) m/sec; 3) helium; 4) hydrogen; 5) oxygen; 6) nitrogen; 7) air; 8) carbon dioxide.

In a sound wave propagating in a gas, the processes of compression and expansion are so rapid that there is practically no heat exchange between that part of the gas through which the sound wave passes and the other parts of the gas, so that the change in state of the gas upon passage of a sound wave through it occurs without the supply or removal of heat, i.e., adiabatically.* Since the changes in the state of the gas are so small that the action of the internal friction is also vanishingly small, sound waves should be regarded as a reversible adiabatic or isentropic process, independently of the law governing the change in the state of the entire gas as a whole.

The velocity of sound is a quantity that is characteristic of the given substance and depends on the state and nature of the substance

$$a = \sqrt{-gv^2\left(\frac{\partial p}{\partial v}\right)_s}. \qquad (8.6)$$

For an ideal gas

$$a = \sqrt{g\kappa RT}. \qquad (8.7)$$

Formula (8.7) for rarefied gases has been well confirmed by experiment. The values of the velocity of sound for various gases in the rarefied state are listed in Table 4.

§2. ISENTROPIC ESCAPE OF GAS FROM NOZZLES

Change in Flow Velocity Along the Nozzle

Let us consider the stationary escape of a gas from a vessel through a fitting (nozzle) of variable cross section under the assump-

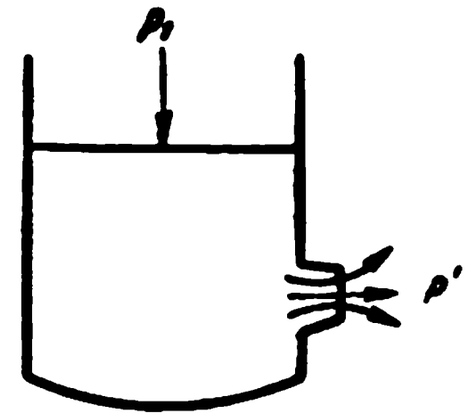

Fig. 77. Diagram of stationary escape of glass through a nozzle.

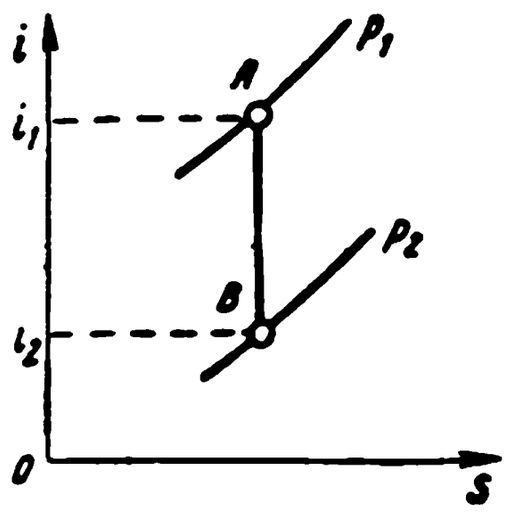

Fig. 78. Graphic representation of isentropic escape of gas.

tion that there is no heat exchange between the surrounding medium and the flowing gas, there are no friction forces, the moving gas performs no "technical work," and the piezometric height of the gas does not change (Fig. 77). We denote the initial parameters of the gas, namely the temperature, specific volume, and pressure, by t_1, v_1, and p_1; the initial gas velocity in the vessel is w_1 and the pressure of the outside medium, into which the gas escapes, is p' (of course, p' is smaller than p_1); the temperature, specific volume, pressure, and velocity of the gas at the outlet of the nozzle (in its outlet section) are denoted by t_2, v_2, p_2, and w_2. Since the escape of the gas is by assumption adiabatic (isentropic) and $l'_m = 0$, it follows on the basis of Eq. (8.1)

$$i_2 + A\frac{w_2^2}{2g} = i_1 + A\frac{w_1^2}{2g}$$

that

$$w_2 = \sqrt{\frac{2g}{A}(i_1 - i_2) + w_1^2}.$$

The difference in the enthalpies in the initial and final states can be determined from the is diagram of the given gas, provided we know the value of the gas pressure p_2 at the outlet section of the nozzle, which, as will be verified below, is not always equal to the external pressure p' and may exceed p'. Drawing from the initial point $A(p_1, t_1)$ of this diagram a vertical line until it crosses the isobar $p = p_2$, we obtain the point $B(p_2, t_2)$, which represents the state of the gas at the outlet of the nozzle (Fig. 78). The line AB along which the entropy does not vary represents graphically the process of isen-

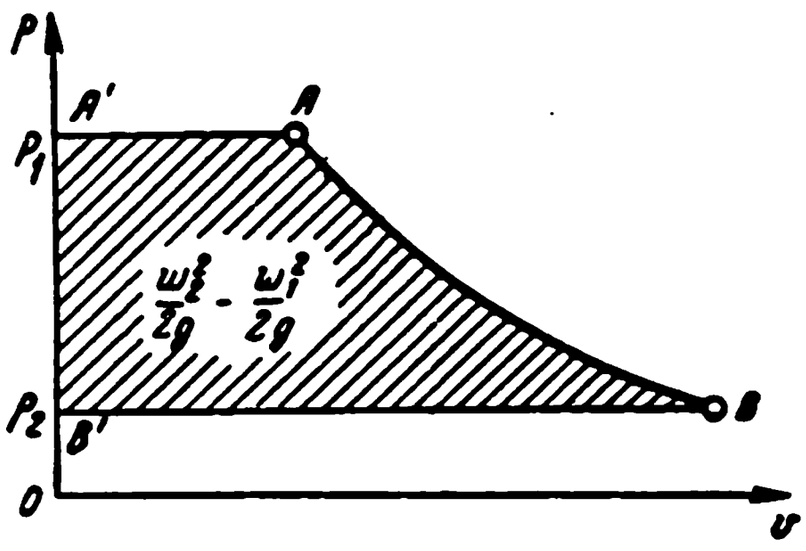

Fig. 79. Graphic representation of the increment in the specific kinetic energy in isentropic flow of gas.

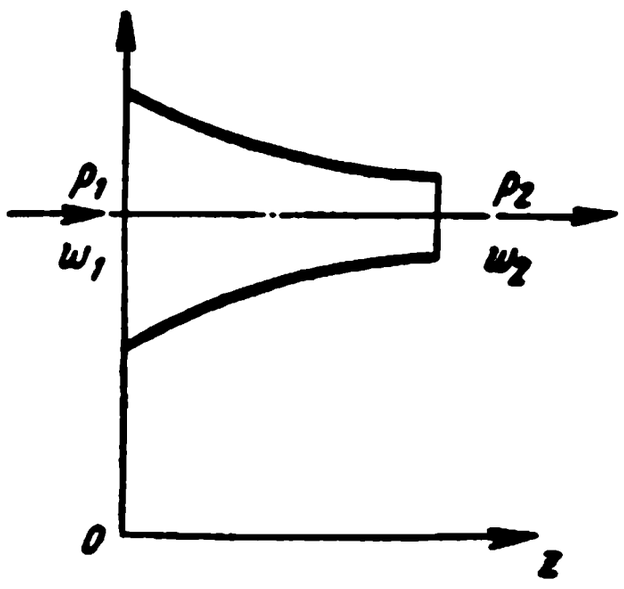

Fig. 80. Change in state of a gas escaping through a nozzle with variable cross section.

tropic flow of gas from the initial state A to the final state B. The length of the segment AB is numerically equal to the difference in the enthalpies, i.e., to the available heat drop $i_1 - i_2$.

According to (8.3), the available work, i.e., the increment in kinetic energy, will in this case be equal to

$$A\frac{w_2^2}{2g} - A\frac{w_1^2}{2g} = -A\int_{p_1}^{p_2} v\,dp.$$

Let us plot the considered isentropic gas-flow process on the pv diagram (Fig. 79). Then the integral $-\int_{p_1}^{p_2} v\,dp$ represents the length of the shaded area A'ABB'.

Let us analyze now the change in the state of the gas as it flows through a nozzle whose cross section varies with distance z from the inlet section in accordance with the law $\Omega = \Omega(z)$ (Fig. 80). For simplicity of argument we assume that the gas is ideal. From Eqs. (8.1), (8.2), and (8.5) we obtain for the isentropic gas flow

$$di + A\frac{w}{g}dw = 0;$$
$$v\,dp = -\kappa p\,dv;$$
$$di = Av\,dp;$$
$$\frac{dw}{w} - \frac{dv}{v} + \frac{d\Omega}{\Omega} = 0.$$

Replacing in the first equation di by Avdp = (ART/p)dp, we obtain

$$\frac{w}{g}dw - \frac{\kappa RT}{v}dv = 0. \qquad (8.8)$$

It follows from this equation, in particular, that an increase in velocity during isentropic flow is accompanied by expansion of the gas

and accordingly by a decrease in its temperature and pressure, while a decrease in velocity is accompanied by compression of the gas and by an increase of p and t. From the continuity equation it is furthermore seen that in flow through a contracting nozzle, when $d\Omega/dz < 0$ $dw/w > dv/v$, i.e., the relative change in the velocity exceeds the relative change in the volume, while in flow through an expanding nozzle, to the contrary, the relative change in velocity is smaller than the relative change in the volume $dw/w < dv/v$.

The rate of flow will increase along the nozzle if the relative increase in the volume of the gas exceeds the relative change in the transverse section of the nozzle $dv/v > d\Omega/\Omega$. This condition is satisfied in contracting nozzles at subsonic flow velocities and in expanding nozzles at supersonic flow velocities. To verify this, let us transform Eq. (8.8), by replacing in it dv/v by $(dw/w) + (d\Omega/\Omega)$. As a result we obtain

$$\frac{1}{w}(w^2 - gkRT)\frac{dw}{dz} = \frac{gkRT}{\Omega}\cdot\frac{d\Omega}{dz}$$

or, since gkRT is the square of the velocity of sound, a^2, then

$$\frac{1}{w}(w^2 - a^2)\frac{dw}{dz} = \frac{a^2}{\Omega}\cdot\frac{d\Omega}{dz}. \qquad (8.9)$$

Equation (8.9) determines the velocity of the gas w at any cross section of the nozzle. In this equation the quantity a represents the velocity of sound at the given section or, as it is called, the local velocity of sound.

In a contracting nozzle $d\Omega/dz < 0$ and therefore the right half of Eq. (8.9) has a negative sign. If the gas velocity at the entrance to the nozzle is smaller than the velocity of sound, then in the initial part of the nozzle $w^2 - a^2 < 0$ and dw/dz will have a positive sign.

Consequently, if the gas velocity in the inlet cross section of a

the velocity of flow along the nozzle increases and the flow of the gas becomes accelerated (the local velocity of sound decreases in this case along the nozzle).

If the velocity of the gas at the inlet to the contracting nozzle exceeds the local velocity of sound $w^2 - a^2 > 0$, then $dw/dz < 0$ and the flow of gas along the nozzle slows down; the gas is then compressed. It is stated then that the nozzle operates like a diffuser.

Critical Escape Velocity

It follows from Eq. (8.9) that in a contracting nozzle it is impossible to go in a continuous fashion through the value of the flow velocity equal to the local velocity of sound. This means, for example, that in the case of subsonic velocity at the entrance to the nozzle it is impossible to attain supersonic velocity at the exit from the nozzle. Indeed, let us assume that such a continuous transition through the velocity of sound is possible within the nozzle (at some intermediate section of the nozzle). Then the motion of the gas to the point of transition and past it should be accelerated and consequently the derivative dw/dz should have the same sign ahead of the point of transition and past it. According to Eq. (8.9) we have to the left of the point of transition $dw/dz > 0$, inasmuch as $w < a$, and to the right of the point of transition, where w should by assumption be larger than a, we have $dw/dz < 0$, from which it follows that in spite of the assumption made the sign of the acceleration is not the same on the two sides of the point of transition.

The reversal of the sign of dw/dz at the point $w = a$ [at the point $w = a$ itself the derivative dw/dz becomes infinite, as can be seen from Eq. (8.9)] denotes that as soon as a flow velocity equal to the local velocity of sound is reached the flow should change over from accelerated into retarded; as a result of this it is impossible

to exceed the velocity of sound (or go through it) in a contracting nozzle. It also follows from this that if the velocity of sound is reached in stationary escape of gas through a contracting nozzle, this is possible only in the outlet section of the nozzle, which is the narrowest.

Thus, upon escape of a gas through a contracting nozzle, the velocity of the gas $\underline{w}$ can nowhere exceed the local velocity of sound $\underline{a}$; in the limiting case the velocity of gas at the exit from the nozzle is equal to the velocity of sound at the exit section of the nozzle. In other words, in the flow of gas through contracting nozzles there exists a critical escape velocity, which we denote by w_{kr}. The critical velocity is the maximal escape velocity which can be attained at the outlet of a nozzle for a given initial state of the gas. This conclusion is valid for all initial gas pressures. No matter how large the initial pressure p_1 of the gas may be compared with the external pressure p' (the pressure of the medium into which the escape takes place), the velocity of the gas at the exit from a contracting nozzle can never exceed the critical escape velocity, equal to the velocity of sound at the exit section of the nozzle.

But it follows from this that the pressure in the exit section of the nozzle is equal to the external pressure only at low escape velocities, smaller than the velocity of sound. Upon escape of gas from a nozzle with sonic velocity, the pressure in the outlet section of the nozzle, depending on the initial pressure of the gas, can be both equal to or larger than the external pressure. To verify this, let us consider the escape of a gas situated in a vessel at a constant pressure p_1 through a contracting nozzle into an external medium, the pressure p' of which can vary. When $p' = p_1$ we have $w_2 = 0$ (there is no escape of gas). When $p' < p_1$ the gas begins to escape, and with de-

creasing pressure p', i.e., with increasing pressure drop $p_1 - p'$ under the influence of which the gas flow is produced, the escape velocity $\underline{w}$ continuously increases until it reaches a critical velocity $w_{kr} = a_2$ at a certain value of the external pressure, which we shall call the critical escape pressure p_{kr}. At that instant, as also previously when $w_2 < w_{kr}$, the pressure p_2 at the outlet section of the nozzle is equal to the external pressure $p_2 = p'$. Further decrease in the pressure of the medium does not lead to an increase in the escape velocity and consequently to a change in pressure in the outlet section of the nozzle, which remains equal to p_{kr} at all times. In other words, in the critical flow mode, the gas does not expand completely to the pressure of the external medium. The dependence of the pressure p_2 of the gas at the outlet section of the nozzle on the external pressure p' is shown graphically in Fig. 81.

The fact that the reduction of the external pressure at values $p' < p_{kr}$ does not lead to a change in the gas pressure in the outlet section of the nozzle has the following physical explanation. The change in pressure, and particularly its decrease, propagates in the medium with a velocity equal to the velocity of sound. At low escape velocities, the decrease in the external pressure is transmitted along the gas jet inside the nozzle and leads to a redistribution of the pressure in the nozzle. As a result, the pressure in the outlet section of the nozzle decreases to the value of the external pressure and the escape velocity increases accordingly. But as soon as the gas velocity at the outlet of the nozzle reaches the velocity of sound, no reduction in the pressure of the surrounding medium in-

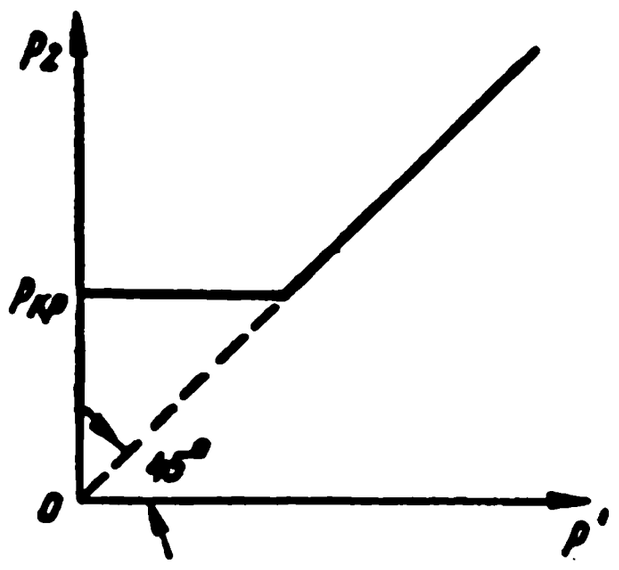

Fig. 81. Gas pressure in outlet section of nozzle as a function of external pressure.

side the nozzle is transmitted further (it is carried away, as it were, by the gas stream escaping from the nozzle with the same velocity with which the perturbation propagates). Therefore the pressure distribution inside the nozzle and consequently also the gas pressure in the outlet section of the nozzle and the escape velocity of the gas remain the same independently of the value of the external pressure.

The value of the critical escape pressure p_{kr}, which is equal to the gas pressure p_2 in the outlet section of the nozzle when the gas escapes with critical velocity $w_{kr} = a_2$, can be determined from Eq. (8.1), which for an ideal gas and under the condition that the initial velocity w_1 is equal to zero has the following form:

$$c_p T_{кр} + A\frac{w_{кр}^2}{2g} = c_p T_1.$$

In accordance with the flow conditions considered here, we designate T_2 by the symbol T_{kr}. Since

$$w_{кр} = \sqrt{g\kappa RT_{кр}}, \quad c_p = \frac{\kappa}{\kappa - 1} AR,$$

we have

$$T_{кр} = \frac{2}{\kappa + 1} T_1.$$

The gas flow under consideration is isentropic, so that on the basis of (4.13) we have

$$T_{кр} = \left(\frac{p_{кр}}{p_1}\right)^{\frac{\kappa-1}{\kappa}} T_1.$$

Substituting this value of T_{kr} into the preceding equation we obtain after simple transformations

$$\frac{p_{кр}}{p_1} = \left(\frac{2}{\kappa + 1}\right)^{\frac{\kappa}{\kappa-1}}. \tag{8.10}$$

Analogously we get

$$\frac{v_{кр}}{v_1} = \left(\frac{2}{\kappa + 1}\right)^{-\frac{1}{\kappa-1}},$$

We see from Formula (8.10) that the ratio of the critical escape pressure p_{kr} to the initial gas pressure p_1 does not depend on the initial gas pressure or on the pressure of the surrounding medium and is only a function of the adiabatic exponent $\underline{k}$. The values of p_{kr}/p_1 for different gases and water vapor are listed in Table 5.

TABLE 5

Value of p_{kr}/p_1 for Different Gases (in the Ideal State)

1 Газ	κ	$p_{кр}/p_1$
2 Одноатомный	1,7	0,482
3 Двухатомный	1,4	0,528
4 Многоатомный, а также перегретый водяной пар при незначительных давлениях	1,3	0,546
5 Сухой насыщенный водяной пар при температуре 100°C	1,18	0,568
150°C	1,24	0,562

1) Gas; 2) monatomic; 3) diatomic; 4) polyatomic, and also superheated steam at slight pressures; 5) dry saturated steam at 100°C.

If the initial gas velocity w_1 is not equal to zero then, as can be readily verified,

$$\frac{p_{кр}}{p_1} = \left(\frac{2}{\kappa+1}\right)^{\frac{\kappa}{\kappa-1}} \left(1 + \frac{\kappa-1}{\kappa} \cdot \frac{w_1^2}{2gR}\right)^{\frac{\kappa}{\kappa-1}}$$

i.e.,

$$p_{кр}/p_1 > \left(\frac{2}{\kappa+1}\right)^{\frac{\kappa}{\kappa-1}}.$$

The quantity $2/(k+1)^{k/(k-1)}$, equal to the ratio p_{kr}/p when $w_1 = 0$, is usually designated by the letter β.

Fundamental Calculation Formulas

Let us derive now the fundamental calculation formulas for the velocity and amount of gas flowing through the nozzle, under the assumption that the gas is ideal and the flow isentropic. We denote, as

before, the initial values of the temperature, specific volume, pressure, and initial gas velocity in the vessel by t_1, v_1, p_1, w_1 and the values of the temperature, specific volume, pressure, and gas velocity at the outlet from the nozzle (in its outlet section) by t_2, v_2, p_2, w_2.

The gas escape velocity from the nozzle is determined by Eq. (8.3):

$$w_2^2 = w_1^2 + 2g\int_{p_1}^{p_2} v\,dp.$$

Since the pressure and the volume are related in the case of isentropic flow by the adiabatic equation $pv^k = \text{const}$, we have

$$\int_{p_2}^{p_1} v\,dp = \frac{\kappa}{\kappa-1} p_1 v_1 \left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{\kappa-1}{\kappa}}\right].$$

Substituting the obtained value of $\int_{p_1}^{p_2} v\,dp$ into the preceding equation, we obtain

$$w_2 = \sqrt{\frac{2g\kappa}{\kappa-1} p_1 v_1 \left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{\kappa-1}{\kappa}}\right] + w_1^2}. \qquad (8.11)$$

The amount of gas escaping through the nozzle per unit time, or the per-second flow of gas G kg/sec, amount to

$$G = \frac{w_2 \Omega_2}{v_2}.$$

Substituting here the value of the escape velocity w_2 and replacing v_2 by $v_1(p_1/p_2)^{1/k}$, we obtain

$$G = \Omega_2 \sqrt{\frac{2g\kappa}{\kappa-1} \cdot \frac{p_1}{v_1}\left[\left(\frac{p_2}{p_1}\right)^{\frac{2}{\kappa}} - \left(\frac{p_2}{p_1}\right)^{\frac{\kappa+1}{\kappa}}\right] + \frac{w_1^2}{v_1^2}\left(\frac{p_2}{p_1}\right)^{\frac{2}{\kappa}}}. \qquad (8.12)$$

Equations (8.9), (8.11), and (8.12) are of general significance and are valid for nozzles of all shapes.

Escape through a contracting nozzle. Let us consider first a pre-critical flow condition in which the velocity w_2 of the gas escaping from the nozzle is less than the critical velocity w_{kr} and the pressure p_2 of the gas in the outlet section of the nozzle is larger than

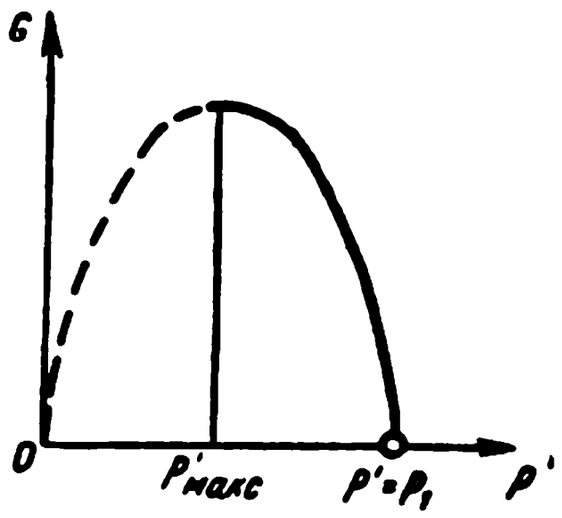

Fig. 82. Dependence of gas flow through a nozzle on the pressure of the medium.

the critical escape pressure p_{kr} and is equal to the pressure of the external medium p' into which the escape takes place ($p_2 = p' > p_{kr}$). Since $p_{kr} = \beta p_1$, we obtain from this the following condition for the realization of a precritical escape mode: $p_1 < p'/\beta$. This means that the precritical escape mode is observed at initial gas pressures p_1 that are slightly higher than the pressure of the external medium p'. In this mode, complete expansion of the gas takes place from the initial pressure p_1 to the external pressure p'.

In order to obtain formulas for the escape velocity and the per-second flow of gas in precritical escape, it is necessary to substitute into Eqs. (8.11) and (8.12), instead of the pressure p_2 in the outlet section of the nozzle, the pressure of the external medium p', which is equal to it. As a result, the velocity w_2 of the gas escaping from the nozzle and the per-second flow of gas G in the precritical flow mode and when $w_1 = 0$ will be equal to:

$$\left.\begin{aligned} w_2 &= \sqrt{\frac{2g\kappa}{\kappa-1} p_1 v_1 \left[1-\left(\frac{p'}{p_1}\right)^{\frac{\kappa-1}{\kappa}}\right]}, \\ G &= \Omega_2 \sqrt{\frac{2g\kappa}{\kappa-1}\cdot\frac{p_1}{v_1}\left[\left(\frac{p'}{p_1}\right)^{\frac{2}{\kappa}}-\left(\frac{p'}{p_1}\right)^{\frac{\kappa+1}{\kappa}}\right]}. \end{aligned}\right\} \qquad (8.13)$$

It is seen from the equation for G that the amount of gas escaping through the nozzle per unit time for a given initial gas pressure p_1 depends on the pressure of the medium p' into which the gas escapes. This dependence is plotted in Fig. 82. When $p' = p_1$ the per-second flow of gas is equal to zero; with decreasing p', the per-second flow G increases, reaches a maximum, and then decreases to a value G = = 0 when p' = 0. At the point of the maximum of the G(p') curve the

derivative dG/dp' vanishes, which leads to the following equation for p'_{maks}:

$$\frac{2}{\kappa}\left(\frac{p'_{макс}}{p_1}\right)^{\frac{2-\kappa}{\kappa}} - \frac{\kappa+1}{\kappa}\left(\frac{p'_{макс}}{p_1}\right)^{\frac{1}{\kappa}} = 0$$

or

$$\frac{p_{макс}}{p_1} = \left(\frac{2}{\kappa+1}\right)^{\frac{\kappa}{\kappa-1}}.$$

Comparing this equation with the equation for the critical escape pressure p_{kr}, we see that $p'_{maks} = p_{kr}$. Consequently, at the point of the maximum of the G(p') curve the external pressure is precisely equal to the critical pressure and therefore the escape velocity reaches a critical value. Therefore in the case of flow through a contracting nozzle with an initial velocity that is smaller than the velocity of sound, the left branch of the curve has no physical meaning, for actually at pressures $p' \leq p'_{maks} = p_{kr}$ the escape mode will not be precritical, as was implied by the indicated curve, since it was assumed in its plotting that $p_2 = p'$, but critical, for which $p_2 > p'$.

In the case of the <u>critical</u> escape mode, when the gas velocity w_2 at the outlet of the nozzle is equal to the critical velocity w_{kr} and the pressure p_2 of the gas in the outlet section of the nozzle is larger than the external pressure p' and is equal to the critical pressure p_{kr}, the expression for the escape velocity can be obtained from the condition $w_{kr} = a_2$. But the velocity of sound in the outlet section of the nozzle $a_2 = \sqrt{gkRT_2}$ in the case of isentropic flow, when $T_2 = (p_{kr}/p_1)^{(k-1)/k} \cdot T_1$, will have the form

$$a_2 = \sqrt{g\kappa RT_1\left(\frac{p_{кр}}{p_1}\right)^{\frac{\kappa-1}{\kappa}}}. \qquad (8.14)$$

It is seen from this formula that the critical velocity of escape increases with increasing initial temperature of the gas.

The per-second flow of gas in the critical escape mode will have the form

$$G = \Omega_2 \sqrt{g\kappa \frac{p_1}{v_1}\left(\frac{p_{kp}}{p_1}\right)^{\frac{\kappa+1}{\kappa}}} \qquad (8.15)$$

Equations (8.14) and (8.15) simplify appreciably if the initial gas velocity w_1 is equal to zero. In this case $p_{kr}/p_1 = [2/(k+1)]^{k/(k-1)}$ and therefore

$$\left.\begin{aligned} w_2 &= \sqrt{\frac{2g\kappa}{\kappa+1}p_1v_1}, \\ G &= \Omega_2\sqrt{g\kappa\left(\frac{2}{\kappa+1}\right)^{\frac{\kappa+1}{\kappa-1}}\cdot\frac{p_1}{v_1}}. \end{aligned}\right\} \qquad (8.16)$$

Substituting into (8.16) the numerical values of $\underline{k}$ and $\underline{g}$ we obtain for monatomic ideal gases

$$w_2 = 3{,}52\sqrt{p_1v_1},$$
$$G = 2{,}29\Omega_2\sqrt{\frac{p_1}{v_1}};$$

for diatomic gases

$$w_2 = 3{,}38\sqrt{p_1v_1},$$
$$G = 2{,}14\Omega_2\sqrt{\frac{p_1}{v_1}};$$

and for polyatomic gases and superheated steam at low pressures

$$w_2 = 3{,}33\sqrt{p_1v_1};$$
$$G = 2{,}09\Omega_2\sqrt{\frac{p_1}{v_1}}.$$

<u>Escape through an expanding nozzle.</u> From Eq. (8.9) it follows that in an expanding nozzle for $w_1 < a$ we have $dw/dz < 0$ (the flow of the gas slows down). Consequently, unlike flow in a contracting nozzle, where the gas expands and the flow velocity increases accordingly, in an expanding nozzle the gas compresses, its pressure increases, the velocity decreases, and the kinetic energy of the gas is converted into potential energy of pressure. Fittings in which the kinetic energy is converted into pressure energy are called <u>diffusers</u>.

The maximum degree of compression in a diffuser in the case when

the initial velocity of the gas is slightly smaller than (and in the limit equal to) the velocity of sound is in accordance with Eq. (8.10)

$$\left(\frac{p_2}{p_1}\right)_{\text{макс}} = \left(\frac{\kappa+1}{2}\right)^{\frac{\kappa}{\kappa-1}}.$$

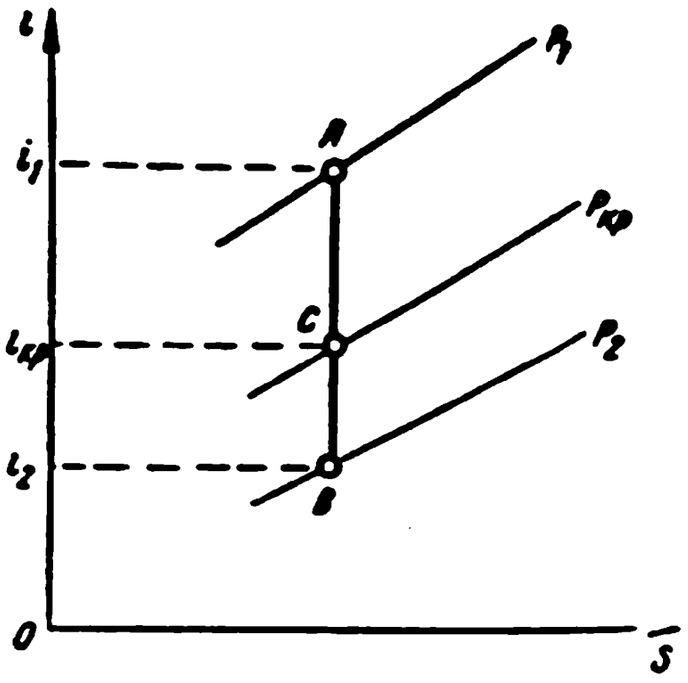

Fig. 83. Determination of the critical escape pressure on the is diagram.

At supersonic initial velocities $w_1 > a_1$, the velocity of the gas escaping increases along the nozzle, the volume of the gas increases, and the pressure and temperature decrease.

Calculation of the escape of real gases and vapors. The equations (8.11)-(8.16) presented above for w_2 and G pertain to ideal gases. For real gases or for vapors the calculation of the escape velocity and of the per-second flow is carried out with the aid of the is diagram of the given substance.

Let us consider briefly the sequence of calculating the escape of gas through a contracting nozzle. We assume that the initial state of the gas is represented on the diagram (Fig. 83) by the point $A(p_1, t_1)$. We shall assume the flow to be isentropic and draw from the point A a vertical line until it crosses the isobar corresponding to the pressure p' of the external medium into which the gas escapes [the point $B(p_2, t_2)$]. If the escape mode is precritical, then the point B represents the final state of the gas at the exit from the nozzle and therefore when $w_1 = 0$ we have

$$\left.\begin{aligned} w_2 &= \sqrt{\frac{2g}{A}(i_1 - i_2)}, \\ G &= \frac{\Omega_2}{v_2}\sqrt{\frac{2g}{A}(i_1 - i_2)}. \end{aligned}\right\} \qquad (8.17)$$

The question arises, however, whether the flow mode near the point B

is precritical or critical. For an ideal gas the answer is obtained by using Eq. (8.10) and comparing the calculated value of p_{kr} with the external pressure p'. If $p_{kr} < p'$, then the escape mode is precritical and if $p_{kr} > p'$, then the escape mode is critical. With a certain degree of approximation we can use this procedure also for a real gas or vapor, but there exists another more accurate method. This method consists in comparing the escape velocity of the gas in the given state, calculated from Eq. (8.17), with the velocity of sound in the same state. The velocity of sound is calculated with the aid of the is or ts diagram by Formula (8.6):

$$a = \sqrt{-gv^2\left(\frac{\partial p}{\partial v}\right)_s}.$$

where the values of the increments of the volume dv and of the pressure dp are taken in the state of the gas under consideration for S = = const, i.e., along the vertical straight line AB. If it turns out that the value of the velocity of sound at the point B exceeds the flow velocity calculated by Eq. (8.17), then the escape mode is precritical and the escape velocity has been correctly determined. If a is smaller than w, then the escape mode is critical and the escape velocity w has been incorrectly calculated. In this case, by comparing the value of the velocity of sound a with the flow velocity w at several intermediate points of the line AB, we determine the point C at which w = a. The isobar drawn through the point C determines the pressure $p_2 = p_{kr}$ in the outlet section of the nozzle and $\sqrt{2g/A(i_1 - i_C)}$ yields the real value of the escape velocity w_2. It is possible to use the foregoing procedure also in the case when the initial velocity w_1 of the gas or vapor is not equal to zero. In this case

$$w_2 = \sqrt{\frac{2g}{A}(i_1 - i_2) + w_1^2}.$$

$$G = \frac{\Omega_2}{v_2}\sqrt{\frac{2g}{A}(i_1 - i_2) + w_1^2}.$$

In the case of the escape of saturated and supersaturated steam, the velocity of sound can be calculated also on the basis of Eqs. (7.41) and (7.42) using the following approximate formula:

$$a = \sqrt{\frac{gpv}{1 - 2\frac{Apv}{r}}}.$$

In this formula the quantity $1/[1 - (2Apv/r)]$ represents the adiabatic exponent of the saturated and supersaturated steam, i.e.,

$$\kappa = \frac{1}{1 - \frac{2Apv}{r}}. \qquad (8.18)$$

§3. INFLUENCE OF FRICTION

So far we have considered isentropic flow of a gas. Real flow of gases or vapors is not isentropic, owing to friction and heat exchange. The action of the friction forces during the flow of gases gives rise to dissipation of mechanical energy (conversion of part of this energy into heat of friction), as a result of which the entropy of the gas increases.

The curve of the process of adiabatic flow of gas with friction can be represented on the is diagram in the form of the line AC (Fig. 84), while the curve of isentropic flow from the same initial state has the form of a vertical straight line AB; owing to the increase in the entropy, the line AC of the actual process lies to the right of the line of the ideal process AB. It is seen in Fig. 84 that for the same pressure drop $p_1 - p_2$, the enthalpy in the final state of flow with friction (the point C) will have a larger value than the enthalpy in the final state in flow without friction (the point B) and the escape velocity is in accordance with Eq. (8.1) smaller than in flow without resistance. The ratio of the escape velocity w_2 of an actual process AC to the escape velocity w_{2teor} of the theoretical process AB

and p_2 is called <u>the velocity coefficient</u> φ

$$\varphi = \frac{w_2}{w_{2\,теор}}. \qquad (8.19)$$

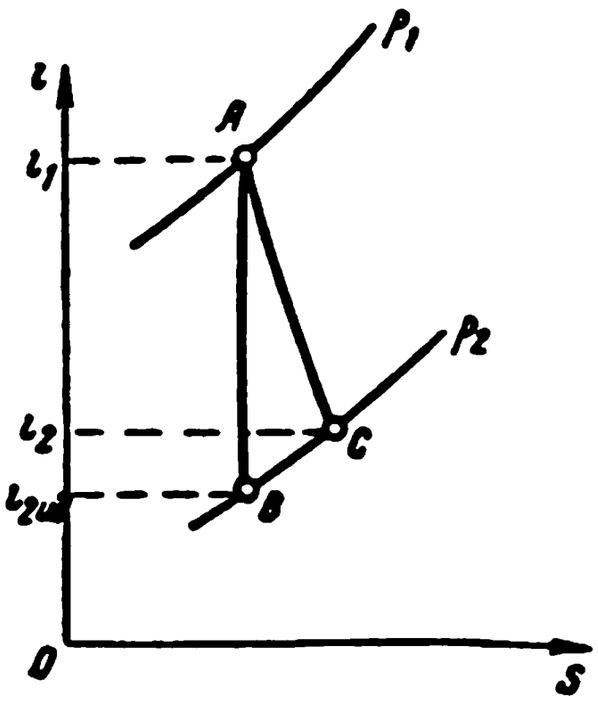

Fig. 84. Plot of adiabatic escape of gas with friction.

Using the definition of the escape velocity, we obtain from (8.1)

$$w_2 = \varphi \sqrt{\frac{2g}{A}(i_1 - i_{2\,теор}) + w_1^2}.$$

The velocity coefficient φ is different for different nozzles; its values are obtained by experiment. For nozzles in modern turbines φ amounts to about 0.93-0.98.

§4. PASSAGE THROUGH THE VELOCITY OF SOUND. LAVAL NOZZLE

<u>Continuous Passage Through the Velocity of Sound</u>

On the basis of the general equation (8.9) for the isentropic flow of a gas in a nozzle of variable cross section

$$\frac{1}{w}(w^2 - a^2)\frac{dw}{dz} = \frac{a^2}{\Omega}\cdot\frac{d\Omega}{dz},$$

we can readily understand how gas flow can exist with a velocity increasing along the nozzle (with $dw/dz > 0$) and continuous passage from subsonic flow velocity to supersonic velocity.

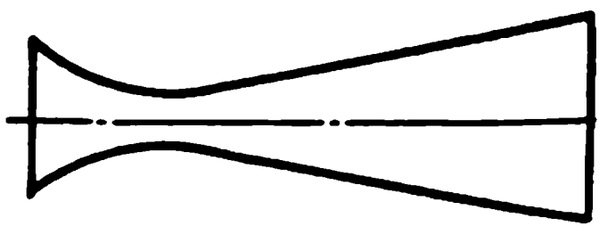

Fig. 85. Diagram of Laval nozzle.

Let the initial part of the nozzle be contracting, and the final one expanding (Fig. 85). If the velocity of the gas at the inlet to the nozzle is smaller than the local velocity of sound, then the gas flow will accelerate in the contracting part of the nozzle. In the narrowest part (throat) of the nozzle, where $d\Omega/dz$ is equal to zero, the left half of Eq. (8.9) vanishes:

$$\frac{1}{w}(w^2 - a^2)\frac{dw}{dz} = 0,$$

which can occur either when $dw/dz = 0$ and $w < a$ or when $dw/dz > 0$ and

$w = a$. Since $\underline{w}$ varies continuously, when $w < a$ in the throat of the nozzle then in the vicinity on the right side of the nozzle (in the expanding part) $\underline{w}$ will also be smaller than $\underline{a}$. Therefore in the former case the derivative dw/dz to the right of the throat of the nozzle, where $d\Omega/dz > 0$, will have a negative sign, i.e., the flow of the gas in the expanding part of the nozzle, unlike the contracting part, will slow down and consequently $w < a$ in any section of the nozzle. This case is realized at relatively low values of pressure drop $p_1 - p'$ at which the gas flows through the nozzle.

If the pressure drop $p_1 - p'$ is appreciable, then the flow velocity at the smallest section of the nozzle reaches the velocity of sound and further flow of gas in the expanding part of the nozzle is with acceleration, i.e., a continuous transition takes place in the flow velocity from sonic to supersonic values. In fact, inasmuch as in the smallest section (the throat of the nozzle) $w = a$ and $dw/dz > 0$, in the initial section of the expanding part the flow velocity will exceed the local velocity of sound ($w > a$). But then, as follows from (8.9) and the condition $d\Omega/dz > 0$, the flow velocity will increase even further, exceeding the velocity of sound in the entire expanding portion of the nozzle.

Thus, by causing the gas to flow at a sufficiently high pressure drop first through a contracting nozzle and then through an expanding one, it is possible to realize a flow with continuously rising velocity and attain at the outlet of the nozzle a flow velocity exceeding the velocity of sound at the output section of the nozzle.

A nozzle consisting of a combination of contracting and expanding nozzles is named after its inventor a Laval nozzle.

To realize a continuous transition of the flow velocity of the gas through the velocity of sound it is necessary that the right half

of Eq. (8.9) reverse its sign at the instant when the flow velocity is equal to the local velocity of sound. When this condition is satisfied, it is possible to realize a continuous transition from subsonic to supersonic velocities not only in a Laval nozzle, but also in other devices, for example in a "thermal" nozzle, in which the heat is first supplied to the gas and then, once the flow velocity has reached the velocity of sound, heat is taken away; in a "mass" nozzle, where the mass of the gas is continuously increased by supplying the initial amounts of gas from the outside and then, starting with $w = a$, the amount is continuously decreased; in a "mechanical" nozzle, where the gas first acquires work from an external work source and then, starting with the instant $w = a$, it gives up useful work to the outside.

Flow Conditions in a Laval Nozzle

The Laval nozzle is widely used to obtain supersonic flows of gases and vapor in steam and gas turbines, in jet engines, etc. Depending on the relation between the initial gas pressure p_1 and the pressure p' of the external medium into which the escape takes place (on the pressure drop $p_1 - p'$), various flow conditions are possible.

Let us assume first that the external pressure p' is so large that in the narrowest section of the nozzle the flow velocity does not reach the velocity of sound ($w < a$). It is seen from Eq. (8.9) that in this case in the narrowest section of the nozzle $dw/dz = 0$, i.e., the gas flow velocity reaches a maximum at the throat of the nozzle. Since the velocity in the contracting part of the nozzle is subsonic, the velocity will remain subsonic also past the narrowest section in the expanding part of the nozzle and with increasing distance from the narrowest section the velocity will decrease while the gas pressure increases.

We now decrease the external pressure p'. As a result of this the

velocity of gas flow in the nozzle will increase and at a certain value of the pressure p', which we denote by p'_a, the gas velocity in the narrowest section of the nozzle reaches the velocity of sound; in the expanding part of the nozzle the flow velocity will have a subsonic value at $p' = p'_a$ and at $p' < p'_a$ on a certain portion past the narrowest section of the nozzle it will be supersonic. The pressure of the gas in the outlet section is, of course, equal to the external pressure. As the external pressure is decreased further below the value p'_a, the gas escape velocity from the nozzle increases until, finally, it reaches a supersonic value.

The continuous flow mode characterized by a supersonic escape velocity and by equality of the external pressure p' to the pressure p_2 in the outlet section of the nozzle is called theoretical. If the external pressure p' is decreased below the theoretical pressure p_2, then the character of motion of the gas inside the nozzle is not disturbed, but the jet of gas leaving the nozzle becomes pulsating. Unlike all other cases, the pressure of the gas in the outlet section of the nozzle is in this case not equal to the external pressure, but exceeds it and has a constant value p_2 independently of the external pressure.

The flow conditions in the case of external pressure p' between p'_a and p_2 arecalled irregular. A distinction is made between two irregular modes. In the first of these the gas jet detaches from the walls of the nozzle at the point where the gas pressure becomes equal to the external pressure p' and leaves the nozzle without touching the walls, in the form of a cylindrical jet. In this case the gas flows as if the section at which detachment of the jet takes place is the regular exit section. In the second mode, which is observed in nozzles

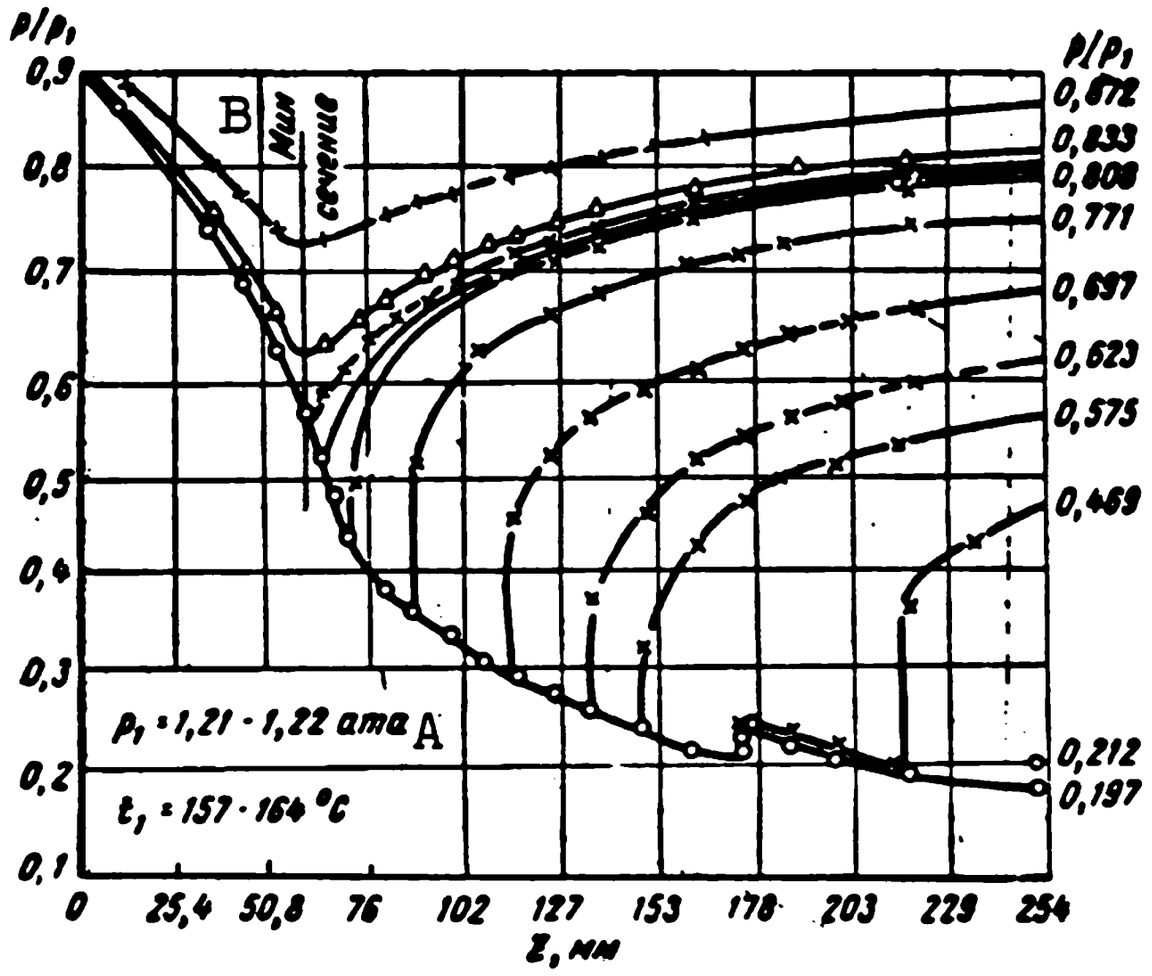

Fig. 86. Escape of steam with formation of shocks. A) atm abs; B) minimum section.

does not become detached from the walls of the nozzle, but at a certain cross section of the diverging part of the nozzle a gas shock is produced (Fig. 86), as a result of which the pressure and the density of the gas increase by a finite amount and the velocity of the gas changes abruptly from supersonic into subsonic. Beyond the shock the flow velocity along the nozzle decreases, and the pressure of the gas increases, assuming at the exit section of the nozzle a value equal to the external pressure. At external pressures slightly larger than the theoretical pressure p_2, there are formed in the diverging part of the nozzle so-called oblique shocks, as a result of which the flow velocity decreases by certain finite amounts, remaining supersonic at the same time.

Different flow conditions in a Laval nozzle for $w_1 = 0$ are shown in Figs. 86 and 87. The curve Oab (or Oa'b') corresponds to flow at large external pressures (back pressures) $p' > \beta p_1$ when the velocity of sound is not reached in the minimum section of the nozzle. The curve Ocm (or O'c'm') describes the flow of gas in the case of back pressure p'_a when the flow velocity in the throat of the nozzle

reaches the velocity of sound. The curve Ocde (or O'c'd'e') corresponds to the theoretical mode of flow, while the curve Ocdfk (or O'c'd'f'k') corresponds to irregular flow with formation of a shock d'f', as a result of which the gas pressure increases by an amount d'f' and the flow velocity decreases by df. The shock is an irreversible process accompanied by an increase in the entropy and causing therefore an additional loss of energy. Therefore, when Laval nozzles are used the tendency is always to ensure the theoretical operating mode.

An increase in entropy is a perfectly obligatory thermodynamic condition for the occurrence of the shock. It follows from this condition, in particular, that in thermally insulated flow of ideal gases no rarefaction shocks, characterized by a decrease in entropy, can occur.

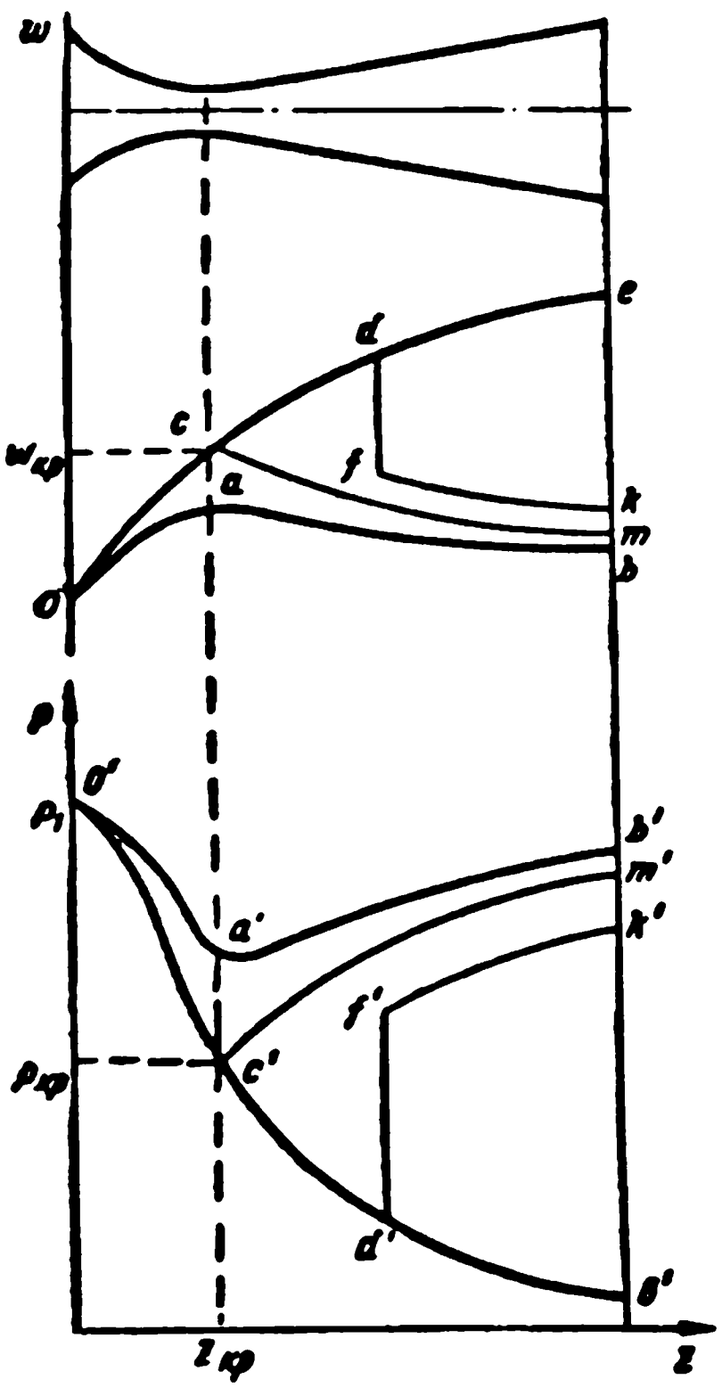

Fig. 87. Gas flow conditions in Laval nozzle.

Design of Laval Nozzle

The design of a Laval nozzle consists of determining the minimum and exit sections of the nozzle in accordance with the conditions of the theoretical flow mode, i.e., for $p' = p_2$, for a specified per-second flow G of gas or steam, for an initial pressure p_1, and a back pressure p'. We assume first that the flow is isentropic and draw on the is diagram (Fig. 88) the straight line AB from the initial point $A(p_1, t_1)$ to the final point $B(p'_2, t_2)$, which will be the graphic representation of the isentropic flow under consideration. At

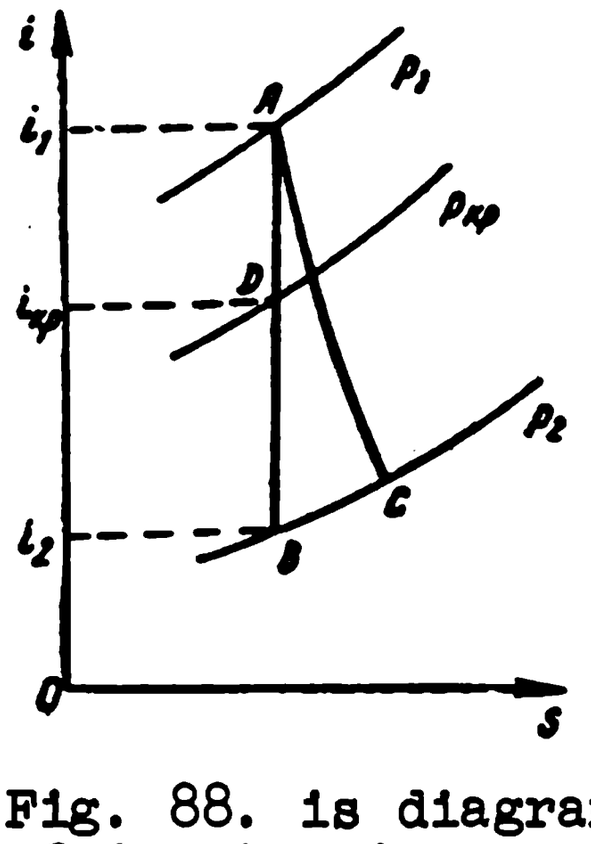

Fig. 88. is diagram of isentropic escape of gas through a Laval nozzle.

flow velocity w is equal to the velocity of sound a. This point can be determined either by comparing the velocity of sound

$$a = \sqrt{-gv^2\left(\frac{\partial p}{\partial v}\right)_s}$$

at different points of the line AB with the velocity of the gas as calculated by the equation

$$w = \sqrt{\frac{2g}{A}(i_1 - i_D) + w_1^2}.$$

or else by using the approximate relation (8.10). Then, bearing in mind that in a Laval nozzle sonic velocity is reached in the narrowest (minimal) section of the nozzle, we determine from the parameters of the point D the values of the pressure p_{kr}, the temperature t_{kr}, and the specific volume v_{kr} in the minimum section of the nozzle. From the obtained values of the velocity in the minimum section of the nozzle

$$w_{кр} = \sqrt{\frac{2g}{A}(i_1 - i_D) + w_1^2}$$

and the parameters p_{kr}, t_{kr}, v_{kr} we calculate the area of the throat Ω_{min} from the continuity equation

$$\Omega_{min} = G\frac{v_{кр}}{w_{кр}}.$$

We then determine from the value of the enthalpy at the final point B the velocity of the gas escaping from the nozzle

$$w_2 = \sqrt{\frac{2g}{A}(i_1 - i_2) + w_1^2}$$

and its specific volume v_2 and then use the equation

$$\Omega_2 = G\frac{v_2}{w_2}$$

to calculate the area of the exit section of the nozzle.

After determining Ω_{min} and Ω_2 and choosing the aperture angle of

the diverging part of the nozzle at not more than 10-12°, we obtain the length of the nozzle. The contracting part of the nozzle is chosen to have arbitrary shape, but such as to result in minimum friction losses.

In the escape of ideal gases, the design of the Laval nozzle can be carried out directly from the conditions $p_2 = p'$ and $w_{kr} = a$ for specified values of G, p_1, v_1, and t_1 with the aid of Eq. (8.10) for p_{kr} and Eqs. (8.12) and (8.15) for the per-second flow of gas.

In the calculation of the Laval nozzle with account of the resistances, one determines on the basis of the empirical values of the velocity coefficient the final point C of the process on the is diagram and the proposed process line AC is drawn. To determine the position of the point C one starts from the following conditions: 1) the pressure at the point C is equal to p_2, i.e., the point lies on the isobar $p = p_2$; 2) the enthalpy at the point C has, in accordance with Eqs. (8.1) and (8.19), the form

$$i_C = (1 - \varphi^2) i_A + \varphi^2 i_B + (1 - \varphi^2) A \frac{w_1^2}{2g}.$$

where i_B is the enthalpy at the final point of the process in the case of isentropic flow. Further, as in the case of isentropic flow, we determine the values of p_{kr}, v_{kr}, t_{kr}, w_{kr} and calculate from the continuity equation the area of the critical section of the nozzle and then the exit section.

In the case of flow with back pressure, the flow velocity, which is equal to the local velocity of sound, is attained not at the throat of the nozzle, but at some distance (usually small) from it in the start of the diverging part of the nozzle. Therefore Ω_{kr} will be somewhat larger (generally very slightly) than Ω_{min}.

Let us make a few remarks concerning the escape of saturated

steam through a nozzle. Experience has shown that the steam which is in the saturated state ahead of the nozzle becomes condensed on flowing through the nozzle with a certain delay (it first goes over into the supersaturated state). The condensation of steam, as a result of which the degree of dryness reaches equilibrium for given conditions, usually occurs (at small initial pressures) beyond the throat of the nozzle, i.e., after critical velocity is attained, and this takes place very rapidly, with formation of the so-called condensation shock. Therefore in designing Laval nozzles for steam it is necessary to take it into account that the steam in the converging part and in the start of the diverging part of the nozzle is supersaturated and at relatively small steam pressure it is necessary to use Formula (8.18) for the adiabatic exponent $\underline{k}$ of saturated water vapor in the design of the throat of the nozzle, assuming this exponent to be constant over the entire converging part of the nozzle.

Nozzles with Oblique Exit

As is well known, in converging nozzles in which the plane of the exit section is perpendicular to the axis of the nozzle, the escape velocity cannot exceed the velocity of sound. However, in addition to nozzles with straight exit, there are also converging nozzles with oblique exit, in which the plane of the exit section is not perpendicular to the axis of the nozzle (Fig. 89). The flow of gas in such nozzles has several distinguishing features which make it possible to attain supersonic escape velocities.

At large back pressures the jet of gas leaving the nozzle is directed along the axis of the nozzle, which makes an angle α with the plane of the exit. In this case the escape of the gas from the nozzle with oblique cut is similar to the escape from a converging nozzle with right exit, and in particular the gas pressure in the narrowest

section AB is equal to the outside pressure.

At small back pressures the direction of the gas flow in the exit part of the nozzle changes. The gas jet is deflected from the nozzle axis by an angle δ, which will be the larger, the smaller the back pressure. As a result of this, the cross section of the jet increases, the pressure decreases, and the escape velocity increases accordingly, reaching supersonic values. The turn that the jet leaving the nozzle makes around the point C (downward) calls for the need of equalizing the pressures in the jet and in the surrounding medium, something attained by expansion of the jet upon turning.

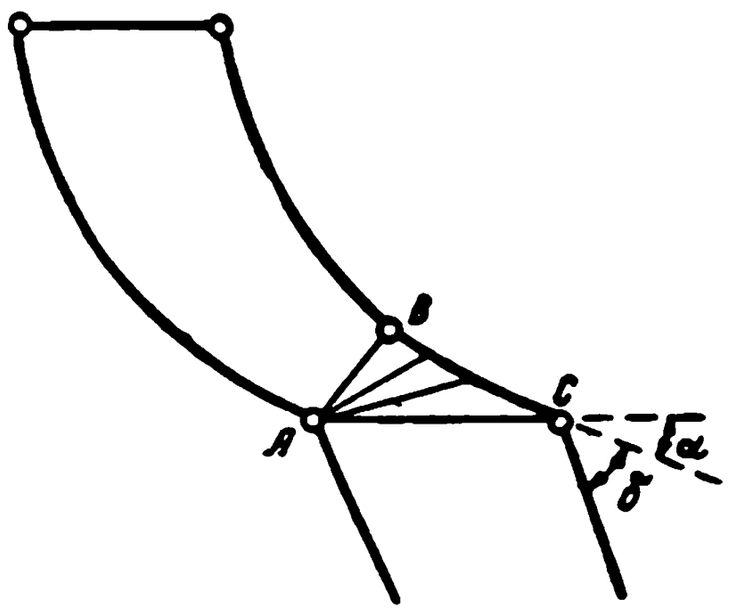

Fig. 89. Gas flow from nozzle with oblique exit.

At a certain back pressure p' smaller than $\beta p'$, the turning angle of the jet reaches a value such that the pressure in the plane of the oblique exit is everywhere equal to the external pressure; this escape mode is called theoretical. The action of nozzles with oblique exits at small back pressures, as a result of the expansion of the nozzle in the oblique section of the nozzle (due to the turning of the jet at the exit from the nozzle), is similar to the action of Laval nozzles, thus explaining the possibility of obtaining supersonic velocities in these nozzles. In the narrowest section of the nozzle with oblique exit (section AB) the gas velocity is smaller than the local velocity of sound (at large back pressure) or is equal to it (at small back pressure); in the latter case the pressure of the gas in section AB is equal to the critical value.

In supersonic escape, the gas velocity w_{kr} in the narrowest section AB and the escape velocity w_2 in the exit section AC of the nozzle are related by the following obvious equation

$$\frac{w_{кр}\,\Omega_{мин}}{v_{кр}} = \frac{w_2\Omega_2 \sin(\alpha+\delta)}{v_2}.$$

with

$$\frac{\Omega_2}{\Omega_{мин}} = \frac{1}{\sin\alpha}.$$

In all other respects the design of nozzles with oblique exits for supersonic velocities is analogous to the design of Laval nozzles.

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[Footnote]

224 This holds true also for liquids, since the coefficient of heat conduction of all liquids is much smaller than the coefficient of kinematic viscosity, as a result of which the heat conduction can be neglected.

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[List of Transliterated Symbols]

221 тр = tr = treniye = friction
223 тор = tor = tormozheniye = stagnation
229 кр = kr = kriticheskiy = critical
235 макс = maks = maksimal'nyy = maximum
239 теор = teor = teoreticheskiy = theoretical
246 мин = min = minimal'nyy = minimum

Chapter 9

CYCLES OF GAS TURBINE INSTALLATIONS AND JET ENGINES

§1. OPERATING PRINCIPLE OF GAS TURBINE INSTALLATION

In the gas turbine installation, a diagram of which is shown in Fig. 90, the useful work is performed at the expense of the kinetic energy of the working medium (air and combustion products of the fuel).

Compressor 1 draws in atmospheric air, compresses it, and directs it to the combustion chamber 4, where liquid fuel is injected through atomizer 3; the fuel is fed by fuel pump 2. Part of the air, equal to the amount necessary for combustion (with slight excess) is fed directly to the atomizer and the remaining part is mixed with the combustion products to cool them and to reduce the temperature of the turbine blades to 650-800°C. The gaseous combustion products are expended in nozzles 5 and are fed to the blades 6 of the gas turbine, after which they are ejected to the atmosphere through the exhaust pipe 7 (the exhaust pressure is in this case equal to atmospheric pressure). On going through nozzles 5, and also through the gaps between the blades of the turbine wheels, the gas stream is accelerated to high velocities. On the turbine blades the kinetic energy of the gas is converted into useful work, which, after subtracting the work needed to drive the compressor, is then taken by the load 8.

The useful external work of the gas turbine installation is equal to the difference in the work of expansion l'_{rassh} of the working medium in the turbine and the work of compression $l'_{ compressor}$ in the compressor:

$$l' = l'_{\text{расш}} - l'_{\text{сж}}.$$

The ratio of the real work performed by the turbine to the theoretical work is the relative internal efficiency of the turbine η_t. The average value of η_t for gas turbines is 0.84, while for steam turbines it is 0.82. The actual work of expansion $l'_{\text{расш}}$ of the working medium in the turbine is equal to $l'^{\text{теор}}_{\text{расш}}\eta_t$.

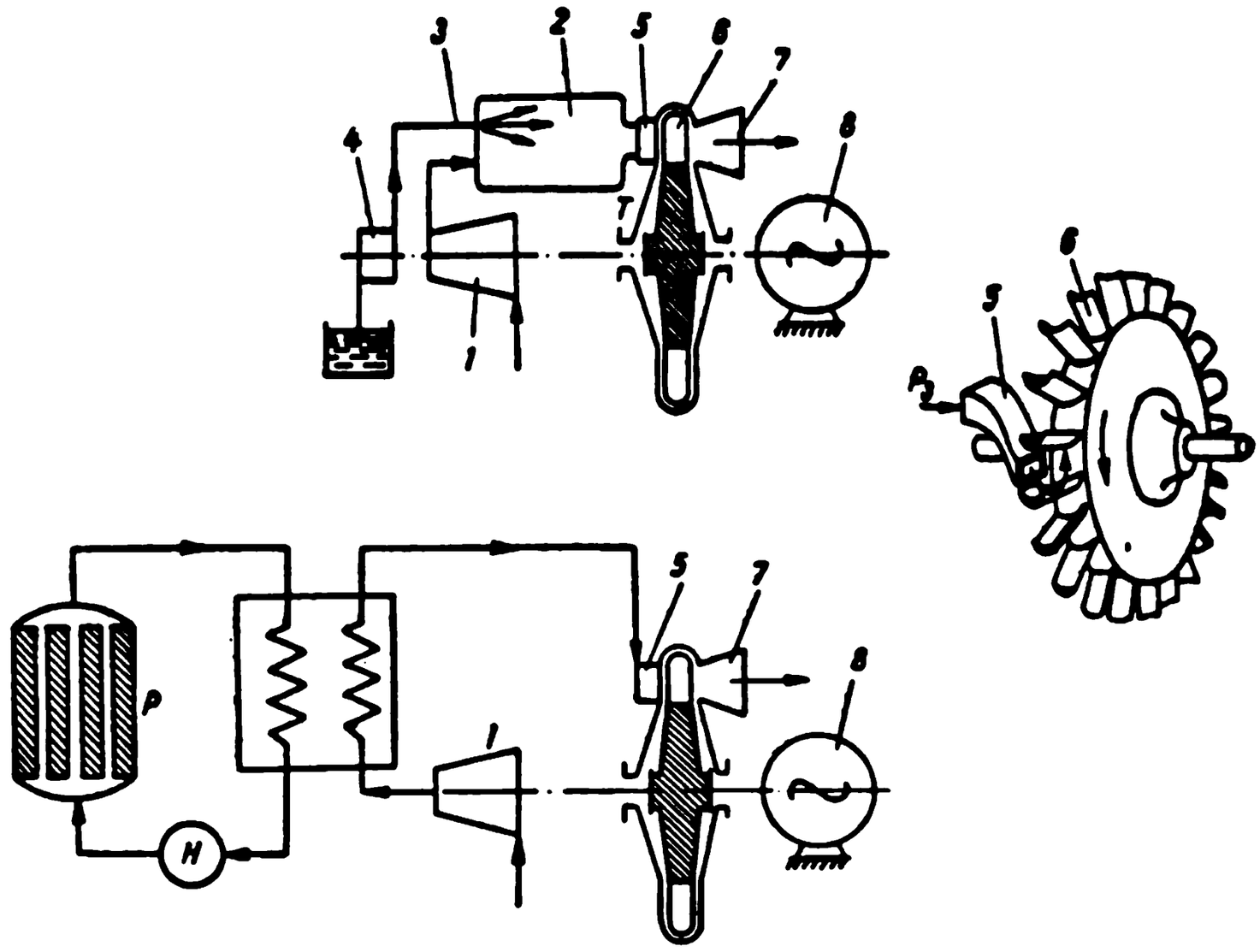

Fig. 90. Diagram of simplest gas turbine installation with heat supplied to the working medium in a combustion chamber and in an atomic reactor.

The actual work of compression $l'_{\text{сж}}$ consists of the work of compressing the air in the compressor, equal to $l'^{\text{теор}}_{\text{сж}}/\eta_k$, and the work of compressing the working medium in the combustion chamber, equal to $\int_{p_2}^{p_3} v\,dp$, where p_2 and p_3 are the pressures at the start and end of the combustion process, that is, past the compressor and ahead of the turbine.

Therefore the real useful external work of the gas turbine installation will be equal to

$$l' = l'^{\text{теор}}_{\text{расш}}\eta_т - \frac{l'^{\text{теор}}_{\text{сж}}}{\eta_к} - \int_{p_2}^{p_3} v\,dp. \qquad (9.1)$$

The works of expansion $\underline{l}'_{rassh}$ and of compression $\underline{l}'_{szh}$, owing to the adiabatic character of the process in the turbine and in the compressor, are equal to the gains in enthalpy:

$$l'_{расш} = -\Delta i_т,$$
$$-l'_{сж} = \Delta i_к.$$

The greater the heat content of the working medium ahead of the turbine (the higher its temperature), the more the useful work of the installation.

The air and the gaseous combustion products have relatively low enthalpy. Therefore to obtain high power there should flow through the installation a large amount of working medium, which entails a considerable work of compression and the loss of an appreciable fraction of the installation's own power to drive the compressor. The relatively small value of useful work per kilogram of working medium and the considerable power consumed for the installation's own needs comprise the main shortcoming of gas turbine installations. The advantage of these installations lies in the high efficiency and large power at small dimensions and weight of the installation.

The combustion of the fuel in the gas-turbine chamber can occur either at constant pressure or at constant volume. Consequently gas turbine installations are subdivided into installations with the heat supplied at constant pressure and those with heat supplied at constant volume. The cycles of these installations differ only in the character of the process of fuel combustion. The remaining processes in the cycle, namely the compression of the air in the compressor (adiabatic or isothermal), the adiabatic expansion of the combustion products, the exhaust of the spent gases (which is regarded as an isobaric process), are all the same.

In addition to a cycle in which the spent gases are exhausted

through the atmosphere, a cycle is possible in which the working medium is not discarded after the adiabatic expansion, but is directed for cooling in a heat exchanger (cooler) and then again to the compressor. Gas turbine installations operating in such a closed cycle are called closed. Their advantage compared with open installations, which use atmospheric air as the working medium, lies in the fact that they make it possible to increase the initial pressure of the working medium and consequently also the amount of the latter and thereby obtain much greater power from the installation without increasing its dimensions. In addition, other conditions being equal, the use of high initial pressure leads to smaller losses of work and consequently increases the effective efficiency of the installation. In closed installations the working medium can be a gas with better thermodynamic properties than air and thus the thermal efficiency of the cycle can be increased. Then the combustion of the fuel need not of necessity take place in the stream of the working medium. The heat can be supplied to the latter by heat exchange through solid walls, which separate the combustion products from the working medium. Because of this, it is possible to use in closed installation low-grade fuel, which is not suitable for open installations.

Gas turbine installations in which the source of heat is an atomic reactor are among the installations in which heat is supplied at constant pressure, inasmuch as the heating of the working medium occurs in them under conditions of constant pressure either directly by having the medium flow past the fuel elements of the reactor or by heat exchange with a primary coolant circulating through the reactor. Consequently all the formulas given in §3 for gas turbine installations with heat supplied at constant pressure hold true also for nuclear gas turbine installations.

§2. CYCLES OF GAS TURBINE INSTALLATIONS WITH HEAT SUPPLY (COMBUSTION OF FUEL) AT CONSTANT PRESSURE

Cycle With Adiabatic Compression

The theoretical cycle of a gas turbine installation with heat supplied at constant pressure and with adiabatic compression of the air is shown in Fig. 91. The amount of heat supplied to 1 kg of working medium in the process 2-3 and removed from it in the process 4-1 is

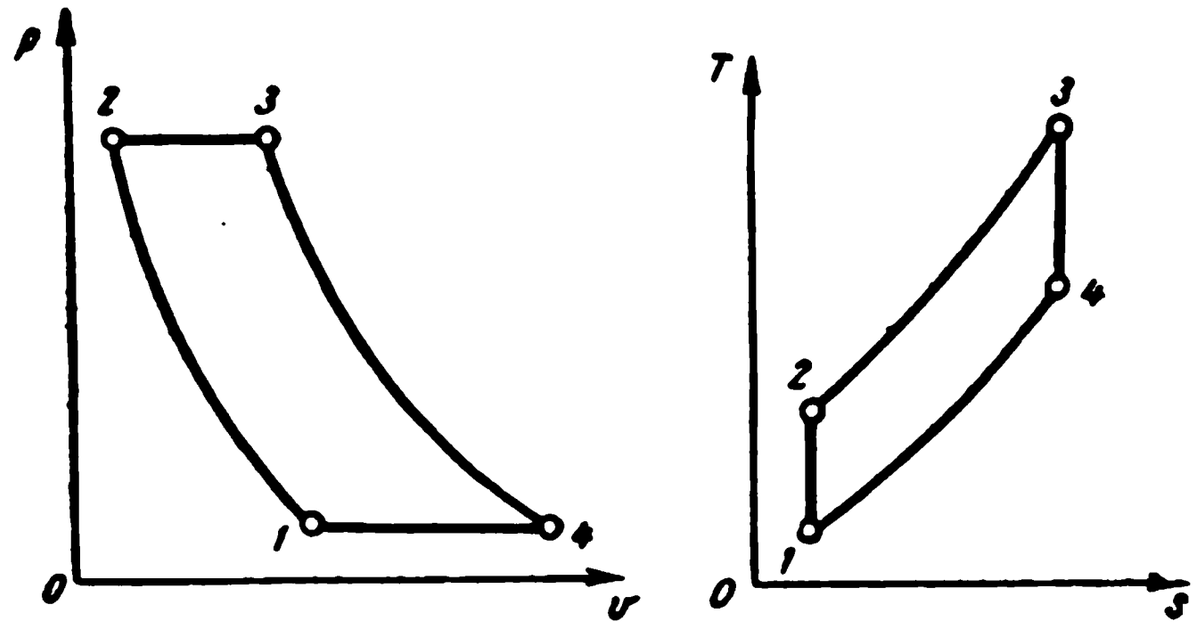

Fig. 91. Cycle of gas turbine installation with heat supplied at constant pressure and with adiabatic compression of the air.

$$q_1 = i_3 - i_2,$$
$$q_2 = i_4 - i_1.$$

The useful work l' is, in accordance with Equation (9.1),

$$l' = (i_3 - i_4) - (i_2 - i_1).$$

If we assume the working medium to be an ideal gas with constant specific heat, then

$$q_1 = c_p (T_3 - T_2),$$
$$q_2 = c_p (T_4 - T_1)$$

The thermal efficiency of the cycle is in this case

$$\eta_t = 1 - \frac{T_1\left(\frac{T_4}{T_1} - 1\right)}{T_2\left(\frac{T_3}{T_2} - 1\right)}.$$

But $T_3/T_4 = T_2/T_1$, since lines 2-3 and 1-4 are isobars with identical c_p. Therefore

$$\eta_t = 1 - \frac{T_1}{T_2}. \qquad (9.2)$$

Owing to the adiabatic character of the air-compression process in the compressor we have

$$T_2 = T_1 \beta^{\frac{\kappa-1}{\kappa}},$$

where $\beta = p_2/p_1$ is the pressure ratio in the compressor, so that

$$\eta_t = 1 - \frac{1}{\beta^{\frac{\kappa-1}{\kappa}}}.$$

The thermal efficiency of a gas-turbine cycle with heat supplied at a constant pressure depends only on the degree of rise in pressure in the compressor, increases with the latter, and does not depend on the temperature interval in which the cycle is realized.

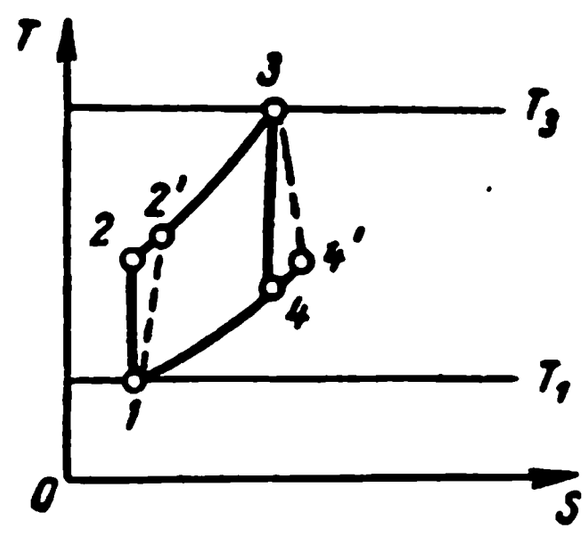

Fig. 92. Actual gas-turbine cycle with heat supplied at constant pressure.

The effective efficiency of the gas turbine installation (Fig. 92) has the form

$$\eta_e = \frac{(i_3 - i_4)\,\eta_T - \frac{i_2 - i_1}{\eta_K}}{i_3 - i_{2'}},$$

with

$$\eta_K = \frac{i_1 - i_2}{i_1 - i_{2'}}, \quad \eta_T = \frac{i_3 - i_4}{i_3 - i_{4'}}.$$

Assuming, as before, that the working medium is an ideal gas, we transform the expression for η_e into

$$\eta_e = \eta_t \frac{\eta_T - \frac{1}{(1-\eta_t)\,\eta_K} \cdot \frac{T_1}{T_3}}{1 - \left(1 + \frac{\eta_t}{(1-\eta_t)\,\eta_K}\right) \cdot \frac{T_1}{T_3}}. \qquad (9.3)$$

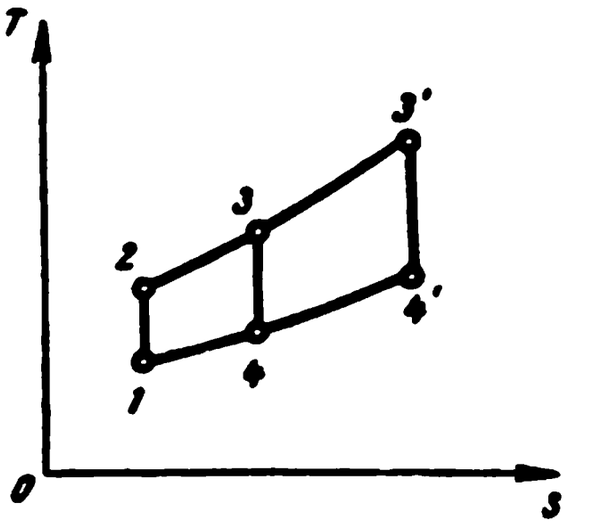

Fig. 93. Ts diagram of cycles with identical ratio T_2/T_1.

It is seen from (9.3) that with increasing maximum temperature T_3 of the cycle (for constant T_1 and T_2) the value of η_e increases. This increase will be even more noticeable if one takes into account the fact that η_t increases with increasing temperature (and pressure) of the working

medium.

At relatively low values $T_3 \approx T_1/[(1 - \eta_t)\eta_k\eta_t] = T_2/\eta_k\eta_t$, the efficiency η_e tends to zero. The smaller η_k and η_t, the higher the temperature T_3 at which the gas turbine installation becomes effective. Calculation has shown that at the presently attainable values of η_k and η_t the temperature of the working medium ahead of the turbine should be not less than 650°C.

If we represent on the Ts diagram (Fig. 93) the aggregate of cycles having identical values of T_2/T_1, and consequently also identical β and η_t, then, in accordance with Equation (9.3), for constant compressor and turbine efficiencies the most effective efficiency will be possessed by the cycle for which the ratio T_3/T_1 is maximal, that is, the "broadest" cycle 1-2-3'-4'. Thus, cycles with equal thermal efficiencies have the greater effective efficiencies the larger the maximum temperature T_3 or, what is the same under the given conditions, the larger the work of expansion as compared with the work of compression.

On the other hand, it follows from (9.3) that for a specified ratio T_3/T_1 there exists that value of the degree of pressure increase β at which the effective efficiency has a maximum value.

Cycle With Isothermal Compression

If the compression of air in a compressor is carried out isothermally (Fig. 94), then the amount of heat removed in the cycle 1-2-3-4 will be larger than in adiabatic compression and for an ideal gas it will amount to

$$q_2 = c_p(T_4 - T_1) + \frac{k-1}{k} c_p T_1 \ln \beta,$$

while the amount of the heat supplied, owing to the condition $T_2 = T_1$, will be

$$q_1 = c_p (T_3 - T_1).$$

Bearing in mind that $T_3 = T_4\beta^{(k-1/k)}$ and $T_3/T_1 = v_3/v_2$, that is, to the degree of preliminary expansion ρ, we obtain

$$\eta_t = 1 - \frac{\beta^{\frac{\kappa-1}{\kappa}}\left(\frac{\kappa-1}{\kappa}\ln\beta - 1\right) + \varrho}{\beta^{\frac{\kappa-1}{\kappa}}(\varrho - 1)}. \qquad (9.4)$$

It is seen from (9.4) that the thermal efficiency of a cycle with isothermal compression goes through a maximum as β is varied, but this maximum is reached at exceedingly large and therefore practically unrealizable degrees of compression.

Comparison of Cycles With Isothermal and Adiabatic Compression

Let us consider two cycles: 1-2-3-4 and 1-2'-3-4 with adiabatic and isothermal compression and with equal degree of pressure rise in the compressor β (Fig. 95). Cycle 1-2-3-4 and the auxiliary cycle 1'-2'-3-4, in which the compression is carried out adiabatically in the same pressure interval, have the same thermal efficiency, equal to

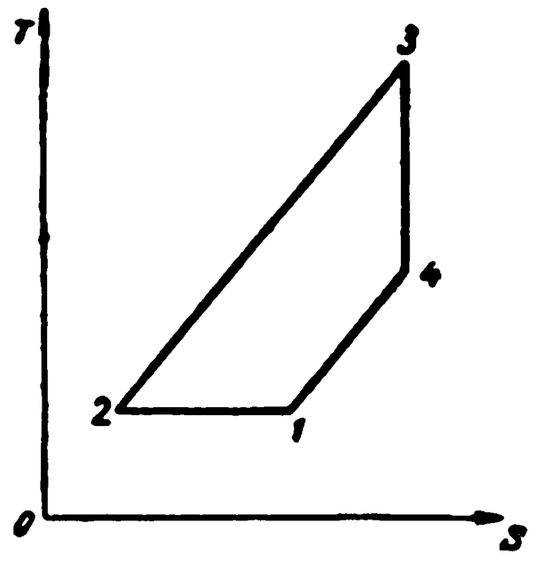

Fig. 94. Cycle of gas turbine installation with heat supplied at constant pressure and with isothermal air compression.

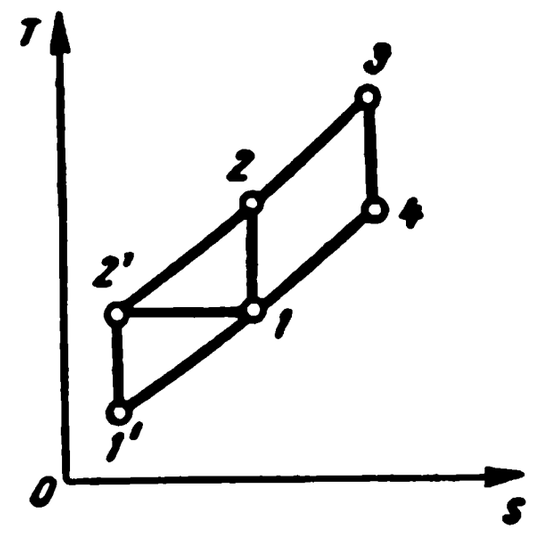

Fig. 95. Cycles with adiabatic and isothermal compression of the air and identical degree of pressure rise in the compressor.

$1 - T_1/T_2$ according to Equation (9.2). The thermal efficiency of the cycle 1-2'-3-4 is less than the thermal efficiency of the cycle 1'-2'-3-4, inasmuch as for equal values of q_1 the value of q_2 in the first cycle is larger than in the second. Accordingly the cycle 1-2'-

3-4 with isothermal compression will have a smaller thermal efficiency than cycle 1-2-3-4 with adiabatic compression.

Regenerative Cycle

If the temperature T_4 of the working medium after expansion in the turbine is larger than the temperature T_2 at the exit from the compressor, then heat can be regenerated in the cycle in sections 2-5 and 6-4 (Fig. 96). Under real conditions, owing to the limited dimensions of the regenerative heat exchangers, the temperature difference between the heated and cooled gas streams will have a certain positive value. Therefore the air heated in the regenerator will have a temperature $T_{5'}$, somewhat smaller than T_4, while the cooled gases will have a temperature $T_{6'}$, somewhat higher than T_2. The completeness of regeneration under actual conditions is estimated by means of the ratio $(T_{5'} - T_{6'})/(T_4 - T_2)$, which is called the degree of regeneration σ.

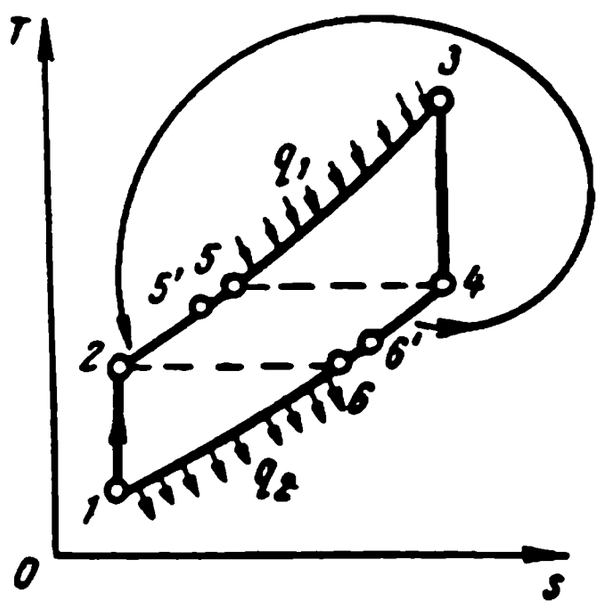

Fig. 96. Diagram showing regeneration of heat in a gas turbine cycle.

The possible diagram of a gas turbine installation with regeneration of heat is shown in Fig. 97. Unlike the ordinary installation, the compressed air from the compressor 1 is fed to the combustion chamber 3 not directly, but after preheating in regenerator 2 by the heat from the exhaust gases. In complete regeneration (when $\sigma = 1$) the amounts of heat supplied and removed are decreased by a quantity $i_4 - i_6$, which for an ideal gas is equal to $c_p(T_4 - T_2)$. Therefore the thermal efficiency of a cycle with heat regeneration and adiabatic compression will have the form

$$\eta_t^{reg} = 1 - \frac{T_2 - T_1}{T_3 - T_4}.$$

Inasmuch as $T_2/T_1 = T_3/T_4$ we have

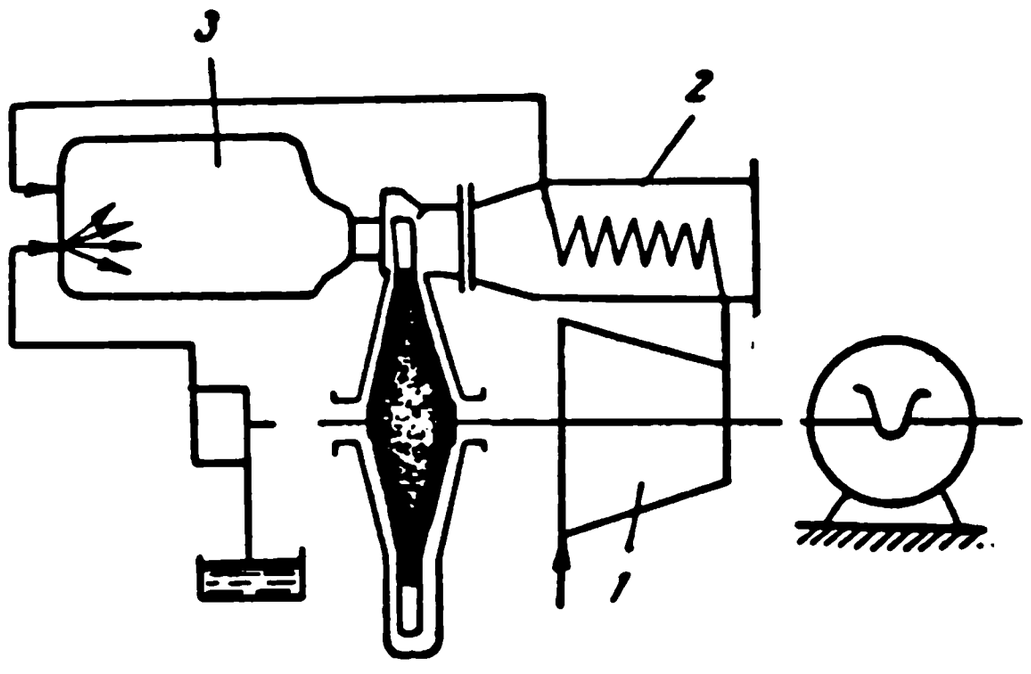

Fig. 97. Diagram of gas turbine installation with heat regeneration.

$$\eta_t^{per} = 1 - \frac{T_1}{T_4}. \qquad (9.5)$$

It is seen from Formula (9.5) that in complete regeneration of the heat the thermal efficiency of the gas turbine installation with heat supply at constant pressure and with adiabatic compression depends only on the temperature of the gas at the end of the adiabatic expansion (at the exit from the turbine).

The use of regeneration is particularly effective at small values of β, that is, at sufficiently low temperatures T_2 at the end of the adiabatic compression, when the portion of the isobar where heat regeneration is possible is relatively large and the average temperatures of heat supply and removal approach T_3 and T_1.

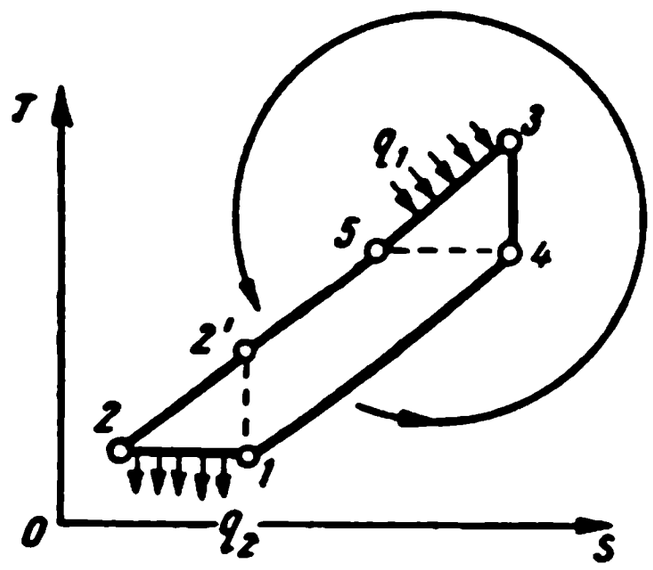

Fig. 98. Cycle with isothermal compression and regeneration.

Regeneration of heat is most conveniently done in isothermal compression of the air (Fig. 98). Indeed, whereas in the cycle with adiabatic compression 1-2'-3-4 the regeneration is possible in the section 2'-5, in the cycle with isothermal compression the regeneration of heat can be effected in the section 2-5, which is considerably larger

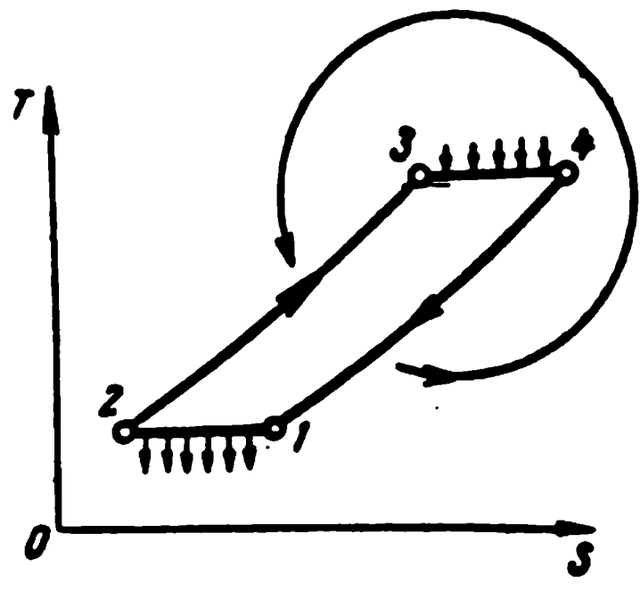

Fig. 99. Cycle with isothermal heat supply and isothermal air compression.

than the section 2'-5. The average temperature of heat supply will be the same here as in the cycle 1-2'-3-4, while the temperature of heat removal drops to T_1.

For equal β and equal maximum temperatures T_3, the regenerative cycle with isothermal compression has greater thermal efficiency than the cycle with adiabatic compression, whereas in cycles without heat regeneration it is the cycle with adiabatic compression, to the contrary, that has the higher thermal efficiency. With increasing temperature T_4 (for constant T_1 and T_3) the thermal efficiency of the regenerative cycle increases, as follows from (9.5). However, at the same time the specific useful work of the installation decreases. Indeed,

$$Al' = \eta_t q_1.$$

For an ideal gas we have

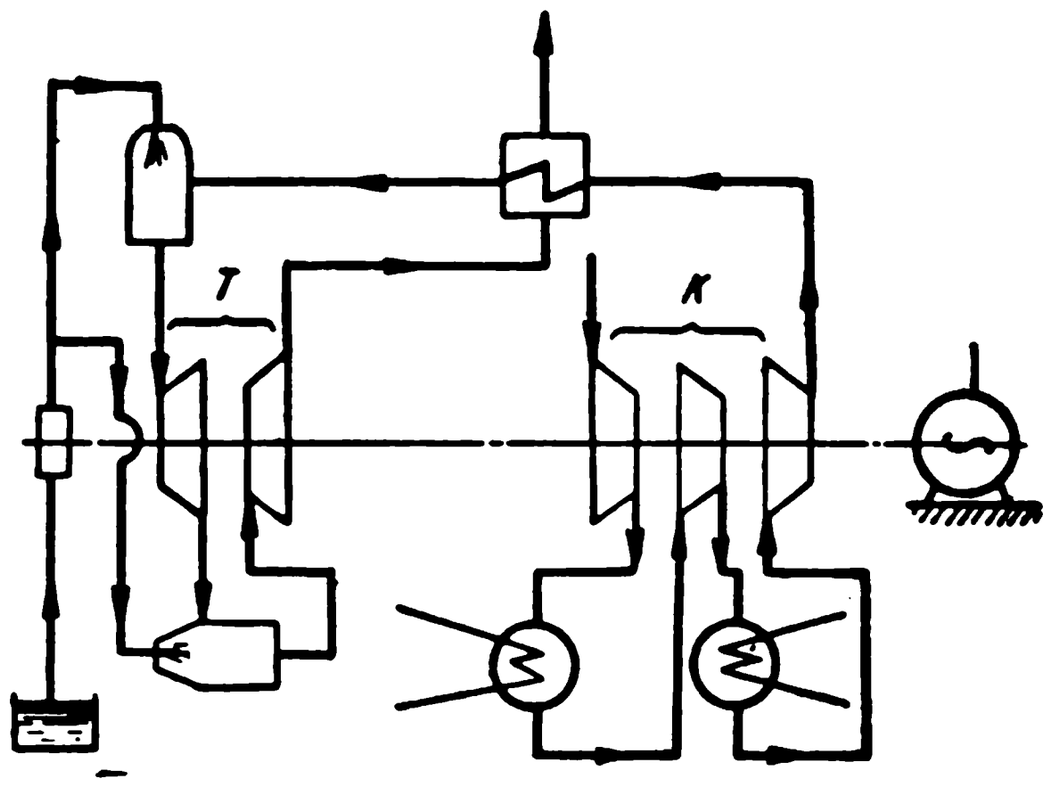

Fig. 100. Diagram of gas turbine installation with stepwise compression of the fuel and compression of the air.

$$Al' = c_p T_4 \left(1 - \frac{T_1}{T_4}\right)\left(\frac{T_3}{T_4} - 1\right).$$

and therefore

$$\frac{dl'}{dT_4} = \frac{c_p}{A}\left(\frac{T_1 T_3}{T_4^2} - 1\right).$$

Inasmuch as in a regenerative cycle $T_2 < T_4 < T_3$, we have $dl'/dT_4 < 0$, that is, l' is a decreasing function of T_4. It follows from this that a regenerative gas turbine installation having the same power should have a larger flow of working medium than a similar installation without heat regeneration. As a result, a greater fraction of the entire turbine power is consumed in compressor drive in the case of regenerative gas turbine installations.

The higher the average temperature of the heat supply in the cycle, the higher the thermal efficiency. Since the limiting temperature in the gas turbine cycle is limited by the strength of the gas-turbine blades, increase in the average heat supply temperature can be realized only by letting the heat supply process approach isothermal maximum permissible temperature. In this case for isothermal compression of the air in the compressor, it is possible to have heat regeneration over the entire isobaric section of the cycle. Such a cycle is shown in Fig. 99. It represents a generalized Carnot cycle 1-2-3-4, the efficiency of which is equal to $1 - T_1/T_3$.

In practice neither strictly isothermal compression of the air in the compressor nor strictly isothermal heat supply can be effected to a full degree. To bring the actual process of air compression close to isothermal, multistage compression is used in the compressors. To bring the actual heat supply process close to isothermal it is possible to employ in gas turbine installations stepwise combustion of the fuel. Figures 100 and 101 show the diagram and the cycle 1-2-3-4 of a gas turbine installation with stepwise ignition of the fuel and stepwise compression of the air, that is, with a multistage gas turbine T and a compressor K. After expansion in the initial stage of the

turbine, the working gas is fed to an intermediate combustion chamber, where its temperature is raised by means of supplementary ignition to the initial value. The larger the number of combustion and compression stages, the closer the gas-turbine cycle of the installation is to the generalized Carnot cycle and the higher its thermal efficiency.

A noticeable increase in the thermal efficiency is attained even using two or three stages; further increase in the number of stages leads to a relatively slow increase in the thermal efficiency and is therefore little effective; the installation itself then becomes more complicated and its cost increases sharply.

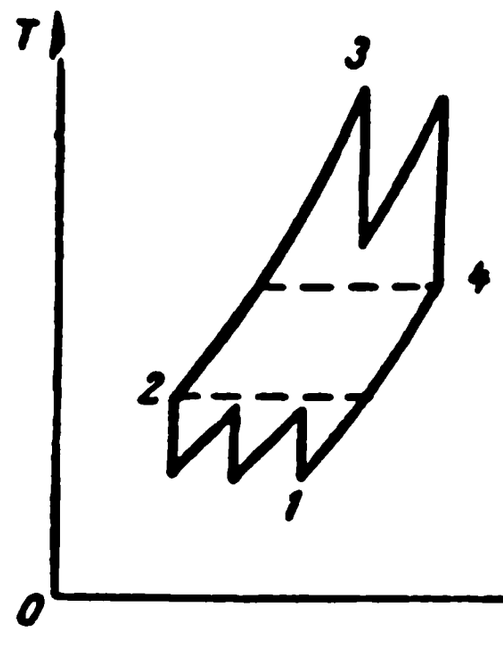

Fig. 101. Cycle of gas turbine installation with stepwise compression of fuel and stepwise compression of the air.

§3. CYCLE OF GAS TURBINE INSTALLATIONS WITH HEAT SUPPLY (FUEL COMBUSTION) AT CONSTANT VOLUME

Cycle With Adiabatic Compression

A diagram of a gas turbine installation with fuel combustion at constant volume is shown in Fig. 102. It differs from the installation with fuel combustion at constant pressure only in the construction of the combustion chamber 3. The latter has not 2 but 3 valves: fuel valve 2, air valve 1, and nozzle valve 4, used to control the feed of the combustion fuel to the turbine 5. The combustion of the fuel occurs

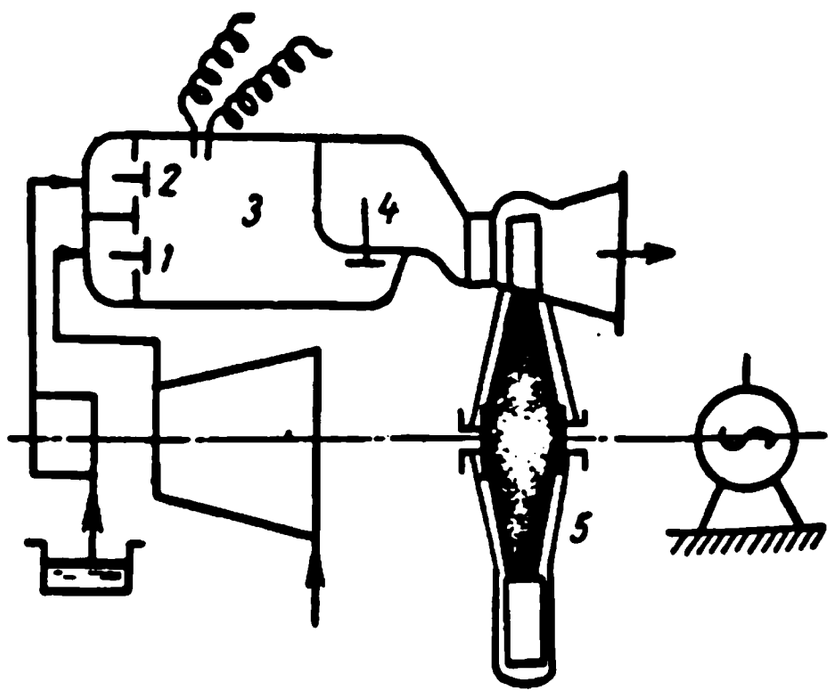

Fig. 102. Diagram of gas turbine installation with heat supplied at constant volume.

with valves 1, 2, and 4 closed, that is, under conditions V = const. When the combustion process is completed, nozzle valve 4 opens and the combustion products are fed to the turbine. The theoretical cycle of a gas turbine installation with heat supplied at V = const and with adiabatic compression is shown in Fig. 103.

The quantity of heat supplied to 1 kg of working medium in process 2-3, the heat removed from the medium in process 4-1, and the useful work will then have the following form:

$$q_1 = u_3 - u_2;$$
$$q_2 = i_4 - i_1;$$
$$Al' = (i_3 - i_4) - (i_2 - i_1) - Av_2(p_3 - p_2).$$

When the working medium is an ideal gas with constant specific

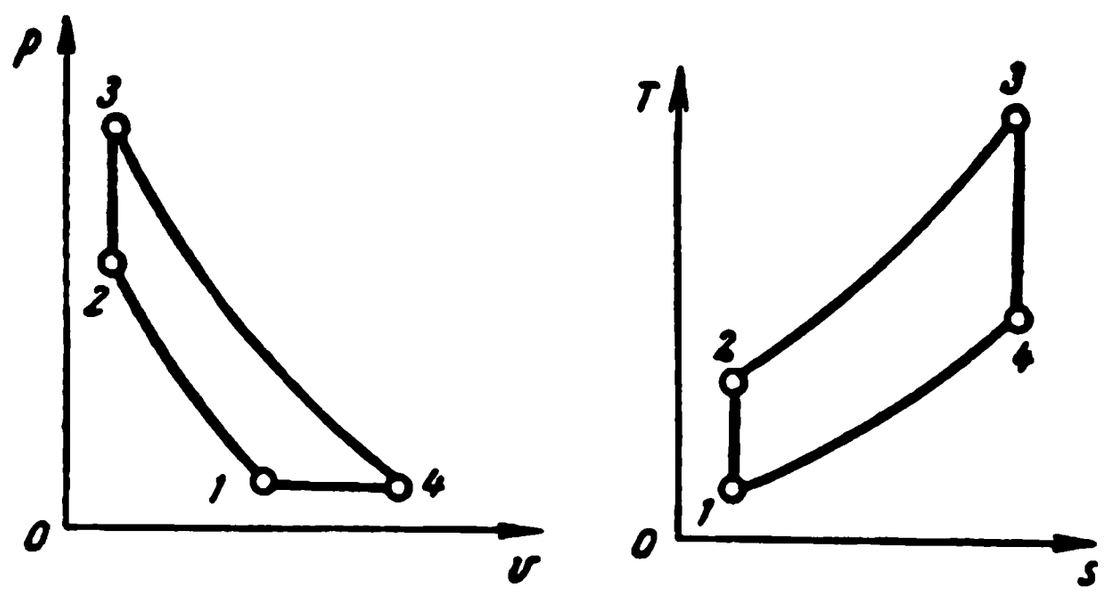

Fig. 103. Cycle of gas turbine installation with heat supplied at constant volume and adiabatic compression of the air.

heat we have

$$q_1 = c_V(T_3 - T_2).$$
$$q_2 = c_p(T_4 - T_1).$$

The thermal efficiency of the cycle is in this case

$$\eta_t = 1 - \kappa \frac{T_4 - T_1}{T_3 - T_2}.$$

Since $\frac{T_3}{T_1} = \beta^{\frac{\kappa-1}{\kappa}}$, $T_3/T_2 = p_3/p_2 = \lambda$, $T_4/T_1 = \lambda^{\frac{1}{\kappa}}$, we have

$$\eta_t = 1 - \frac{\kappa\left(\lambda^{\frac{1}{\kappa}} - 1\right)}{\beta^{\frac{\kappa-1}{\kappa}}(\lambda - 1)}. \qquad (9.6)$$

It follows from (9.6) that the thermal efficiency of a gas turbine cycle with heat supplied at constant volume increases with the increasing degree of pressure rise in the compressor and degree of additional pressure increase in the combustion chamber.

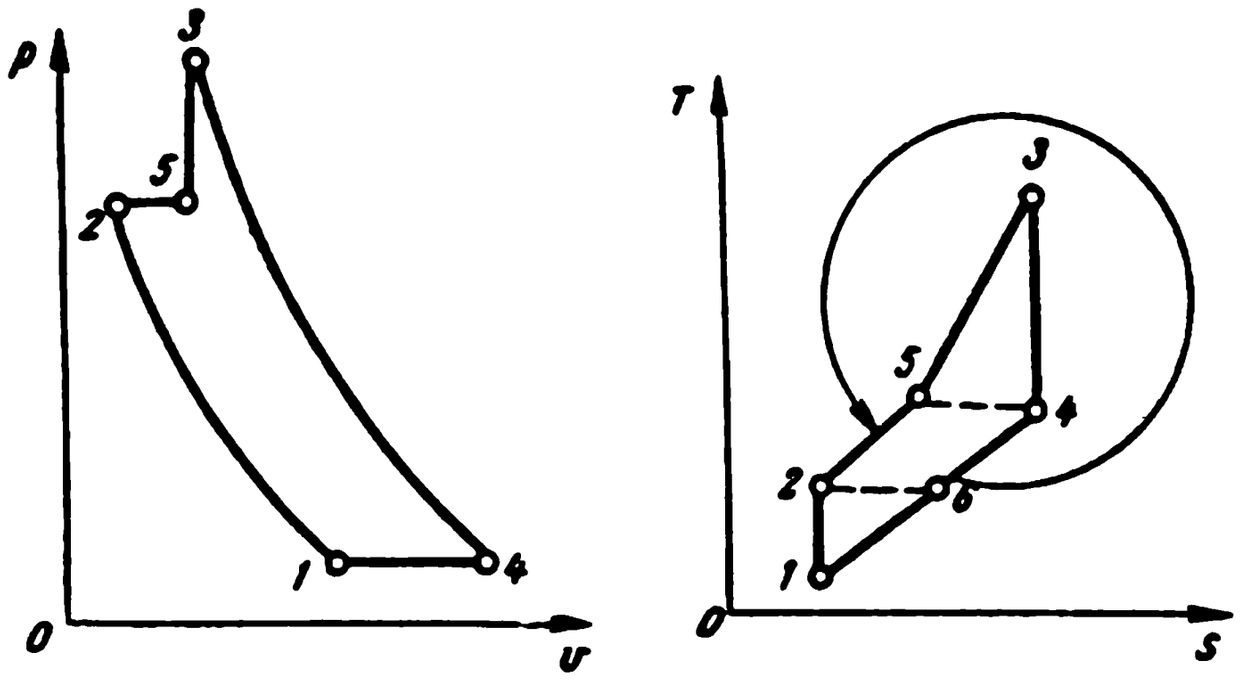

Fig. 104. Diagram with regeneration of heat in a cycle with $T_4 > T_2$.

If $T_4 > T_2$, then heat regeneration is possible in the cycle (Fig. 104). Since the regenerative heating of the air by the spent gases is carried out at constant pressure, the initial form of the cycle is somewhat modified: the initial portion of the isochore 2-3 is replaced by the isobaric section 2-5, while the isochoric fuel combustion process 5-3 begins at a higher temperature.

Cycle With Isothermal Compression

A cycle with isothermal air compression is shown in Fig. 105. The amounts of heat supplied and removed in this cycle have the following form

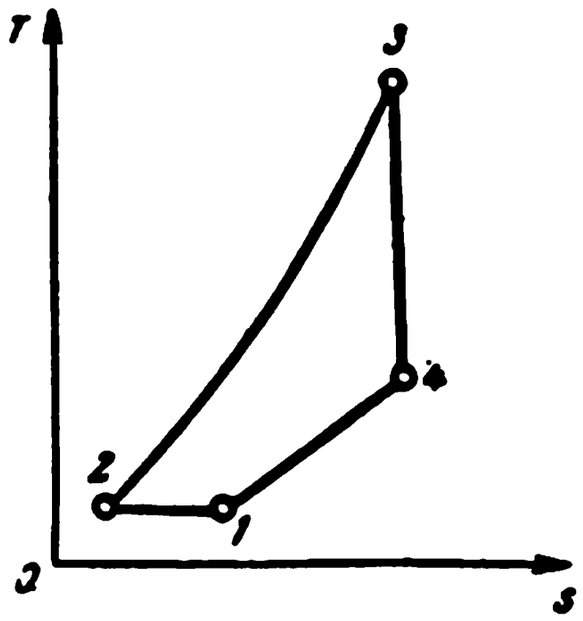

Fig. 105. Cycle with heat supplied at constant volume and isothermal compression of air.

$$q_1 = c_V(T_3 - T_2);$$
$$q_2 = c_p(T_4 - T_1) + ART_1 \ln\frac{p_3}{p_1}.$$

The thermal efficiency of the cycle is

$$\eta_t = 1 - \frac{(\varkappa-1)\ln\beta + \varkappa\left(\frac{\lambda^{\frac{1}{\varkappa}}}{\beta^{\frac{\varkappa-1}{\varkappa}}} - 1\right)}{\lambda - 1}. \quad (9.7)$$

It follows therefore that with increasing degree of additional pressure rise λ, the thermal efficiency increases; with increasing degree of pressure rise β, the thermal efficiency first increases and then decreases.

Comparison of Cycles With Adiabatic and Isothermal Compression

The thermal efficiency of a cycle with isothermal compression is smaller than the efficiency of the analogous cycle with adiabatic compression. To verify this, let us consider the cycles 1-2'-3-4, 1-2-3-4 (Fig. 106) and the auxiliary cycle 1'-2'-3-4, in which the heat supply

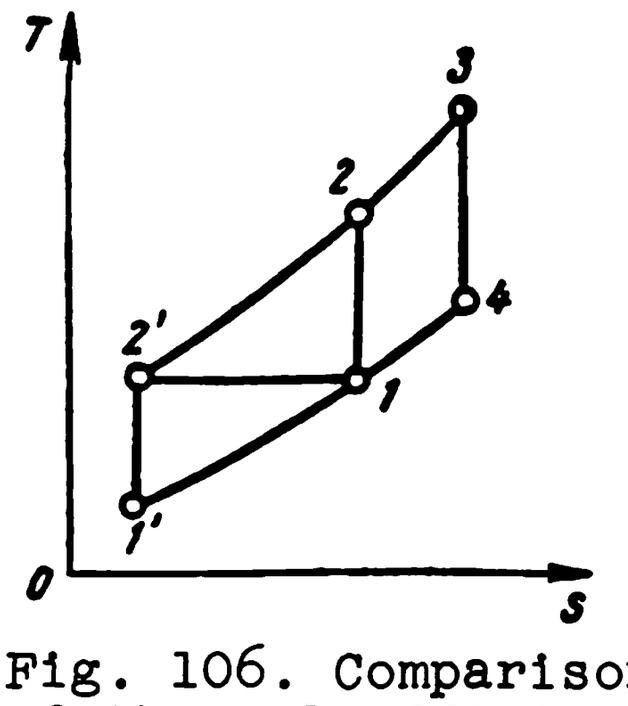

Fig. 106. Comparison of thermal efficiencies of cycles with isothermal and adiabatic air compression.

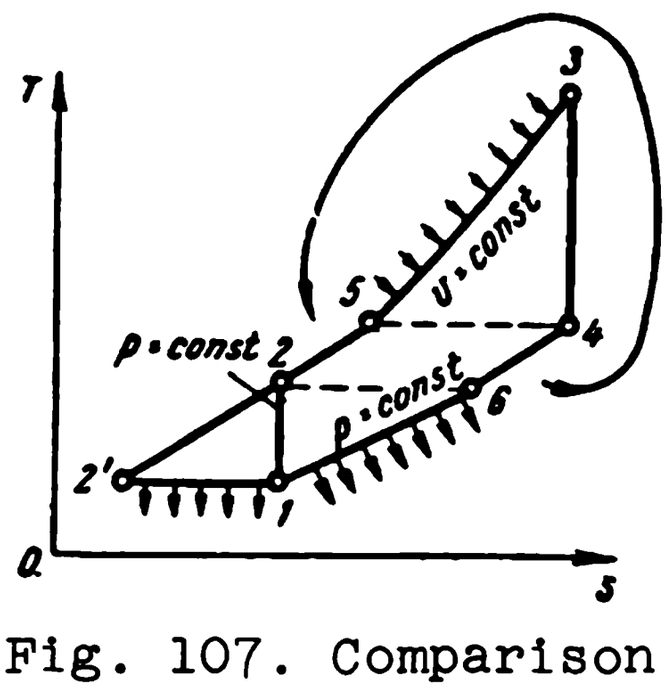

Fig. 107. Comparison of regenerative cycles for equal degrees of compression and maximum temperatures.

is along the isochore 2'-3, and the heat removal is along the isobar 1'-4. If the specific heat is constant, then

$$s_4 - s_{1'} = c_V \ln\frac{T_3}{T_{2'}} = c_p \ln\frac{T_4}{T_{1'}},$$

and the average temperature, for example on portion 1'-4, amounts to $c_p(T_4 - T_{1'})/(S_4 - S_{1'})$. Therefore the ratio of the average heat removal and heat supply temperatures in the cycle 1'-2'-3-4 which is equal to

$$\frac{T_4}{T_3} \cdot \kappa \cdot \frac{1 - e^{-\frac{S_4 - S_{1'}}{c_V}}}{1 - e^{-\frac{S_4 - S_{1'}}{c_p}}},$$

will be larger than in the cycle 1-2-3-4, while the thermal efficiency will be smaller. But the thermal efficiency of the cycle 1-2'-3-4 is smaller than the efficiency of the cycle 1'-2'-3-4, for at equal values of q_1 the value of q_2 in cycle 1-2'-3-4 is larger. Since $\eta_t^{1'-2'-3-4} < \eta_t^{1-2-3-4}$, certainly $\eta_t^{1-2'-3-4} < \eta_t^{1-2-3-4}$.

Let us compare now the regenerative cycles for identical degrees of compression and maximum temperatures (Fig. 107). It is obvious that in both cycles the amount of heat supplied q_1 will be the same, since the heat is supplied on one and the same section 5-3; for the same reason, the average heat supply temperatures will be equal.

In the cycle 1-2'-3-4 with isothermal compression, the heat is removed on the isothermal section 1-2'; consequently the average temperature of heat removal is T_1. In cycle 1-2-3-4 with adiabatic compression, the heat is removed on the portion of the isobar 6-1 and therefore q_2 and the average heat-removal temperature will be larger in this cycle. Since for the same value of q_1 the value of q_2 is smaller in the cycle with isothermal compression than in the cycle with adiabatic compression, the thermal efficiency of the former cycle will be larger. Thus, as in the cycle with heat supplied at constant pressure, the use of regeneration in isothermal compression turns out to be more effective.

§4. COMPARATIVE EFFICIENCY OF GAS TURBINE INSTALLATIONS WITH FUEL COMBUSTION AT CONSTANT PRESSURE AND AT CONSTANT VOLUME

A comparison of cycles is best carried out for identical β and T_{maks}. Figure 108 shows the following: the cycle 1-2-3-4 with heat supplied at constant pressure and the cycle 1-2-3'-4' with heat supplied at constant volume, both cycles having the same maximum temperature $T_{3'} = T_3$ and the same value of β.

From a comparison of both cycles we see that the average heat-supply temperature is the same in both cycles (provided the specific heats of the working medium are constant) and the average temperature of heat removal is larger in cycle 1-2-3-4 than in cycle 1-2-3'-4'. Therefore the cycle with heat supply at constant volume has under the chosen conditions of comparison a higher thermal efficiency. It is appropriate to note that the maximum pressure in this cycle with isochoric heat supply is higher.

If both cycles are compared at identical maximum temperatures and pressures then, as can be seen from Fig. 109, the cycle with heat supply at a constant pressure turns out to be more economical, for in this cycle $T_{sr.podv}$ and q_1 are larger, while $T_{sr.otv}$ and q_2 are the same. The same ratio between the thermal efficiencies of the two cycles holds for isothermal compression.

For a complete evaluation of the effectiveness of a gas turbine installation it is necessary to take into account in addition to the thermal efficiency of the cycle also the technical-economic characteristics of the real installation. In particular, it must be borne in mind that a gas turbine installation which works in a cycle with isochoric heat supply is characterized by a complicated distribution system and the relative internal efficiency of the turbine is smaller, owing to the losses in the valves. Because of these and a few other

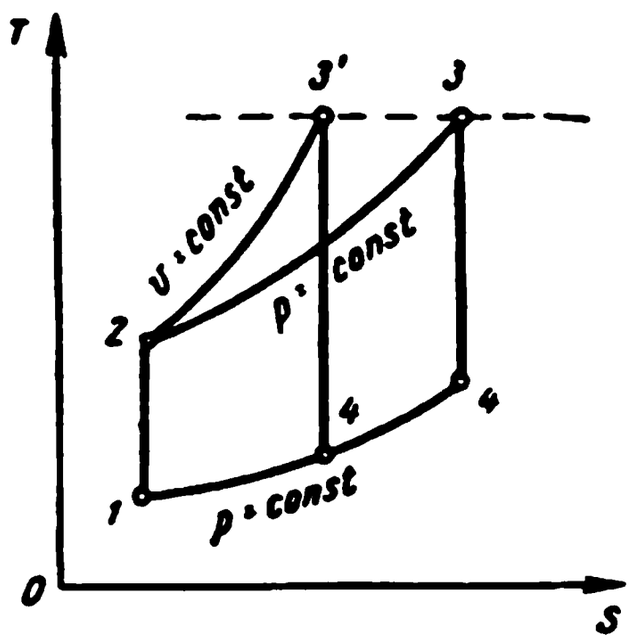

Fig. 108. Comparison of cycles with β = = idem and T_{maks} = = idem.

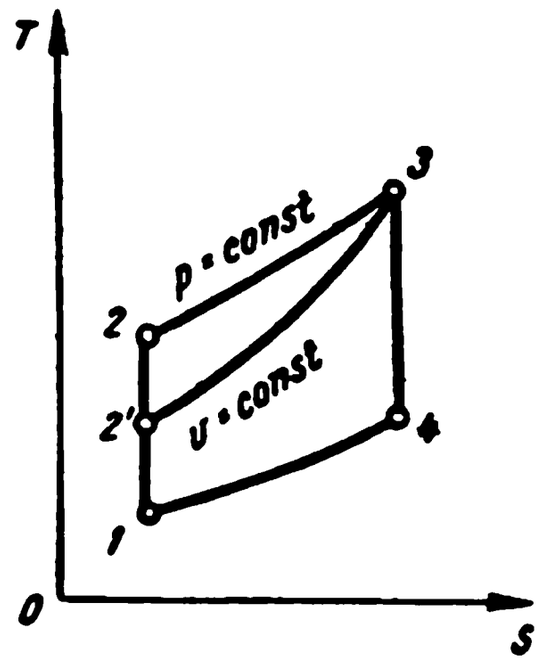

Fig. 109. Comparison of cycles with t_{maks} = = idem and p_{maks} = idem.

circumstances, the most widely used gas turbines are those operating with a cycle where the heat is supplied at constant pressure.

§5. JET ENGINE CYCLES

In a jet engine the heat obtained from the combustion of the fuel (or as a result of the fissionary action) is converted into the kinetic energy of gaseous combustion products (or air), which are ejected to the outside with high velocity. The reaction force developed thereby produces the thrust of the engine; therefore jet engines, unlike engines with propeller thrust, are called direct-reaction engines.

Unlike a piston-type propeller installation, the thrust developed by a jet engine depends little on the flying speed and has the form

$$P = \frac{G}{g}(w - W),$$

where G is the weight of the gaseous combustion products formed per unit time; w is the velocity of their escape from the engine exhaust, usually amounting to 2100-2400 m/sec; W is the speed of the flying craft.

Jet engines are capable of developing a thrust sufficient for flight at high speed, including supersonic, with weights and dimensions

smaller by many times than for installations with piston engines.

All the existing jet engines are subdivided into three principal groups:

1. Solid-fuel rockets, that is, jet engines using a powder which contains the oxygen necessary for the combustion.

2. Liquid-fuel jet engines, in which a liquid oxidizer (liquid oxygen, nitric acid, hydrogen peroxide, etc.) is used to ignite the liquid fuel.

3. Air jet engines in which atmospheric air is used to ignite the liquid fuel.

Air jet engines are in turn divided, depending on the method of compressing the air fed to the combustion chamber, into ram jet engines (with compression only due to the velocity head of the incoming air stream) and compressor types.

Jet Engines Using Liquid Oxidizers

The diagram of a liquid-fuel jet engine is shown in Fig. 110. The liquid fuel and the liquid oxidizer are fed to the combustion chamber 2 by means of pumps 1. The fuel combustion occurs at constant pressure (for this purpose it is sufficient that exhaust 3 of the engine be always open).

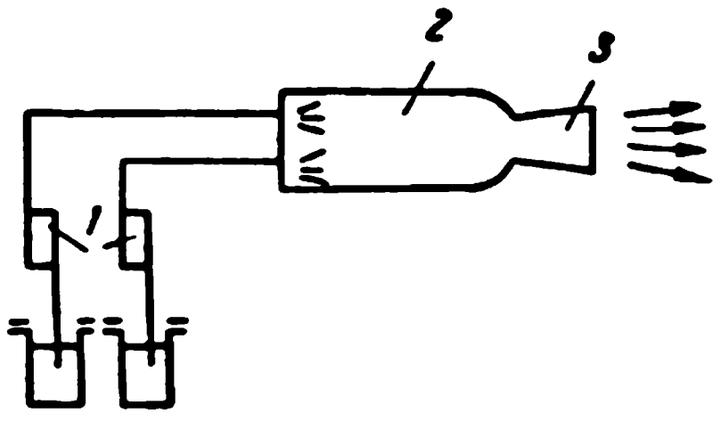

Fig. 110. Diagram of liquid-fuel jet engine.

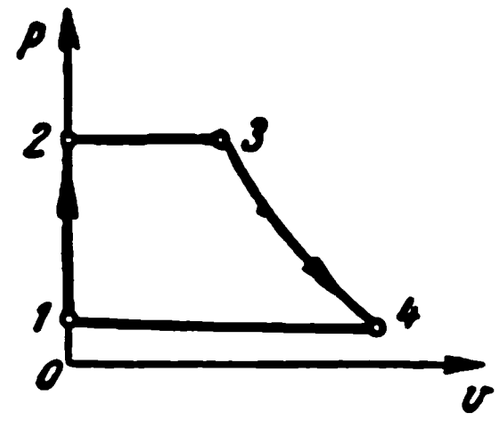

Fig. 111. Cycle of liquid-fuel jet engine in pv coordinates.

The cycle of the liquid-fuel jet engine in pv coordinates is shown in Fig. 111, where line 1-2 corresponds to adiabatic compression

(injection) of the liquid components with pumps. In view of the small compressibility and negligibly small volume of the liquid compared with the volume of the combustion products, the injection can be also regarded as an isochoric process with volume v close to zero; therefore line 1-2 coincides with the coordinate axis. The isobar 2-3 corresponds to the fuel combustion process, while line 3-4 corresponds to adiabatic expansion of the combustion products in the exhaust. The isobaric process 4-1, which closes the cycle, corresponds to the cooling of the gases expelled from the exhaust to the temperature of the surrounding medium at constant pressure, equal to atmospheric pressure.

The useful external work of the cycle (in this case, the available kinetic energy of the gas streams) has in accordance with Equation (9.1) the following form

$$Al' = (i_3 - i_4) - (i_2 - i_1),$$

with

$$i_2 - i_1 = A(p_2 - p_1)v_1.$$

The thermal efficiency of the cycle is

$$\eta_t = \frac{i_3 - i_4 - A(p_2 - p_1)v_1}{i_3 - i_2}.$$

If we neglect the work concerned in driving the liquid pumps (which is perfectly permissible, since the work of the pump is negligible owing to the small specific volume of the liquid), we obtain

$$\eta_t = \frac{i_3 - i_4}{i_3 - i_1}.$$

Advantages of the liquid-fuel jet engine are: the independence of its operation of the state of the surrounding medium, the possibility of flight in airless space, simplicity of construction, and low specific weight (weight of the installation per kilogram of thrust).

Ram Jet Engines

Ram jet engines are subdivided into direct flow engines (the fuel is consumed at constant pressure) and pulse jets (the fuel is consumed at constant volume). Both types of engines operate only with an incoming air stream, so that flying craft with such engines call for forced startup.

Direct flow ram jet (with fuel combustion at constant pressure). For preliminary compression of the air in a ram jet engine, use is made of the velocity head produced by the translational motion of the flying craft.

Depending on the velocity of the flying craft, two schemes of ram jet engines are possible: for supersonic speeds and for subsonic speeds. The ram jet scheme for supersonic flying speeds and the character of the variation of the pressure p and the velocity of the gas stream w inside the engine are shown in Fig. 112a. In section I the air enters the engine with initial supersonic velocity; the state of the air in this section is determined by the state of the atmosphere at the given altitude. As is well known (see Chapter 8) to compress a gas that moves with a velocity larger than the velocity of sound, the channel in the stream direction should first converge and then diverge. Accordingly, the channel of the ram jet engine converges in section I-II; the stream velocity in this section decreases to that of sound and the pressure increases to critical value. In the expanding part of the channel, II-III, which serves as a diffuser, further decrease in the stream velocity takes place, as does an increase in the air pressure. In section III the compression terminates. In this section the stream has minimum velocity. In portion III-IV of the channel (combustion chamber) the fuel injected into the air is ignited and heat is released. The specified pressure in the combustion

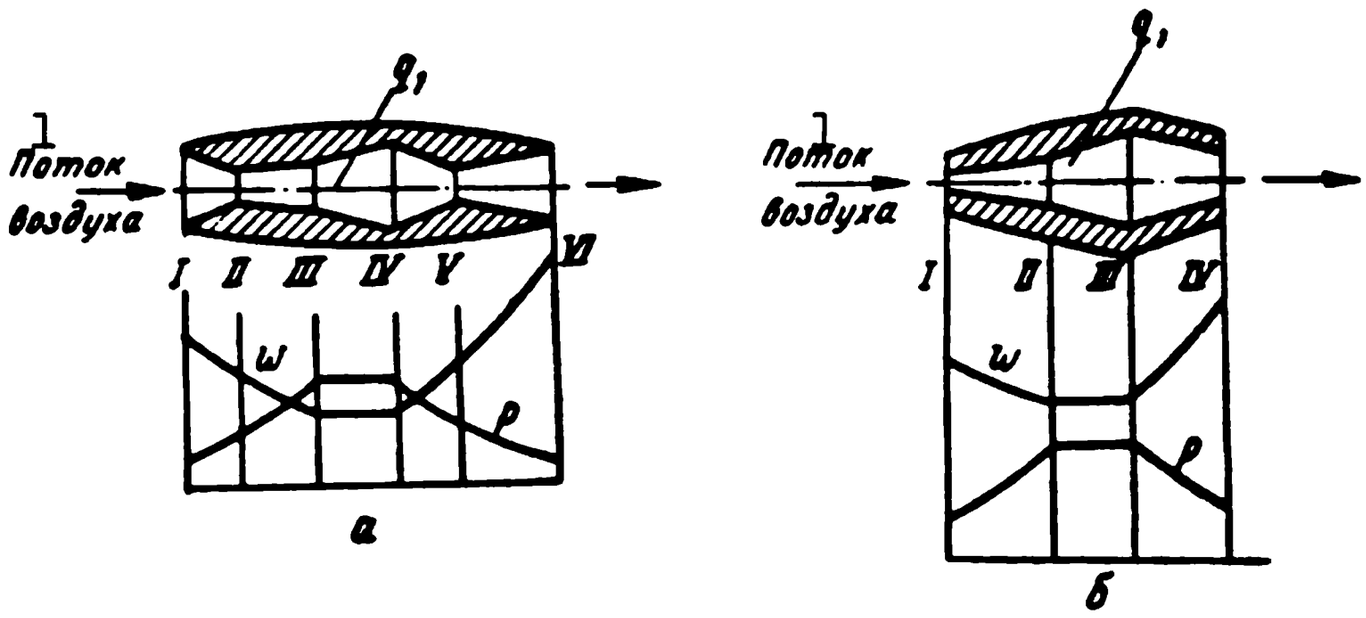

Fig. 112. Diagram of ram jet engine.
1) Air stream.

chamber is ensured by suitable dimensions of the section of the channel in portion III-IV. Beyond section IV the gas expands and the stream velocity increases. Since the stream is subsonic at the exit from the compression chamber, the channel of the engine is made further convergent (portion IV-V) and then divergent (section V-VI). In section V the stream has a velocity equal to the local velocity of sound. In the exit portion of the nozzle, V-VI, further decrease in pressure takes place from critical to that of the surrounding medium in section VI. Therefore, the velocity rises from sonic in section V to supersonic in section VI.

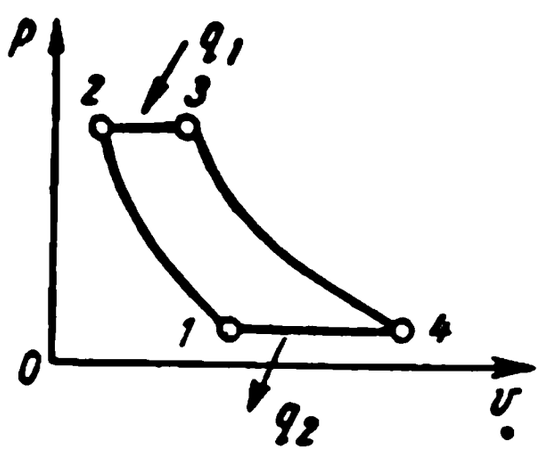

Fig. 113. Cycle of direct-flow ram jet.

A diagram of a ram jet engine for subsonic velocities and the character of gas-pressure and stream-velocity variation are shown in Fig. 112b. The need for the first (converging) portion of the channel is obviated in this case, since the gas compression begins with a velocity smaller than the velocity of sound.

The theoretical cycle of a ram jet engine is shown in Fig. 113. Line 1-2 corresponds to adiabatic compression of the incoming air stream in the diffuser, 2-3 corresponds to an isobaric combustion of

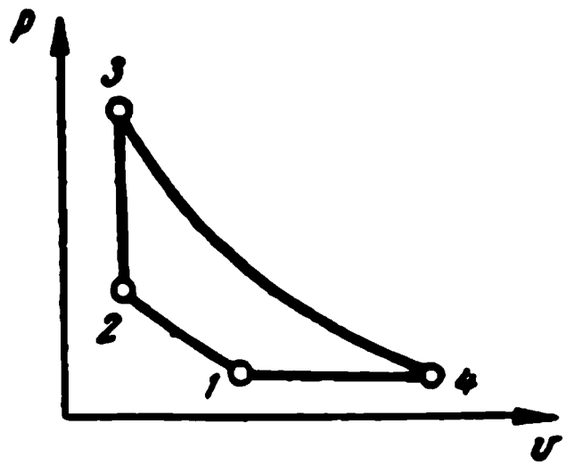

Fig. 114. Cycle of pulse jet engine.

the fuel; 3-4 corresponds to adiabatic expansion of the combustion products in the exhaust nozzle and line 4-1 corresponds to cooling of the combustion products which have been expelled to the outside.

As can be seen from Fig. 113, a ram jet engine with combustion at constant pressure operates in accordance with the same cycle as a gas turbine installation with isobaric fuel combustion. Therefore, the thermal efficiency of the air-regenerative engine with combustion of fuel at constant pressure will have in accordance with (9.2) the form

$$\eta_t = 1 - \frac{1}{\beta^{\frac{\kappa-1}{\kappa}}},$$

where β is the degree of pressure rise of the air in the diffuser.

The degree of pressure rise β, and consequently also the economy of the engine, increases with increasing flying speed. At speeds of 600-800 km/hour the thermal efficiency of a direct-flow engine does not exceed 2-4%. At supersonic speeds the efficiency and economy of the engine increase appreciably. At large flying speeds, exceeding the velocity of sound by a factor of 2 or more, the most economic is the direct-flow ram jet engine.

The advantage of the ram jet engine lies in the simplicity of construction and small weight.

Pulse jet engine (with fuel combustion at constant volume). To realize the combustion of the fuel at constant volume it is necessary to have a system of distribution valves, with which it is possible to disconnect the combustion chamber from the diffuser and from the exhaust nozzle. In all other respects the operation of a ram jet engine with fuel combustion at constant volume is analogous to that with

fuel combustion at constant pressure.

Figure 114 shows the operating cycle of a ram jet engine with fuel combustion at constant volume. Line 1-2 corresponds to the compression of air in the diffuser as the flying craft moves. In state 2, when the combustion chamber is disconnected from the diffuser (by closing the valve), the fuel is ignited (with a spark plug). The line 2-3 corresponds to isochoric combustion of the fuel. When the fuel combustion terminates, the valve separating the combustion chamber from the exhaust nozzle is opened and the combustion products begin to expand adiabatically in the nozzle (line 3-4). Line 4-2 corresponds to isobaric cooling of the combustion products ejected from the nozzle to the outside.

The cycle of a pulse ram jet engine with combustion of fuel at constant volume does not differ from the cycle of a gas turbine with isochoric fuel combustion, so that its thermal efficiency has the form

$$\eta_t = 1 - \frac{\varkappa(\lambda^{1/\varkappa} - 1)}{\beta^{\frac{\varkappa-1}{\varkappa}}(\lambda - 1)},$$

where β is the degree of increase in air pressure in the diffuser and λ is the degree of additional pressure rise in the combustion chamber. The pulse jet engine has a higher thermal efficiency compared with the direct-flow engine, so that it can be used at lower flying speeds.

Direct Flow Compressor-Type Jet Engines

The most widely used type of compressor jet engine is the turbojet engine, widely used at present in high-speed aviation.

Its diagram is shown in Fig. 115. In this engine the compression of the air is both by velocity head and with the aid of an axial compressor driven by a gas turbine with which it has a common shaft.

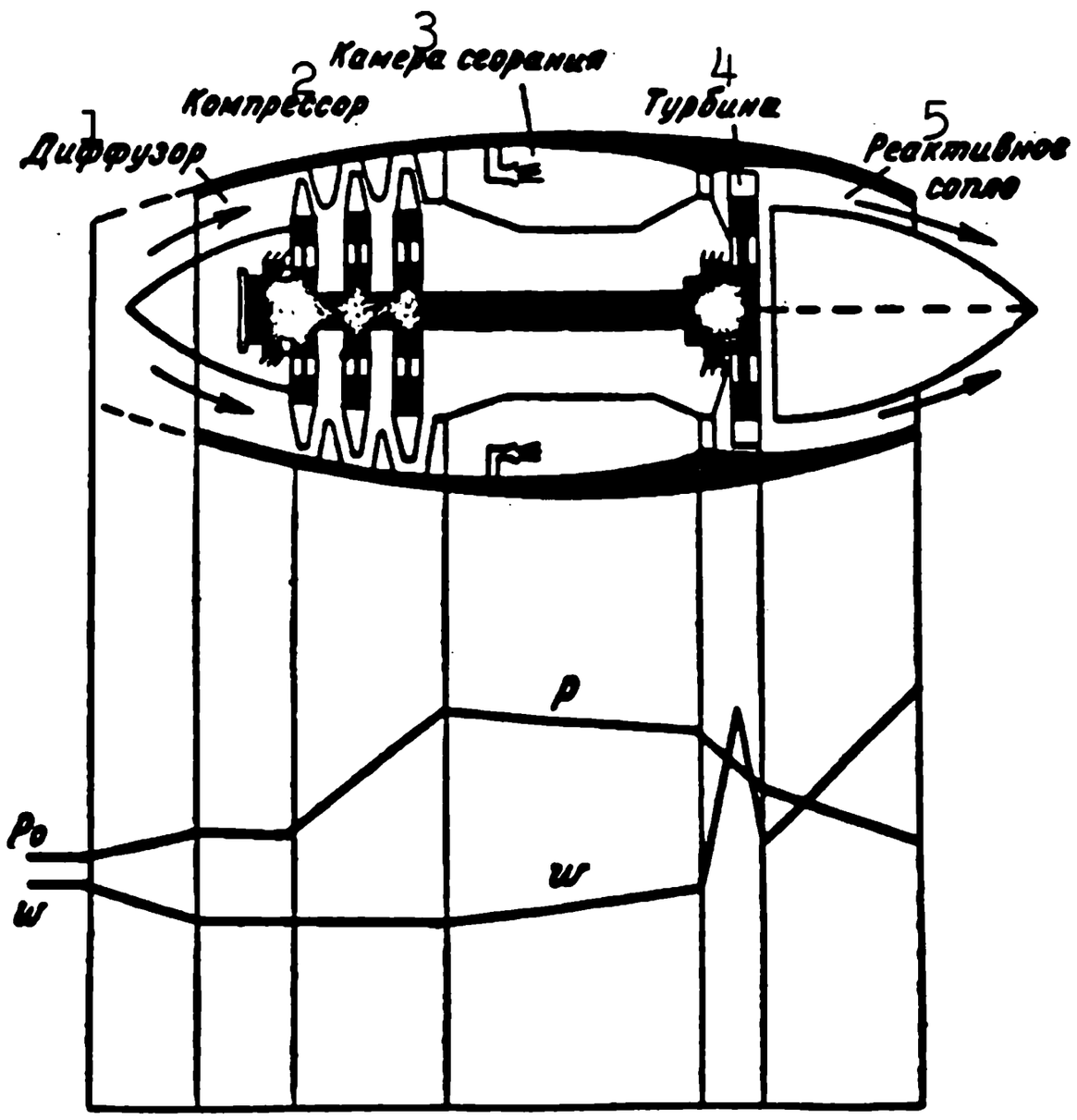

Fig. 115. Diagram of turbojet aviation engine. 1) Diffuser; 2) compressor; 3) combustion chamber; 4) turbine; 5) exhaust nozzle.

The turbine operates with the hot gases formed in the combustion chamber. The gas stream past the turbine is guided into the nozzle of the engine.

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[List of Transliterated Symbols]

251 расш = rassh = rasshireniye = expansion

251 сж = szh = szhative = compression

252 т = t = turbina = turbine

252 теор = teor = teoreticheskiy = theoretical

252 к = k = kompressor = compressor

259 рег = reg = regeneratsiya = regeneration

268 макс = maks = maksimal'nyy = maximum

268 ср.подв = sr.podv = sredniy podvod = average supply

268 ср.отв = sr.otv = sredniy otvod = average removal

Chapter 10

CYCLES OF STEAM POWER INSTALLATIONS

The working medium in a steam power installation (Fig. 116) is usually water which is converted in steam boiler 1 into saturated steam and in the steam superheater into superheated steam. From the superheater the steam enters turbine 2, where it expands and performs useful work. The spent steam is condensed in condenser 3 and the condensate is again delivered to the boiler by means of feed water pump 4.

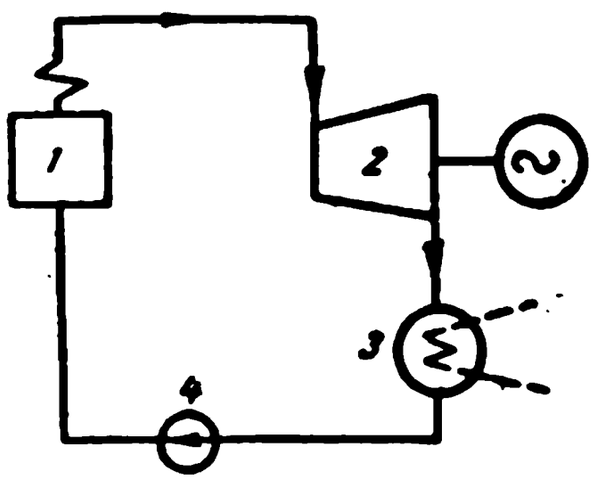

Fig. 116. Diagram of steam power installation.

The heat necessary to convert the working medium from the liquid phase into steam is obtained either by combustion of fuel or as a result of the fission of nuclei or fissionable elements. In the former case the heat supply comes from the gaseous combustion products, in the latter from the fuel elements of the reactor or from the primary coolant which circulates past the fuel elements.

§1. CYCLE OF STEAM POWER INSTALLATION

The presence of phase transformations of the working medium is a characteristic feature of the cycle of steam-power installations. A theoretical cycle of a steam-power installation is shown in Fig. 117. Here line 2'-3 corresponds to adiabatic compression of water in the pump; 3-4 corresponds to isobaric heating of water in the steam boiler to the boiling temperature; 4-5 corresponds to evaporation of the water in the boiler until dry saturated steam is obtained; 5-1 cor-

responds to isobaric superheating of the steam in the superheater, while line 1-2 corresponds to adiabatic expansion of the steam in the turbine (or in the steam engine) to the lowest temperature and pressure of the cycle; the process 2'-2, which closes the cycle, corresponds to total condensation of the steam, carried out in a special condenser. The smaller the degree of dryness of the wet steam at the exit from the condenser (the more complete the condensation), the smaller the work of compression in portions 2'-3 and the larger the useful work of the cycle. This can be readily verified by recognizing that the work of compression is given by area a 32" b (Fig. 118). Since the volume of the water is very small, the work consumed in compression of the water will be many times (by a factor of 100 or more) smaller than the work of compression of the wet steam. For this reason one uses in all steam-power installations a cycle with complete condensation of the steam. In view of the small compressibility of the water, the adiabatic process 2'-3 of the compression of the water in the pump at slight pressures at the point 3 can also be regarded as isochoric.

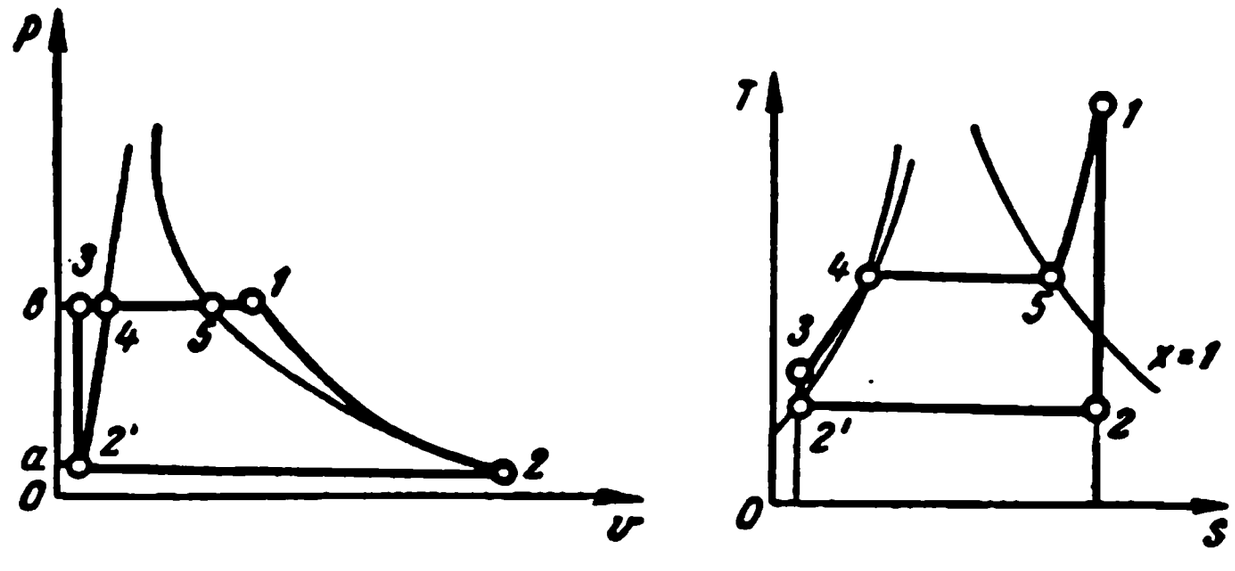

Fig. 117. Cycle of steam-power installation.

At small pressures (approximately smaller than 1/3 p_k) all the isobars in the region of the liquid pass quite close to one another and practically merge with the left boundary curve, as a result of which the point 3 almost coincides with point 2' on the Ts diagram

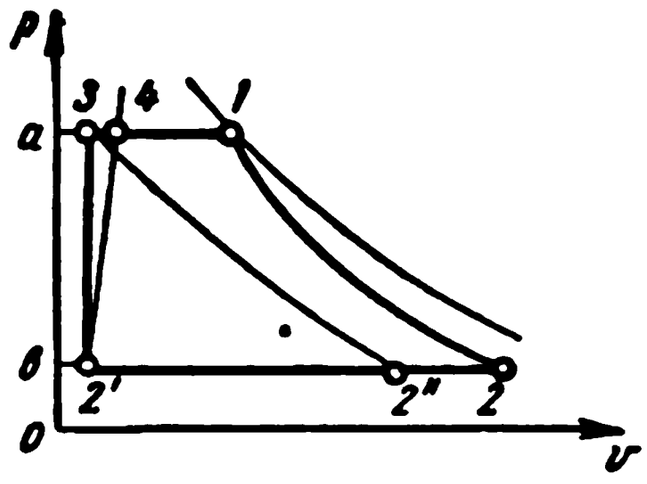

Fig. 118. Graphic representation of the work of compression.

or with point 4 on the pv diagram. Therefore the cycle of a steam power installation for small steam pressures is represented on the Ts and pv diagrams usually as shown in Fig. 119.

A cycle with total condensation of the steam without its superheating is called the Rankin cycle.

Heat is supplied to the steam-power installation cycle in the portion 3-4-5-1 of the isobar corresponding to the pressure p_1, and heat is removed on the portion 2-2' of the isobar with pressure p_2.

Therefore

$$q_1 = i_1 - i_3,$$

$$q_2 = i_2 - i_{2'}.$$

The theoretical useful work is

$$l'_{\text{теор}} = l'^{\text{теор}}_{\text{расш}} - l'^{\text{теор}}_{\text{сж}},$$

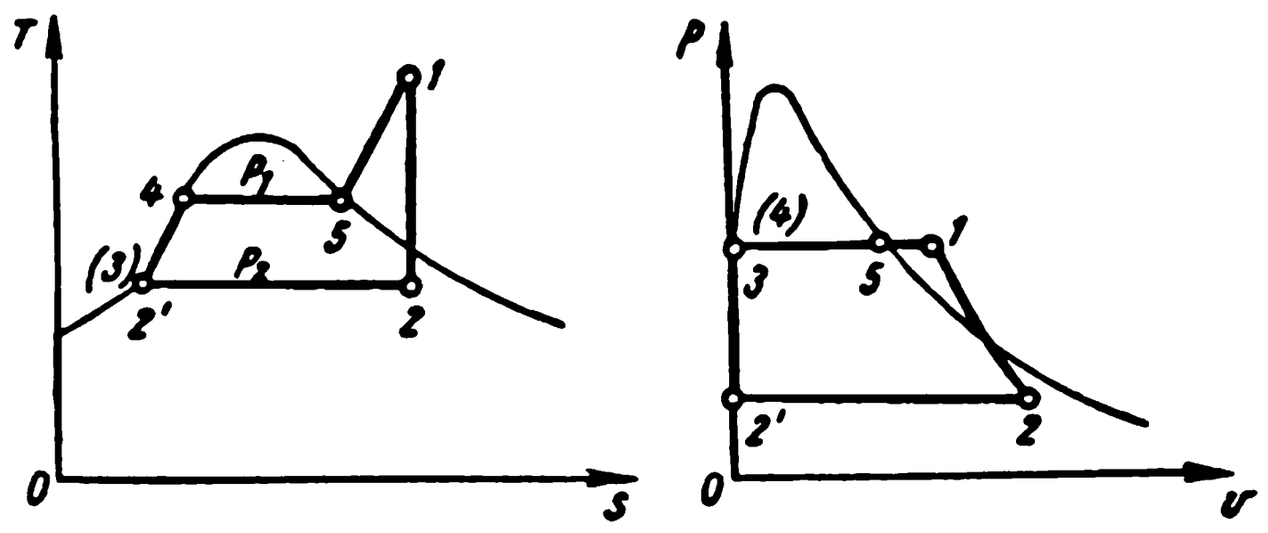

Fig. 119. Cycle of steam-power installation for small steam pressures.

where $l'^{\text{teor}}_{\text{rassh}} = i_1 - i_2$ is the theoretical work of 1 kg of steam in adiabatic expansion in a turbine and $l'^{\text{teor}}_{\text{szh}} = i_3 - i_{2'}$ is the theoretical work consumed in compressing 1 kg of water in the pump.

The thermal efficiency of the steam cycle is

$$\eta_t = 1 - \frac{i_2 - i_{2'}}{i_1 - i_3}. \qquad (10.1)$$

In an actual steam cycle (Fig. 120) each of the processes comprising the cycle is irreversible to some degree or another. Thus, for example, processes 2'-3 and 1-2 are irreversible owing to losses due to friction, while processes 3-1 and 2-2' are irreversible because of heat exchange at a finite temperature difference. The degree of irreversibility of the process 2'-3 (compression of water in the pump) is small, while that of the process 1-2 (expansion of the steam in the turbine) is appreciable, inasmuch as the flow of steam occurs at high speed; the degree of irreversibility increases here with increasing wetness of the steam. Accordingly, the process of compressing the water in the pump will terminate at the point 3e, which lies on the Ts diagram above and to the right of the point 3, while the process of expansion of the steam in the turbine will terminate at the point 2e; the initial state of the steam ahead of the turbine is determined, owing to the losses in the steam header, not by the point 1 but by the point 1e. The irreversibility of the processes in the steam-power installation leads to a loss of work and accordingly to a decrease in the useful work. The work lost in the pump, steam header, turbine, condenser, and steam boiler are represented in Fig. 120 by the shaded areas a, b, c, d, e, and f.

The actual useful work of the steam-power installation is

$$Al' = (i_{1\partial} - i_2)\,\eta_т - \frac{i_3 - i_2'}{\eta_к}.$$

In a real steam-power installation, in addition, of all the heat q' released by complete combustion of the fuel, the working medium receives only $q_1 = q'\eta_{kot}$, while the remainder $q'(1 - \eta_{kot})$ is uselessly lost.

Therefore the effective efficiency of the entire steam-power installation as a whole, without account of mechanical losses on the

turbine shaft and in the reduction mechanism (if we take into consideration the fact that $q_1 = i_1 - i_{3д} = i_1 - i_{2'} - [i_3 - i_{2'}]/\eta_n$) has the form

$$\eta_t = \frac{(i_{1д} - i_2)\,\eta_т - \dfrac{i_3 - i_{2'}}{\eta_н}}{i_1 - i_{2'} - \dfrac{i_3 - i_{2'}}{\eta_н}}\,\eta_{кот}, \qquad (10.2)$$

where η_t and η_n are the relative internal efficiencies of the turbine and the pump, while η_{kot} is the efficiency of the boiler.

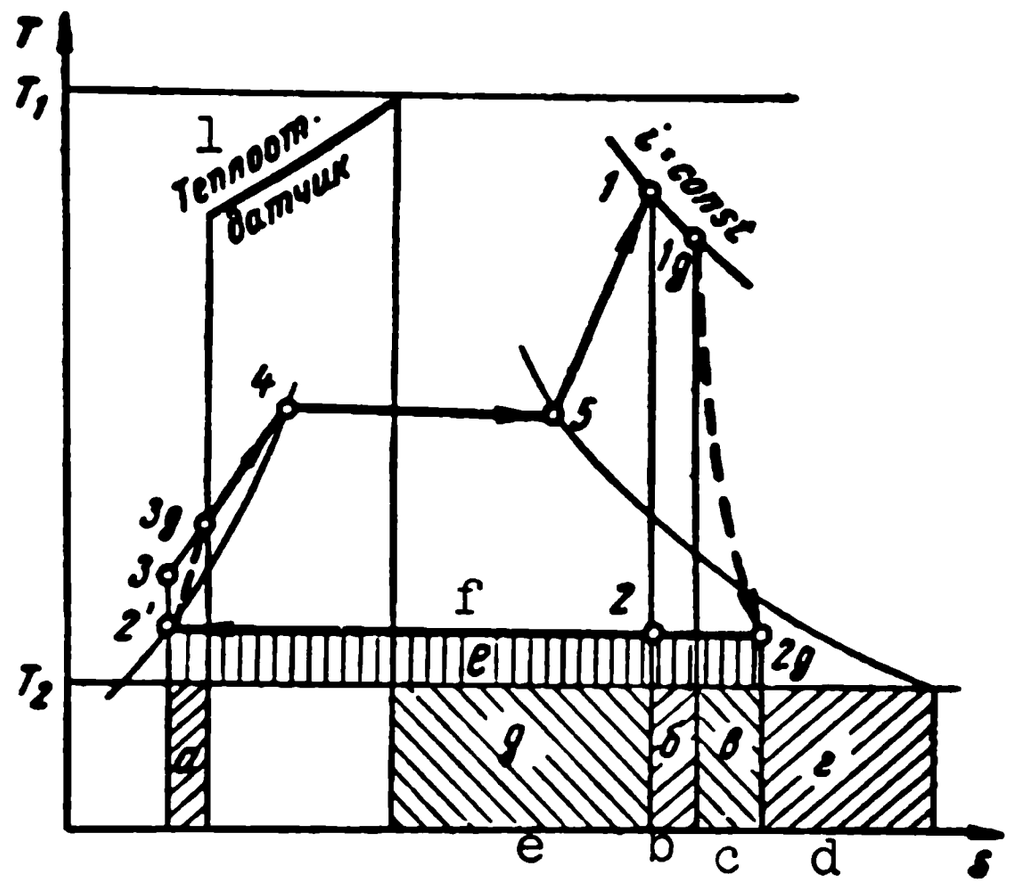

Fig. 120. Actual cycle of steam-power installation. 1) Heat exchanger.

§2. EFFECT OF STEAM PARAMETERS ON THE THERMAL EFFICIENCY OF THE CYCLE

Initial Steam Pressure

If for a final pressure p_2 and for one and the same maximum cycle temperature t_1 we raise the initial pressure of the steam p_1, the resultant increase in saturation temperature causes an increase in the average heat-supply temperature, and consequently also in the thermal efficiency of the cycle (Fig. 121).

The greatest increase in thermal efficiency of the cycle with increasing initial pressure is observed at $p_1 < 100$ atm abs.

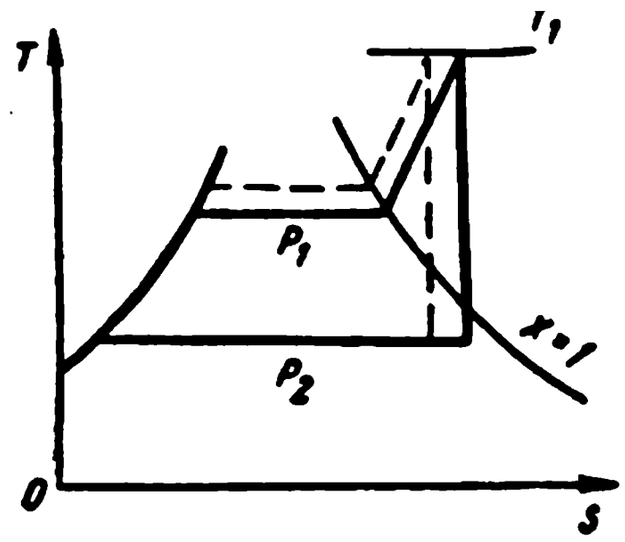

Fig. 121. Increase in average heat-supply temperature with increasing initial pressure.

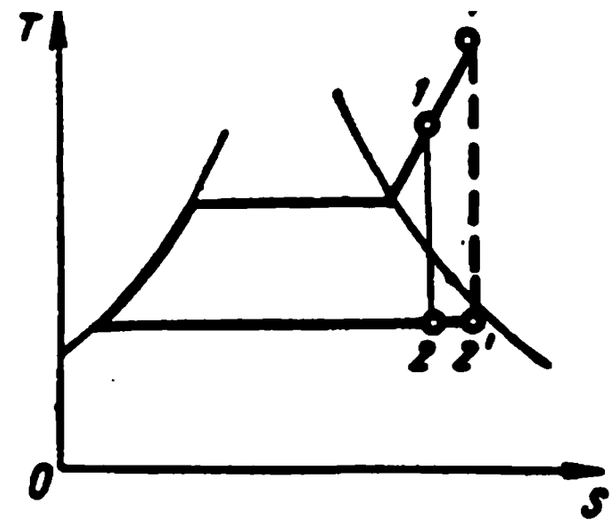

Fig. 122. Increase in average heat-supply temperature with superheating of the steam.

Superheating of the Steam

An increase in the initial temperature of the steam from t_1 to t_1' causes an increase in the average heat-supply temperature for a constant heat-removal temperature and consequently increases the thermal efficiency of the cycle (Fig. 122). The steam is also superheated in order to reduce the final moisture content of the steam. With the increasing moisture content of the steam, as was already indicated, the losses occurring during the expansion of the steam in the turbine increase. Furthermore, the water drops contained in wet steam cause erosion of the turbine blades, as a result of which a moisture content more than 13-15% in the turbine steam is not tolerated. Consequently, superheating of the steam is essential in the case of large initial pressures. In some cases it is necessary to resort to reheating of the steam (after expansion of the steam in the initial stage of the turbine, Fig. 123). The degree of superheat of the steam in a steam-power installation is limited by the properties of the metal used in the fabrication of the superheaters.

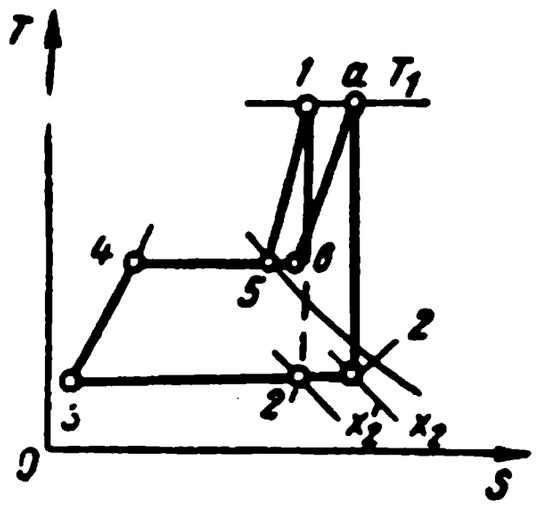

Fig. 123. Cycle with intermediate reheating of the steam.

At the present time the superheat temperature amounts to 600-650°C.

Final Steam Pressure

A decrease in the final pressure p_2 causes a decrease in the steam condensation temperature t_2 and consequently also in the heat-removal temperature (at a rather insignificant decrease in the average temperature of the heat supply), as a result of which the thermal efficiency of the steam-power installation increases (Fig. 124).

The theoretical limit to which the pressure can be decreased in the cycle is determined by the following condition: the saturation temperature at the final pressure p_2 must not be lower than the temperature of the surrounding medium. Otherwise the heat released during the condensation of this steam cannot be transferred to the surrounding medium. In practice, however, it is impossible to attain this limit. This is caused above all by the fact that in order to ensure heat exchange between the condensing vapor, which gives up the heat, and the medium, which receives this heat, there should be a finite temperature difference between them. The second limitation is connected with the fact that the specific volumes of the steam increase rapidly with decreasing final pressure and this may lead to exceedingly large dimensions of the installation.

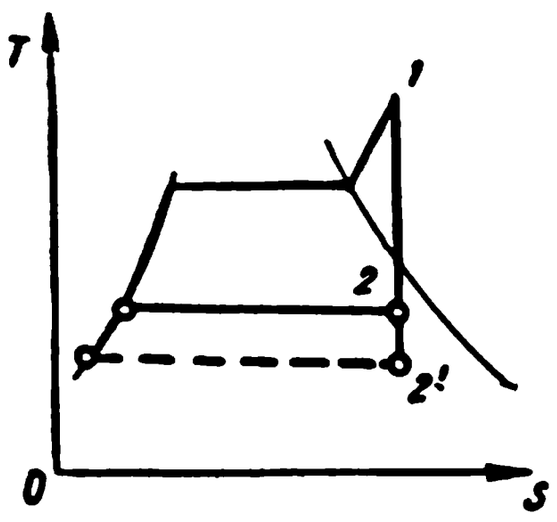

Fig. 124. Decrease in average temperature of heat removal with decreasing final pressure.

In water and steam-turbine installations the pressure in the condenser is 0.035-0.040 atmospheres, which corresponds to a saturation temperature 26-29°C.

§3. REGENERATIVE CYCLES OF STEAM-POWER INSTALLATIONS

To realize heat regeneration it is necessary that there exist in the cycle certain portions on which the supply and removal of heat take place at identical temperatures. In ordinary steam-power installation cycles, when all the heat is carried away at the lowest cycle temperature t_2, there are no such portions. Therefore, in order for regeneration in a steam-power installation to be possible, it is necessary to modify the cycle, namely replace the lower part of the adiabat 1-2 by a line 6-2" which is equidistant to the line 2'-4 of water heating in the steam boiler (Fig. 125).

In the regenerative cycle the heating of the feed water to a temperature t_v (portion 2'-7) is carried out by the heat released during the cooling and condensation of the steam (portion 6-2"). Because of this, the consumption of heat from the heat source in the portion 2'-7 of the cycle is eliminated and the amounts of delivered and removed heat q_1 and q_2 decrease compared with the nonregenerative cycle by the respective amounts $\int_{2'}^{7} Tds$ and $T_2(s_2 - s_{2''})$, with $s_2 - s_{2''} = s_7 - s_{2'}$.

The thermal efficiency of the regenerative cycle is

$$\eta_t^{per} = 1 - \frac{i_2 - i_{2'} - T_2(s_7 - s_{2'})}{i_1 - i_{2'} - \int_{2'}^{7} Tds}. \qquad (10.3)$$

Regenerative heating of water leads to a decrease in the work of expansion of the vapor in the turbine, as a result of which the useful specific work in the regenerative cycle is smaller than in the ordinary cycle. However, the amount of heat supplied q_1 decreases to an even greater degree, so that the efficiency of the installation increases (see §4, Chapter 4). The value of the latter depends on the temperature t_v to which the water has been regeneratively heated; the limiting value of t_v is equal to the boiling temperature of water

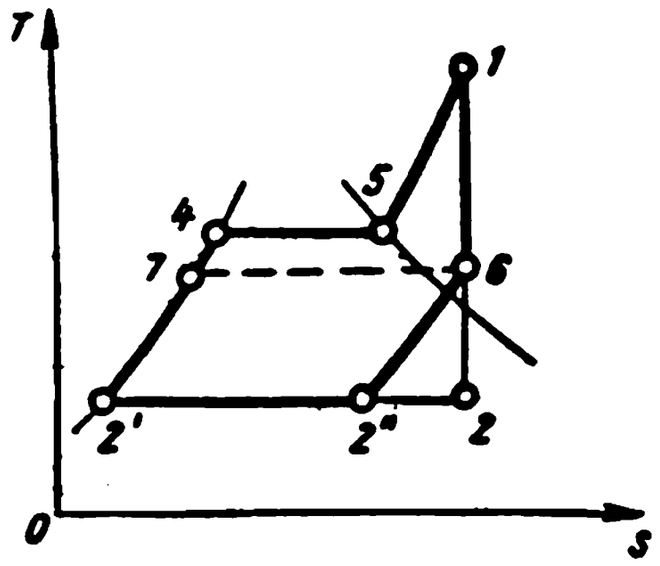

Fig. 125. Steam-power cycle with regeneration of heat.

t_4 at a specified steam pressure p_2 in the boiler. Therefore the greatest value of the thermal efficiency of the regenerative cycle with saturated steam (without superheating of the latter) amounts to

$$\eta_{t,\text{макс}}^{\text{per}} = 1 - \frac{T_2}{T_1}.$$

It is impossible in practice to carry out continuous removal of heat from expanding steam to the feed water along the line 6-2". In place of this, one uses a stepwise removal of heat by extracting a certain amount of steam from the last stages of the turbine. The extracted steam is fed into regenerative heaters, where it is condensed; the heat of condensation heats the feed water.

Regenerative heaters can be of the mixing type and of the surface type. In the former the steam is mixed with the heated feed water; in the latter the heat of condensation released heats the feed water via the walls of the heat-exchanger tubes and the formed condensate of the heating steam is fed in some way or another into the common feed water stream (Fig. 126).

Owing to the branching of the steam stream and the diversion of part of it for heating of the feed water, the specific useful work $\underline{l}'_{reg}$ will be less than the useful work $\underline{l}' = 1/A(i_1 - i_2)$ of the same cycle without regeneration, by an amount $\Sigma g_j(i''_j - i_2)$, where g_j is a fraction of the steam extracted in the j-th heater, while i''_j and i'_j are the enthalpies of the steam at the entrance to this heater and of the water leaving it. Consequently, to obtain the same power it is necessary to feed to the turbine a somewhat larger amount of steam (by 10-15%), that is, a regenerative steam-power installation con-

sumes more working medium per unit power.

The amount of steam extracted for the regenerative heaters is determined from the heat-balance equations: for the heaters

$$g_j(i'_j - i'_j) = i'_j - i'_{j+1} \quad (1 < j < n)$$

and for the collecting reservoir of the condensate of the heating steam

$$\sum g_j i'_j = i',$$

where i' is the enthalpy of the water entering the last (n-th) heater. It is obvious that $i' = i_{2''}$; $i''_1 = i_6$. The more steam extraction points, the closer the heat removal process comes to the ideal process 6-2. Good approximation is obtained even with n = 4-6.

The optimum temperature of regenerative water heating t^{v}_{opt}, for a finite number of steam extractions, will be smaller than t_4. The numerical value of t^{v}_{opt} depends on the number of steam extractions and is determined by the condition that the effective efficiency of the steam-power installation be maximized. The temperatures at the extraction points (for extraction of saturated steam and an identical degree of regeneration in all the regenerative heaters) are chosen from the condition

$$T_j/T_{j+1} = \sqrt[n]{T^{s}_{опт}/T_{s}},$$

corresponding to the minimum losses of work capacity in the regenerative heating system.

The thermal efficiency of the regenerative cycle and the effective efficiency of the installation are respectively

$$\left.\begin{aligned} \eta_t^{reg} &= \frac{i_1 - i_2 - \sum g_j (i'_j - i_2)}{i_1 - i_{2'} - \sum g_j (i'_j - i'_j)} \\ \eta_e^{per} &= \frac{[i_1 - i_2 - \sum g_j (i'_j - i_2)]\,\eta_т}{i_1 - i_{2'} - \sum g_j (i'_j - i'_j)\,\eta_{то.j}}\,\eta_{кот} \end{aligned}\right\}, \qquad (10.4)$$

where η_{to} is the efficiency of the regenerative heat exchanger.

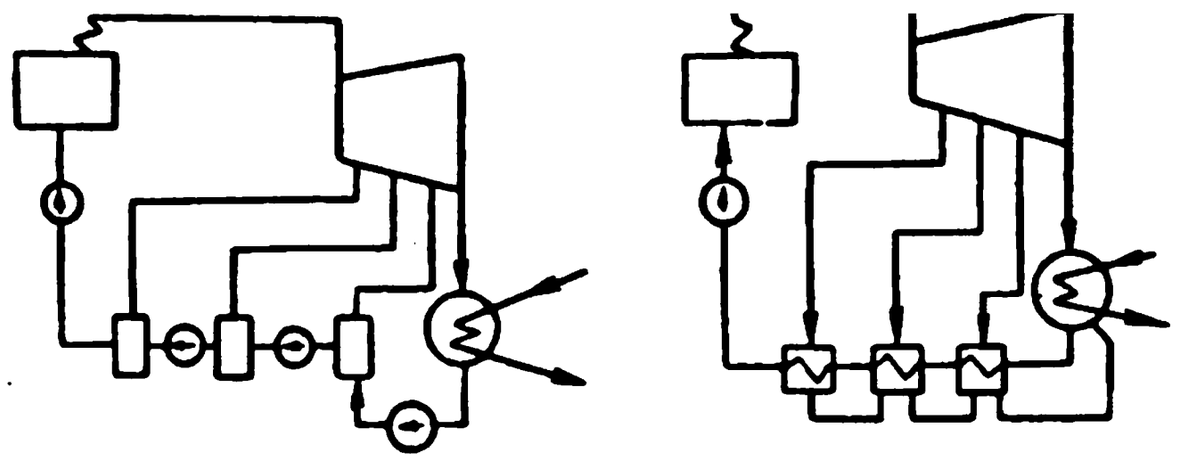

Fig. 126. Diagram of regenerative feed water heating.

In these formulas, no account is taken of the work consumed in driving the pump and of the mechanical losses in the installation.

In regeneration, the increase in the effective efficiency and the resultant fuel economy reach 7-15%. Therefore regenerative heating of feed water is used in all large steam-power installations. Regeneration of heat assumes particular significance in modern stations with high initial parameters.

§4. NUCLEAR STEAM-POWER INSTALLATIONS

The process of converting energy in a nuclear power installation (Fig. 127) consists in the following: a nuclear reactor 1 delivers, as a result of the fissioning of the nuclei of the fissionable elements (atomic fuel), a certain amount of heat Q at a certain temperature t_r*. This heat is carried away from the reactor by the stream

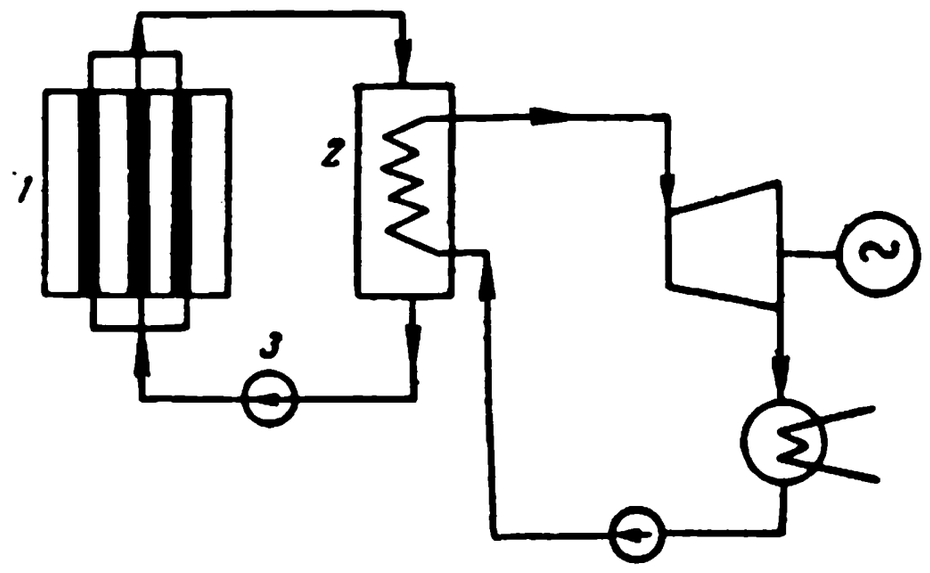

Fig. 127. Diagram of nuclear power installation.

of coolant into steam generator 2 and is there transferred to the

working medium of the thermodynamic cycle. This cycle is analogous to that of an ordinary steam power installation (the fact that the formation of the steam occurs in a steam generator and not in a steam boiler with heating by fire is immaterial). In some cases the working medium may also be the primary coolant itself.

As a result of this supply of heat from the primary coolant to the working medium, the temperature of the latter increases from the ambient temperature t_2 to t_1. It is obvious that the average temperature of the working medium on the heat-supply portion, t^{sr}_{podv}, is smaller than the average temperature t_{sr} of the primary coolant, while the latter is smaller than the temperature t_r of the reactor.

In the steam-power part of the installation, the heat Q is used (assuming that there are no losses of work in the thermodynamic cycle) to perform the useful work

$$AL' = Q\left(1 - \frac{t_2}{t^{ср}_{подв}}\right).$$

Some fraction of this work is consumed in the installation's own needs, particularly pumping the primary coolant by means of pump 3; this fraction depends on the nature of the primary coolant and on the scheme chosen for the installation.

If we take into account the leakage of heat in the installation, amounting to $Q\alpha$, the expression for L' assumes the form

$$AL' = Q\left(1 - \frac{t_2}{t^{ср}_{подв}}\right)(1-\alpha). \qquad (10.5)$$

The thermal losses both in the reactor and in the associated equipment are as a rule insignificant.

The amount of heat carried away from the reactor is the larger, the greater the temperature difference between the fuel elements of the reactor and the coolant. The temperature of the primary coolant always exceeds the temperature of the working medium and on the aver-

age amounts to $t^{sr}_{podv} + \Delta$, where Δ is a positive quantity representing the mean value of the temperature difference between the primary coolant and the working medium during the process of supplying heat to the latter. Therefore, the temperature difference between the fuel elements and the primary coolant is on the average $t_r - t^{sr}_{podv} - \Delta$, and consequently Q should decrease with increasing t^{sr}_{podv}. Since L' is equal to the product of the two factors $Q(1 - \alpha)$ and $1 - (t_2/t^{sr}_{podv})$, one of which decreases with increasing t^{sr}_{podv} while the other increases, for specified values of t_r and Δ the value of L' goes through a maximum with increasing t^{sr}_{podv}, while the temperature of the reactor t_r is determined by the maximum possible temperature of the nuclear fuel and represents the mean value of the temperature at the center of the fuel elements. The average temperature difference of the coolant and of the working medium Δ depends principally on the choice of working medium of the thermodynamic cycle.

Optimal Mode

The optimal operating mode of a nuclear power installation depends on the specific conditions of its utilization and also on economic factors. Unlike steam electric stations, the fuel part of the cost of the generated electricity is in atomic electric stations much smaller than the remaining parts and above all, much smaller than the capital expenditures per unit installed capacity*. Therefore, an atomic electric station will be most economical if its power is maximum, for then the capital expenditures per unit installed capacity will be minimized and the cost of generation of electricity will be minimal. For other nuclear power installations, the requirement of maximum power is even more significant. Consequently, one can assume that optimum operating conditions for a nuclear power installation are those conditions under which the ratio of the useful work performed

by the nuclear power installation to the capital expenditures is maximum, that is, the rating of the installation is maximal.

The useful work performed by a nuclear power station and consequently the power rating of the installation are, for specified values of t_r and Δ, functions of the average temperature t^{sr}_{podv}. This function reaches a maximum value for a certain optimum value of the temperature t^{sr}_{podv}, which will be henceforth designated t*. The exact value of the optimal temperature t* can be calculated for each nuclear power installation by starting from the specific operating conditions. There exists, however, an approximate formula with the aid of which it is possible to determine the approximate value of the optimal temperature t*.

Let us assume that all the coefficients in the expression for L', which are obtained after substituting the value of $Q = K(t_r - t^{sr}_{podv} - \Delta)$, are constant. Such an assumption is valid for specified t_r and Δ, inasmuch as the heat transfer coefficient K is a slowly varying function of the temperature. Then

$$AL' = K(1-\alpha)\left(t_p - t^{ср}_{под} - \Delta\right)\left(1 - \frac{t_2}{t^{ср}_{под}}\right).$$

where $K(1 - \alpha)$ is regarded as a constant quantity, while t_2, t_r and Δ have specified values. From the maximum condition $\partial L'/dt^{sr}_{podv} = 0$ we obtain the optimal temperature

$$T^* = \sqrt{T_2(T_p - \Delta)}. \qquad (10.6)$$

The quantity Δ, which is negligible compared with t_r, can be neglected. Then

$$T^* = \sqrt{T_2 T_p}.$$

Thermal Efficiency of the Cycle and Effective Efficiency of the Installation

Using Formula (10.6) we can readily find the value of the effective efficiency of the nuclear power installation. The effective

efficiency η_e^{ts} of the heat-power portion of the installation is the ratio of the useful work performed L' to the amount of heat Q released in the reactor. According to Equations (10.5) and (10.6), the limiting value of η_e^{ts} (under the assumption that all the processes in the thermodynamic cycle, with the exception of the heat supply process, are reversible), the thermal efficiency of the thermodynamic cycle is equal to

$$\eta_t = (1-\alpha)\left(1-\sqrt{\frac{T_2}{T_p}}\right). \tag{10.7}$$

If we disregard leakage of heat, then η_t can be expressed in the following fashion:

$$\eta_t = 1-\sqrt{\frac{T_2}{T_p}}. \tag{10.8}$$

It is clear that the value of the effective efficiency of the heat-power part of the installation at the optimal temperature T* is not maximal and, unlike the useful work or the power of the installation, L' does not go through a maximum with increasing average temperature of the working medium, but increases monotonically with the latter. Therefore, the largest value of η_t would occur at the highest possible temperature of the working medium in the thermodynamic cycle, that is, at a temperature approaching the temperature of the reactor. In this case, however, the useful power of the installation would be equal to zero.

The effective efficiency of the nuclear power installation as a whole will be smaller than η_e^{ts} by an amount equal to the ratio of the work consumed for pumping the primary coolant to Q.

From Formulas (10.6) and (10.7) it follows that with the increasing temperature in the reactor (with increasing maximum permissible temperature of the fuel elements) the optimum temperature t* of the working medium and the effective efficiency of the installation in-

crease. In this case the average temperature of the working medium is increased slowly, while the effective efficiency increases relatively rapidly.

The effective efficiency of a nuclear steam-power installation can be increased by regeneration of the heat. However, the role of heat regeneration in a nuclear power installation is more modest than in an ordinary steam-power installation, owing to the smaller values of the temperature t_r (see §4, Chapter IV).

§5. CENTRAL HEATING INSTALLATIONS

In a steam-power installation only a small fraction of the heat (about 35%) released upon combustion of the fuel is converted into useful work. The greater part of the heat is given up to the water that cools the condenser at temperatures that exceed only slightly (by 10-15%) the atmospheric temperature and is therefore uselessly lost.

For domestic and technological needs, however, it is frequently necessary to have along with electricity also heat at relatively low temperature (80-100°C). The source of such heat can be the steam that has previously been expanded in a turbine, or else the water used to cool the condenser, provided the pressure to which the steam is expanded in the turbine (the total amount or only part of it) is raised. In this case the heat of the burned fuel will be used both to obtain useful work and to obtain the required low-temperature heat.

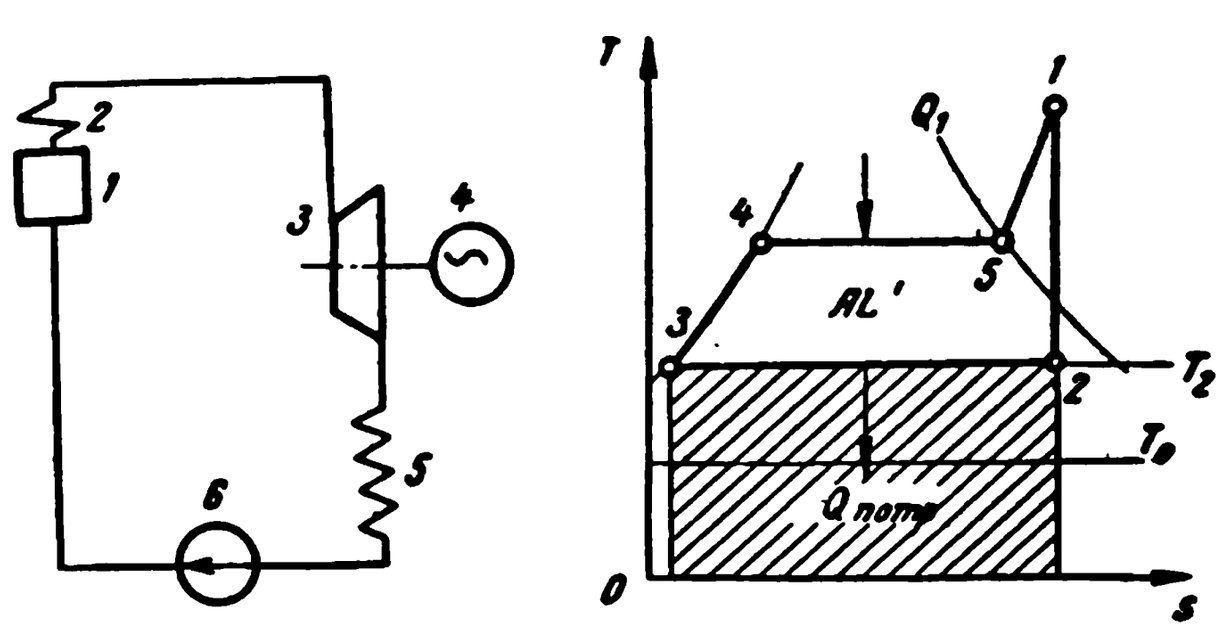

Fig. 128. Diagram of simplest central-heating installation.

Simultaneous generation of electricity and low-temperature heat in one and the same steam-power installation is much more convenient than separate generation of electricity and of heat.

Combined generation of electricity and heat in thermal electric stations for technological and domestic needs (by extraction and using the spent steam) on the basis of centralized heat supply is called central heating.

Figure 128 shows the principal diagram of the simplest central-heating installation with back pressure, consisting of a steam boiler 1 with steam superheater 2, turbine 3, generator 4 for electricity mounted on the same shaft as the turbine, heat load 5, and pump 6.

In this installation the useful work performed L' and in addition heat Q_{potr} with temperature $T_{potr} > T_2$ is produced for domestic and technological needs.

The ratio AL'/Q_{potr} is called the specific generation of electricity on the base of heat consumption. The larger AL'/Q_{potr}, the higher the economy of the central-heating installation.

In practice, in order to satisfy better the needs for both heat and electricity, most central-heating installations are equipped with condensing turbines with extraction of steam at the pressure necessary for the external consumption. These installations make it possible to vary the thermal and electric loads over a wide range. Quite promising from the economic point of view are atomic central-heating electric stations.

§6. REQUIREMENTS IMPOSED ON THE WORKING MEDIUM OF STEAM-POWER INSTALLATIONS

In steam-power installations, the ratio of the average heat-supply and removal temperatures, which determine the efficiency of the cycle, depend on the properties of the working medium. Consequently,

by using the most suitable working media, it is possible to increase greatly the ratio of the average heat-supply and removal temperatures and to make the thermal efficiency of the cycle of the steam-power installation closer to the efficiency of the Carnot cycle in the same temperature interval.

From the thermodynamic point of view, the best working medium for a steam-power installation is a substance in which the critical temperature is much higher than the limiting temperatures used in steam-power installations. In this case, the average heat-supply temperature and, consequently, the thermal efficiency of the cycle would have the greatest value and therefore it would be possible to realize a cycle without superheating the steam with the most convenient isothermal heat supply.

In addition to this main requirement, which is dictated by thermodynamic considerations, many other requirements are imposed on the working medium and follow from technical, technical-economic, and operational conditions. These requirements pertain above all to the values of the pressures in the cycle. The working medium should have such properties that, on the one hand, under ordinary lowest cycle temperatures (25-28°C) the saturated-steam pressure, which determines the pressure in the condenser, not be too low and, on the other hand, at the highest temperatures of the cycle the pressure of the saturated steam not be too large. The technical requirements concern also the humidity of the steam at the end of the adiabatic expansion, which should not be larger than a certain limit. The degree of wetness at the end of the expansion process depends on the specific heat of the saturated steam c". It is most convenient when c" has a small negative value. Then, on the one hand, the wetness at the end of the expansion is small and, on the other hand, the entire heat-

removal process occurs in the region of wet steam.

The specific heat of the liquid c' on the left boundary curve has also considerable importance. At a small value of c', the compression and heating of the water are nearly adiabatic and the cycle itself approaches a Carnot cycle. In general, the smaller the ratio c'/r, the higher the average heat-supply temperature and the better the working medium. Water has a rather large specific heat in the liquid state and consequently the heat consumed in heating the water influences the average heat-supply temperature and decreases it. This adverse property of water is partially offset by the use of heat regeneration. Along with the foregoing requirements, it is also necessary that the working medium be sufficiently abundant, inexpensive, nonpoisonous, have good heat-transfer properties, not exert appreciable corrosion action on the metals, be chemically stable at high temperatures, be relatively nonviscous, etc. All these requirements are difficult to satisfy, so that until now no substance has been found to satisfy completely all the foregoing requirements.

It must be noted that the main thermodynamic requirement (high critical temperature) remains in force also for working media in nuclear power installations. For the primary coolant circulating through the reactor, it is desirable in addition to have a relatively low critical pressure; together with a high critical temperature, this ensures a low pressure of the heat coolant in the reactor. The primary coolant must likewise not have too large a neutron capture, be stable under neutron-irradiation conditions, and not lead to the appearance of induced radioactivity.

§7. BINARY CYCLES

The absence of a substance capable of simultaneously satisfying all the main requirements imposed on the working medium of steam-

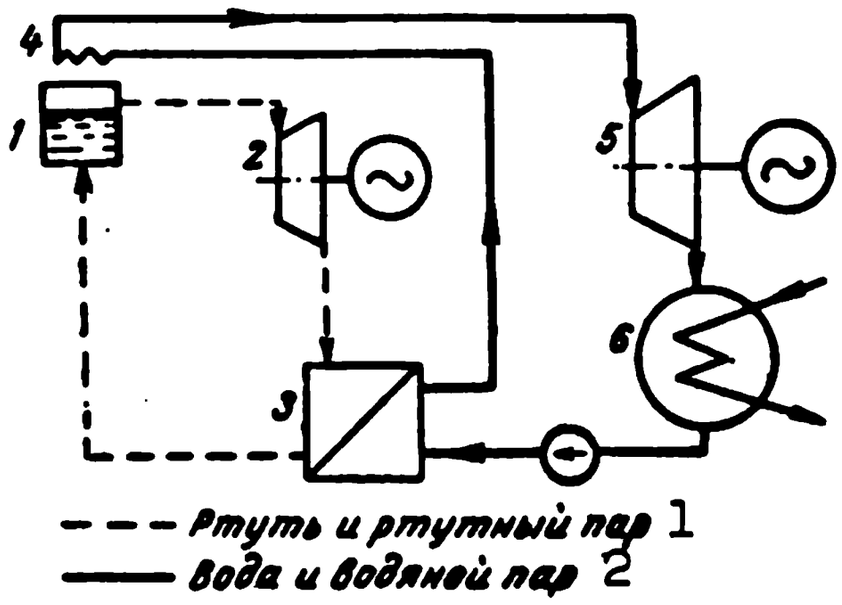

Fig. 129. Diagram of mercury-water binary installation. 1) Mercury and mercury vapor; 2) water and steam.

power installations has led to the use of two working media in one installation, each within a definite temperature interval, with respect to which the pressure of the saturated vapor of this substance is most acceptable. Cycles with two working media are called binary cycles. In practice, only mercury-water binary cycles have been realized so far. The diagram and the cycle of the mercury-water binary installation are shown in Figs. 129 and 130.

Mercury vapor, produced in a mercury boiler 1, is fed from the boiler into turbine 2 and after expansion in the turbine it is directed to a so-called condenser-operator 3, where it is condensed, and the heat released during condensation is used to evaporate the water and to form water vapor. Liquid mercury from the condenser-evaporator is again fed to the mercury boiler and the steam is fed to steam superheater 4, after which it is fed to the steam turbine 5, where it performs useful work. The spent steam gives up its heat to the cooling water in condenser 6; the condensate of the steam is again fed to the condenser-evaporator.

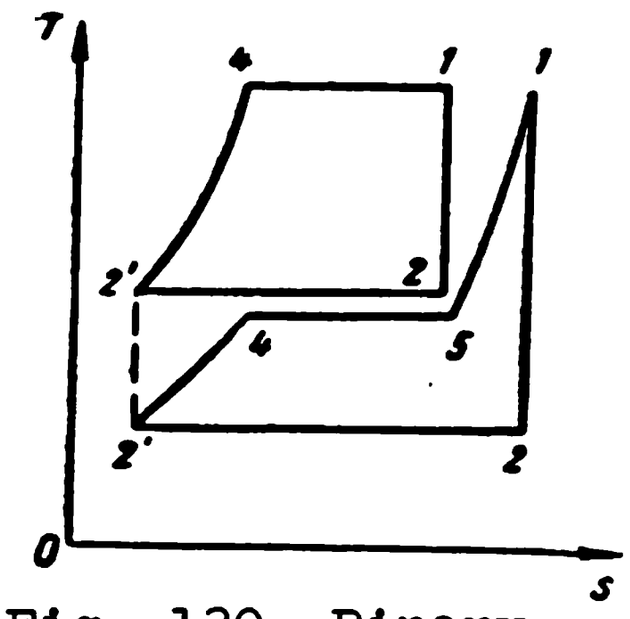

Fig. 130. Binary mercury-water cycle.

Inasmuch as the heat of evaporation of mercury is relatively small, to evaporate 1 kg of water it is necessary to condense several kilograms of mercury. The ratio of the weight of the mercury to the weight of the water in the binary cycle is called the multiplicity of the mercury and is designated by m. From the heat-balance equation it fol-

lows that

$$m = \frac{i_1'' - i_{2'}''}{i_1'' - i_{2'}''}.$$

When mapping the binary cycle on the Ts diagram, the lower stage is constructed for 1 kg of water vapor (cycle 1-2-2'-4-5) and the upper for $\underline{m}$ kg of mercury vapor (cycle 1-2-2'-4) in such a way that the point corresponding to the final state of the mercury during the condensation process is on the same vertical line as the point of the start of the water-heating process in the condenser-evaporator.*

The thermal efficiency of the binary cycle with superheating of the water vapor (disregarding the work of the pumps) is

$$\eta_t^{\text{бин}} = \frac{m\left(i_1'' - i_2''\right) + \left(i_1' - i_2'\right)}{m\left(i_1'' - i_{2'}''\right)}. \qquad (10.9)$$

The thermal efficiency of binary cycles reaches 0.90-0.95 of the thermal efficiency of the Carnot cycle realized in the same temperature interval and is the largest compared with all other known cycles. Binary cycles can be used in nuclear steam-power installations. Thus, for example, in installations with reactors using diphenyl as the coolant (in the primary loop or in the intermediate loop, as can occur in reactors with liquid-metal coolant), the diphenyl-water binary cycle leads to an appreciable increase in efficiency and is quite promising.

§8. STEAM-GAS CYCLES

In §2 of Chapter 9 it was indicated that in gas-turbine installations the energy consumed for the installation's own needs amounts to a considerable fraction of the useful work of the installation. This fraction depends on the specific enthalpy of the working medium ahead of the turbine and decreases with increase in the latter. The increase in the heat content of the working medium by increasing the temperature

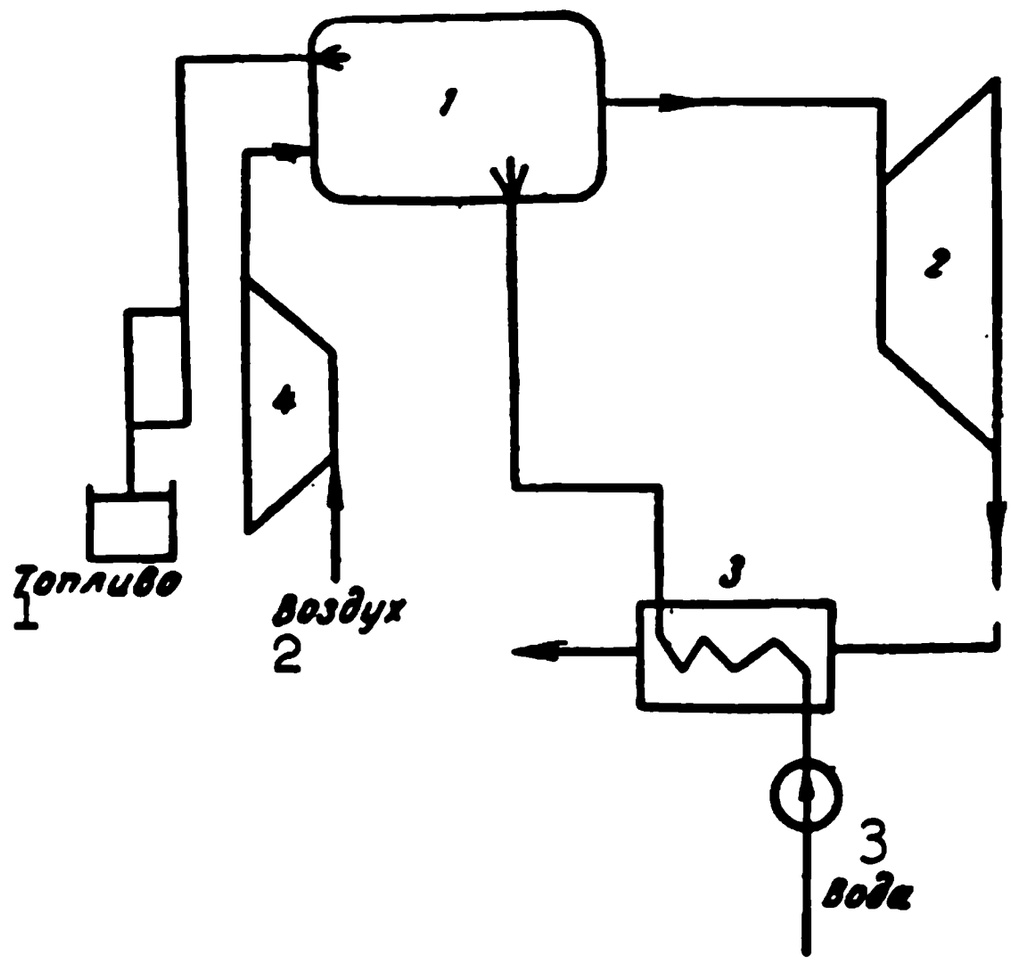

Fig. 131. Diagram of simplest steam-gas installation. 1) Fuel; 2) air; 3) water.

is limited by the capabilities of modern metallurgy, which sets a limit on the maximum temperature of the cycle at 700-800°C. Another way is to use a working medium with large specific enthalpy. In gas-turbine installations operating in an open cycle, this is attained by adding to the air a substance which increases the heat content of the working mixture. Such a substance may be ordinary water, which, as is well known, has a considerable specific enthalpy.

A gas turbine installation in which the working substance is a mixture of air (or combustion products) and water vapor is called a steam-gas installation and its cycle is called the steam-gas cycle. A diagram of a very simple steam-gas installation and its cycle are shown in Figs. 131 and 132.

The steam-gas generator 1 of the installation represents a combination of a combustion chamber and an evaporator, comprising as it were the end portion of the combustion chamber. Into the evaporator one injects water that has been previously heated in regenerator 3 (the regenerative water heating process is represented on the Ts dia-

gram by the line 3-4). The air needed for the combustion is fed into the combustion chamber under pressure from compressor 4. The steam formation (line 4-5) and the superheating of the steam (line 5-1) occur in the steam-gas generator, by heat exchange with the gaseous combustion products. Lines 1-2 and 3'-4' represent the processes of adiabatic expansion of the water vapor and of the gaseous combustion products in the turbine 2, while the line 4'-1' represents cooling of the gaseous combustion products in the regenerator and the atmosphere. Condensation of water vapor (line 2-2') occurs in the atmosphere; it is also possible to use a special condenser.

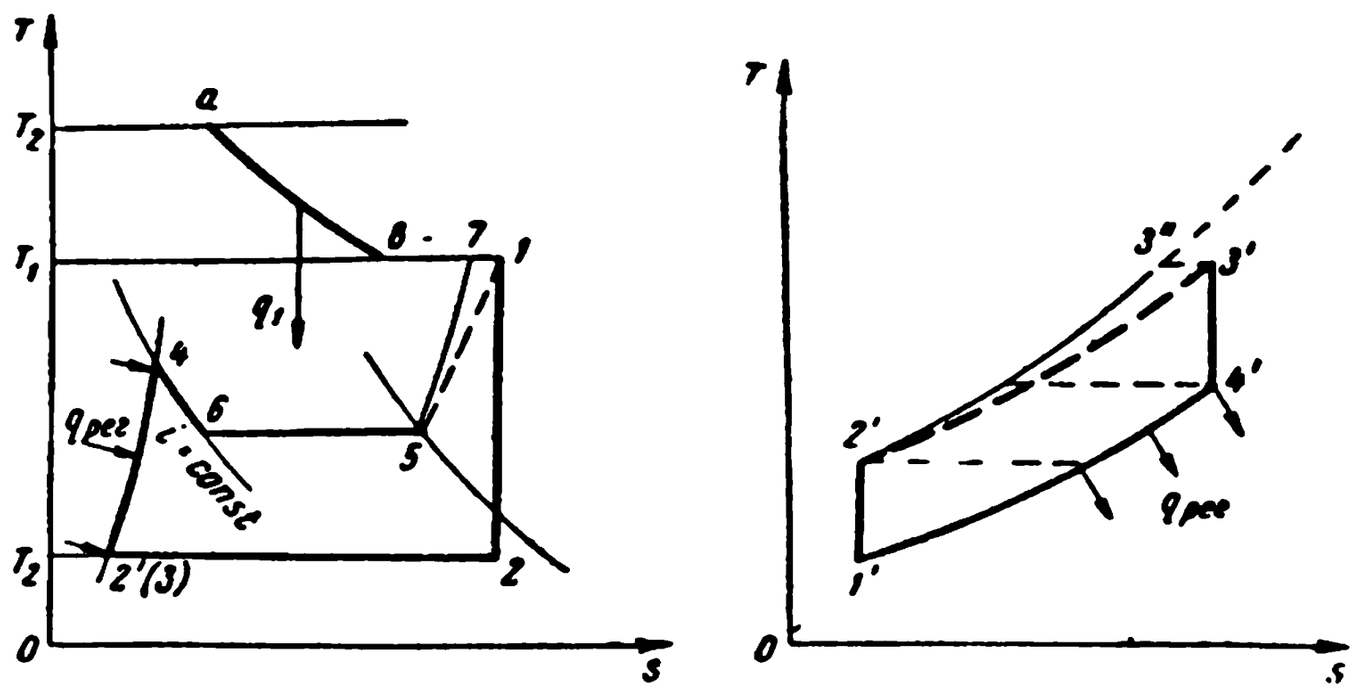

Fig. 132. Cycle of steam-gas installation.

Thus, the steam-gas cycle represents a combination of two cycles – steam 1-2-2'-4-6-5 and gas-air 1'-2'-3'-4'. If the consumption of water in the steam-gas installation is G_v kg and that of air is G_g kg, then the steam cycle is constructed for 1 kg and the gas-air cycle for $m = C_g/G_v$ kg of working medium.

A feature of the steam-gas cycle is the irreversible character of the processes 4-1 and 3"-3', owing to heat exchange at a finite temperature difference between the steam and the gaseous combustion products and their mixing. Line 3-4 in the steam cycle represents regenerative heating of the feed water by the heat from the spent gases released on portion 4'-1'. The water is fed to a regenerative heat

exchanger after compression in a pump. If the pressure to which the water is compressed exceeds the pressure in the combustion chamber, then upon injection of the water into the steam-gas generator the water pressure drops from p_3 to p_2, which is equal to the pressure in the combustion chamber. This process, which occurs without performance of useful external work and heat exchange (in view of the high speed of the process) with the hot gases, can be regarded in accordance with Equation (2.11) as adiabatic throttling, as a result of which $i_4 = i_6$ (from this condition it is easy to determine the position of the point 6 on the Ts diagram). Owing to the irreversible process 4-6, a loss of useful work of the cycle $A\Delta l'_{dr}$ takes place, equal in accordance with (3.21) to $T_2(s_6 - s_4)$.

The processes 6-1 of evaporation of water and superheat of the formed steam and the process a-b of cooling of the gaseous combustion products are accompanied by mixing of the steam with the hot gases and are therefore not isobaric for each of the components. Since the pressure in the steam gas generator is constant after mixing, as a result of which the pressure of each of the components decreases to partial pressures $p_v = p_1$ and $p_g = p_{3'}$, the following relation should be satisfied:

$$p = p_1 + p_{3'}.$$

If the superheated steam and the gaseous combustion products are not regarded as ideal gases, then the right half of this relation should be taken in accordance with Equation (1.9).

The steam and combustion-product temperatures are the same after mixing: $t_1 = t_{3'}$. Owing to the reversibility of the processes 6-1 and 3"-3', useful external work is lost. To calculate this loss let us assume that the heat-exchange and mixing processes occur not simultaneously, but in sequence, that is, the steam is first heated to a

temperature t_1 without mixing along the isobar 6-5-7 and is then mixed with the gaseous combustion products at a constant temperature t_1. Such an analysis is perfectly rigorous if the enthalpies of the mixing bodies are functions of the temperature only and not of the pressure, that is, if the superheated steam and the gaseous combustion products can be regarded as ideal gases.

Indeed, according to (4.23), we have for mixing

$$G_s i_6 + G_r i_a = G_s i_1 + G_r i_{3'}.$$

On the other hand, from the heat-balance equation we have for the heat exchange between the steam and the hot gases in the absence of mixing

$$G_s i_6 + G_r i_a = G_s i_7 + G_r i_{3''}.$$

and at the points 7, 1 and 3', 3" the temperature is the same while the pressure is different. If the enthalpy does not depend on the pressure then $i_7 = i_1$, $i_{3''} = i_3$ and both equations coincide, from which it follows that the actual process of heating and mixing can be broken up into two stages – isobaric heating and isothermal mixing.

The loss of useful work due to the irreversible heat exchange on the portion 1-5-7 can be readily determined with the aid of the iT and Ts diagrams. Let us construct for this purpose on the miT diagram of the gaseous combustion products the isobar (Fig. 133) corresponding to the pressure p, from the point a with initial temperature equal to the combustion temperature T_g to the point b with temperature $t_{3'}$; along the isobar a-b there is cooling of m kg of hot gases by heat exchange with 1 kg of water. We then transfer with the aid of the relation ds = di/T this isobar to the Ts diagram (see Fig. 132) for water and draw the line of isobaric heating of the water 6-5-7, taking the value of the entropy at the point 6 to be the initial value of the entropy of both curves. Since all the heat given up by

the hot gases is transferred to the water, we have

$$m\int_{a}^{b} T\,ds_r = \int_{b}^{7} T\,ds_w$$

or, in a different form

$$i_7 - i_b = m\,(i_a - i_{3'}).$$

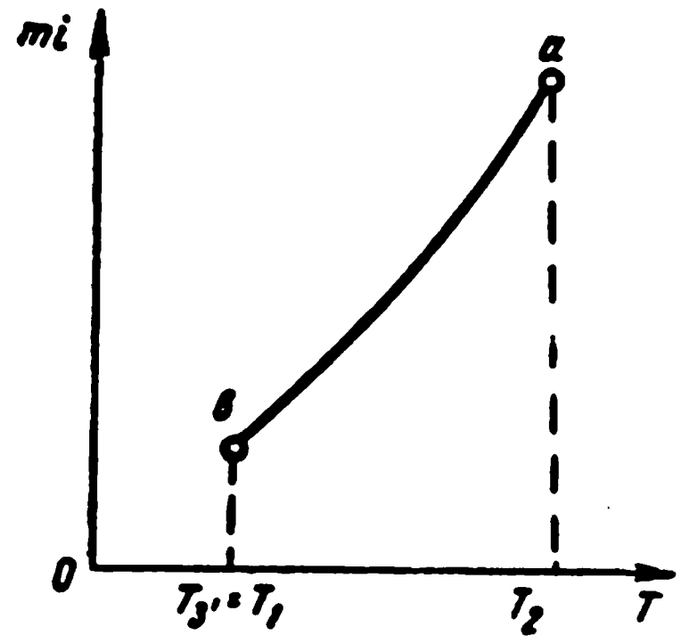

Fig. 133. miT diagram of gaseous combustion products.

With the aid of this equation we can readily determine the temperature at the points 7 and 3", and consequently also the position of the final points 7 of the isobaric heating of the water and 3" – the isobaric cooling of the gaseous combustion products. The entropy difference at the points 7 and b represents the increment in the entropy of the system ΔS^*_{to} due to the irreversibility of the heat-exchange process, while the product $T_2(s_7 - s_v)$ represents the loss of useful work $A\Delta l'_{to}$ occurring thereby. After mixing, the state of the water vapor is represented by the point $1(p_1, t_1)$, while that of the gases by the point $3'(p_{3'}, t_1)$. The partial pressures p_1 and $p_{3'}$ can be determined from the known values of p and t_1 from the relation $p = p_1 + p_{3'}$, from the equations of state of the water vapor and of the gases (in particular from the Klapeiron-Mendeleyev equation $p_1 = \frac{G_v R_v T_1}{V_v}$, $p_{3'} = \frac{G_r R_r T_1}{V_r}$), and from the obvious condition $V_v = V_g$. From the known temperature t_1 and from the obtained values of p_1 and $p_{3'}$ we determine the position of the points 1 and 3'.

The loss of useful work as a result of mixing of the gaseous combustion products and the water vapor (on portions 7-1 and 3"-3') amounts to

$$A\Delta l'_{см} = T_2(s_1 - s_7)$$

and

$$A\Delta l'^{\,\Gamma}_{\text{см}} = T_2 (s_{3'} - s_3).$$

The thermal efficiency of the steam-gas cycle is

$$\eta_t^{\text{п.г}} = \frac{(i_1^{\text{в}} - i_2^{\text{в}}) + (i_{3'}^{\Gamma} - i_{4'}^{\Gamma}) - (i_3^{\text{в}} - i_{2'}^{\text{в}}) - (i_{2'}^{\Gamma} - i_{1'}^{\Gamma})}{i_a^{\Gamma} - i_{2'}^{\Gamma}}. \qquad (10.10)$$

The effective efficiency of the steam-gas installation (without account of mechanical losses) is

$$\eta_e^{\text{пг}} = \frac{(i_1^{\text{в}} - i_2^{\text{в}})\,\eta_{\text{т}}^{\text{в}} + (i_{3'}^{\Gamma} - i_{4'}^{\Gamma})\,\eta_{\text{т}}^{\Gamma} - \dfrac{i_3^{\text{в}} - i_{2'}^{\text{в}}}{\eta_{\text{н}}^{\text{в}}} - \dfrac{i_{2'}^{\Gamma} - i_{1'}^{\Gamma}}{\eta_{\text{к}}^{\Gamma}}}{i_a^{\Gamma} - i_{2'}^{\Gamma}}\,\eta_{\text{кс}}. \qquad (10.11)$$

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[Footnotes]

287 The energy released in the fission of 1 kg of U^{235} is $23 \cdot 10^6$ kw·hr.

289 According to the materials of the Second International Conference on the Peaceful Use of Atomic Energy, the fuel component of the cost of generated electricity in an atomic electric station is 30-40%, as against 60% in a steam electric station.

297 Under this condition the parameters of the cycle will be determined on the Ts diagram with account of the loss of useful work due to the irreversible heat exchange between the mercury and the water.

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[List of Transliterated Symbols]

279 теор = teor = teoreticheskiy = theoretical
279 расш = rassh = rasshireniye = expansion
279 сж = szh = szhatiye = compression
280 т = t = turbina = turbine
280 н = n = nasos = pump
280 кот = kot = kotel = boiler
284 в = v = voda = water
284 рег = reg = regeneratsiya = regeneration
285 макс = maks = maksimal'nyy = maximum
286 опт = opt = optimal'nyy = optimum
286 то = to = teploobmennik = heat exchanger
287 р = r = reaktor = reactor
288 ср = sr = sredniy = average
288 родв = podv = podvod = supply
291 тс = ts = teplosila = heat power
293 потр = potr = potrebleniye = consumption

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[List of Transliterated Symbols (continued)]

297 р = r = rtut' = mercury

297 бин = bin = binarnyy = binary

299 г = g = gaz = gas

300 др = dr = drosselirovaniye = throttling

302 см = sm = smesheniye = mixing

303 п = p = par = steam

303 к = k = kompressor = compressor

303 кс = ks = kamera sgoraniya = combustion chamber

Chapter 11

THERMODYNAMICS OF CHEMICAL PROCESSES

§1. CHEMICAL EQUILIBRIUM

Chemical Reactions

Any mixture consisting of substances that react chemically with each other goes sooner or later as a result of the chemical reaction occurring in the mixture into a state of equilibrium, which is called chemical equilibrium. In the chemical equilibrium state, the amount of each of the substances participating in the reaction does not change with time. Any chemical reaction can occur both in the forward and in the reverse direction. Chemical equilibrium is a mobile or dynamic equilibrium, which is established when the rates of the direct and reverse reactions are equal. The state of chemical equilibrium is determined only by the parameters of the system and does not depend on the manner in which the reaction takes place. For the direct and reverse reactions, the state of the chemical equilibrium is one and the same.

Heat of Reaction

Chemical reactions are accompanied by the release or absorption of heat. In the former case, that is, upon release of heat, the reaction is called exothermal, in the latter case it is called endothermal. By thermal effect of a reaction is meant the amount of heat Q released (or absorbed) during a given reaction, when it occurs when two parameters of state $\underline{t}$ and V or $\underline{t}$ and $\underline{p}$ are constant, under the condition that there is no work connected with the change in volume, that is,

$L_V = 0$. The quantity Q is usually referred to 1 mole of the produced reaction products and is assumed positive if the heat is released during reaction and negative if the heat is absorbed.

According to the first law of thermodynamics, the heat of reaction at constant t and V should be equal to

$$Q_V = U_1 - U_2,$$

and at constant t and p it should be

$$Q_p = I - I_2.$$

Since U and I are functions of state, the heats of reaction Q_V and Q_p are completely determined by the initial and final states of the system and do not depend on the character of the process occurring during the reaction. It follows from this that the heat of a complicated reaction, consisting of several successive intermediate reactions or stages, does not depend on the intermediate stages through which the reaction has passed, and is equal to the sum of the heats of all the component reactions.

The independence of the heat of reaction on the path of the transition was established first by the Russian academician H.H. Hess (1840); it is called Hess' law. The thermal effect of the reaction, depending on the parameters of state, has a different value which generally is not equal to zero (including also when $T \to 0$). In order to find the temperature dependence of Q, let us differentiate the expressions for Q_V and Q_p with respect to T with V = const and p = const respectively:

$$\left(\frac{\partial Q_V}{\partial T}\right)_V = C_V^{\text{исх}} - C_V^{\text{кон}},$$

$$\left(\frac{\partial Q_p}{\partial T}\right)_p = C_p^{\text{исх}} - C_p^{\text{кон}}.$$

It follows therefore that the thermal effect increases or decreases with the temperature, depending on the sign of the difference of the

specific heats of the system C_V and C_p in the final and initial states. The specific heats C_V and C_p represent in these equations the total specific heats of the entire system, equal to the algebraic sum of the specific heats of all the reagents:

$$C_V = \sum C_V^{(i)}; \quad C_p = \sum C_p^{(i)}.$$

The thermal effects of the most important reactions are listed in Table 6.

TABLE 6

Standard Heat of Formation and Combustion of Several Compounds (at p = 1 atmosphere and t = 25°C)

1 Вещество	2 Q, ккал/кг·моль	
	3 образования	4 сгорания
5 C (алмаз)	+204	—
CO (газ) 6	+26416	—
CO_2 (газ)	+94052	—
H_2O (газ)	+57798	—
H_2O (жидкость) 7	+68317	—
NH_3 (газ)	10980	—
CH_4 (газ)	+17889	+212790
C_2H_2 (газ)	−54228	+310600
CH_3OH (жидкость)	+57000	+173600
C_2H_5OH (жидкость)	+66350	+326660

1) Substance; 2) Q, kcal/kg·mole; 3) formation; 4) combustion; 5) diamond; 6) gas; 7) liquid.

In addition to the release or absorption of heat as a result of chemical reaction, with suitable conditions a work L_V not involving a change in volume can occur. In this case the amount of heat released during the reaction will be smaller than the thermal effect of the reaction by the amount of the useful external work performed.

Chemical reactions lead to a change in the chemical energy of the reacting substances. The chemical energy of a given substance is equal to the heat of formation from its component parts (simple substances). The change in the chemical energy of some substance is

weight dG.

Chemical Affinity

The application of thermodynamics to chemical reactions makes it possible to establish the direction of the reaction and the conditions of chemical equilibrium. The main question which thermodynamics must answer first of all is to ascertain whether a given reaction is possible. Let us agree to call the aggregate of all the chemical and physical conditions which make possible the realization of the chemical reaction between several substances in a given system their chemical affinity. In other words, the chemical affinity is called the ability of different substances to combine with one another.

In order to establish a measure of chemical affinity, we recall that in any isothermal system, when equilibrium is established, the free energy or the isobaric potential of the system, depending on whether the volume or the pressure of the system remains constant, decreases and reaches a minimum in the state of equilibrium. Therefore, in any system which has constant values of t and V or t and p, only those chemical reactions are possible, which lead to a decrease in the free energy or the isobaric potential of the system.

Consequently, a measure of the chemical affinity of the substances that enter into reaction, in a system whose temperature t and volume V do not change, is the loss of free energy F of the system, that is, the difference of the values in the free energies of the system before and after the reaction $F_{iskh} - F_{kon}$, and in a system which is under a constant pressure p and has a constant temperature t it is the loss of the isobaric potential Φ of the system, that is, the difference in values of the isobaric potential of the system Φ before and after the reaction, $\Phi_{iskh} - \Phi_{kon}$.

But in a reversible isothermal-isochoric process the loss of

free energy of the system, and in the case of a reversible isothermal-isobaric process the loss of the isobaric potential of the system, is equal to the maximum useful external work AL' (see §6 of Chapter 3).

We can therefore state that a measure of the chemical affinity of substances participating in a reaction is the maximum useful work which can be performed on an external object of work as a result of the chemical reaction between these substances when this reaction is performed reversibly*.

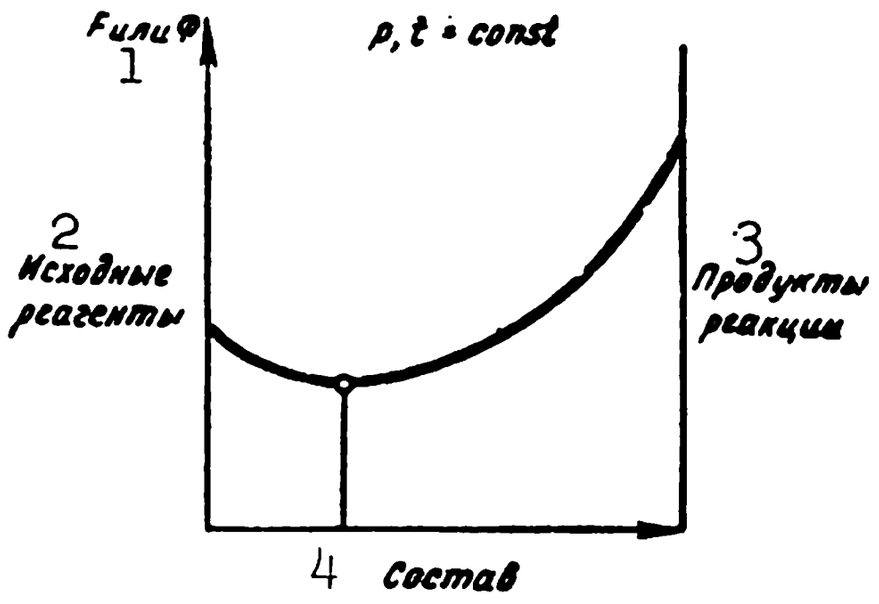

Fig. 134. Dependence of the free energy of the initial reacting substances on the composition of the mixture. 1) Or; 2) initial reagents; 3) products; 4) composition.

If the difference $F_{iskh} - F_{kon}$ or $\Phi_{iskh} - \Phi_{kon}$ is positive, then a direct reaction for specified t and V or t and p is realizable, and the ability of the substances to react with one another will be the larger, the greater the value of this difference; if $F_{iskh} - F_{kon}$ or $\Phi_{iskh} - \Phi_{kon}$ is negative, then the more probable would be the reverse reaction, but the direct reaction also takes place under these conditions. This is clear from the fact that F (or Φ) in the state of chemical equilibrium is smaller than F_{iskh}, F_{kon} (or Φ_{iskh}, Φ_{kon}) and therefore, if we take pure initial or final substances the reaction will occur between them.

The results obtained can be represented graphically (Fig. 134). For this purpose we mark the vertical left axis with the values of the free energy F or of the isobaric potential Φ of the initial reacting substances and the final reaction products (on the right-hand axis) for given constant values of t and V or t and p, while the horizontal axis is marked by the composition of the mixture of the reacting substances (the initial reagents and the reaction products).

The dependence of the free energy or of the isobaric potential on the composition of the mixture is represented by a certain curve which passes through a minimum. The point of minimum on this curve corresponds to chemical equilibrium, and the composition of the mixture at this point corresponds to the equilibrium composition of the mixture of the chemically reacting substances under given conditions. Other points of this curve characterize the nonequilibrium states of the mixture.

Thus, the state of equilibrium of a mixture of reacting substances is characterized by the fact that none of the concentrations of the reacting substances, both the initial ones and those formed as a result of the reaction, equals zero.

The change in the isobaric potential $\Phi_{kon} - \Phi_{iskh}$ as a result of formation of 1 mole of a given compound made up of simple substances, under the condition that the initial simple substances and the compound produced have a pressure of 1 atmosphere, while the reaction takes place at constant temperature of 25°C, is called the standard isobaric potential of a given substance and is designated by $\Delta\Phi^0_{298}$ (Table 7).

TABLE 7

Standard Isobaric Potentials of Substances

1 Вещество	$\Delta\Phi^0_{298}$, 2 ккал/моль	Вещество	$\Delta\Phi^0_{298}$, ккал/моль
CO_2 (газ) 3	−94,260	NH_3 (газ)	−3,94
H_2O (газ)	−54,635	NH_3 (жидкость)	−6,30
H_2O (жидкость) 4	−56,690	O_2 (газ)	−71,7
C_2H_2 (газ)	50,034		

1) Substance; 2) kcal/mole; 3) gas; 4) liquid.

Conditions of Chemical Equilibrium

After the possibility of realizing a reaction has become clear, it is necessary to establish the conditions for the chemical equili-

brium. Any chemical reaction is expressed by the stoichiometric formula

$$\sum \nu_j B_j = 0, \qquad (11.1)$$

which expresses the fact that different substances react chemically with one another in strictly equivalent quantitative relationships determined by their valence. Here B_1, B_2, ..., B_n are the chemical symbols of the reacting substances, while the coefficients ν_1, ν_2, ... ν_n are positive or negative integers, equal (accurate to a common constant factor) to the loss in the number of moles of the reacting substances:

$$\frac{M_1^{исх} - M_1}{\nu_1} = \frac{M_2^{исх} - M_2}{\nu_2} = \ldots = \frac{M_n^{исх} - M_n}{\nu_n}, \qquad (11.2)$$

where M_1^{iskh} is the initial number of moles (prior to the reaction) of the j-th substance, and M_1 is the number of moles of this substance after the reaction, that is, in the state of chemical equilibrium.

The state of chemical equilibrium does not depend on how the reaction took place; therefore in the derivation of the conditions for the chemical equilibrium one can make arbitrary assumptions concerning the course of the reaction. We assume for the sake of being definite that the reaction took place at constant temperature $\underline{t}$ and pressure $\underline{p}$. Then the isobaric potential of the system $\Phi(p, t, M_1, M_2, \ldots, Mn)$, which is a function of the temperature $\underline{t}$, of the pressure $\underline{p}$, and of the number of moles M_1, M_2, ..., M_n of all $\underline{n}$ reacting substances, will have in the equilibrium state a minimum, that is, $d\Phi = 0$, or, inasmuch as $dp = 0$ and $dt = 0$, while all dM_j are proportional to ν_j, we have

$$\sum \left(\frac{\partial \Phi}{\partial M_j}\right)_{p,t} \nu_j = 0. \qquad (11.3)$$

Equation (11.3) is the sought condition for chemical equilibrium.

§2. LAW OF EFFECTIVE MASSES

Equilibrium Constant

The condition for chemical equilibrium (11.3) and the n – 1 equa-

tions (11.2) contain as knowns the quantities t, p, M_1^{iskh}, M_2^{iskh}..., M_n^{iskh}, along with the n unknown values of the moles M_1, M_2, ..., M_n of the reacting substances in the state of chemical equilibrium. Thus, the thermodynamic equations determine completely the chemical equilibrium and, in particular, the concentration of the reacting substances in the equilibrium state. This, however, is all that thermodynamics can yield. Thermodynamics is incapable of determining the state of the reaction or other kinetic characteristics; these questions are the concern of chemical kinetics.

In order to calculate from Equations (11.2) and (11.3) the concentrations or the number of moles M_1, M_2, ..., M_n of the reacting substances in chemical equilibrium, it is necessary to know the expression for the isobaric potential Φ of the system as a function of t, p, and the number of moles of each substance participating in the reaction. Let us consider the simplest case, when the reacting substances are ideal gases. The isobaric potential of a mixture of ideal gases under a pressure p and having a temperature t (with C_p = const) has the form

$$\Phi = \sum [(C_{p.\mu}^{(j)} T + U_\mu^{0.(j)}) + T(C_{p.\mu}^{(j)} \ln T - A\mu R \times \ln p^{(j)} + S_\mu^{0.(j)})] M_j.$$

Bearing in mind that $p^{(j)} = z^{(j)} p$, we get

$$\Phi = \sum \Phi_\mu^{(j)}(p, T) M_j + A\mu RT \sum M_j \ln z^{(j)},$$

where $\Phi_\mu^{(j)}(p, T)$ is the molar thermodynamic potential of the j-th gas contained in the mixture and taken at a pressure equal to the total pressure of the mixture p and its temperature t.

From this we obtain

$$\left(\frac{\partial \Phi}{\partial M_i}\right)_{p.t} = \Phi_\mu^{(i)}(p, T) + A\mu RT(1 + \ln z^{(i)}) A\mu RT \sum z^{(i)}$$

or, inasmuch as $\sum z^{(i)} = 1$,

$$\left(\frac{\partial \Phi}{\partial M_i}\right)_{p.t} = \Phi_\mu^{(i)}(p, T) + A\mu RT \ln z^{(i)}. \qquad (11.4)$$

Substituting the obtained value of $(\partial\Phi/\partial M_j)_{p,t}$ into Equation (11.3) we obtain after some transformations

$$\sum \nu_j \ln z^{(j)} = -\frac{1}{A\mu RT}\sum \nu_j \Phi_\mu^{(j)}(p, T). \qquad (11.5)$$

The right half of this equation does not depend on the concentration of the reacting substances and is constant for given p and t; it is denoted ln K, where K is the constant of chemical equilibrium:

$$\ln K = -\frac{1}{A\mu RT}\sum \nu_j \Phi_\mu^{(j)}(p, T). \qquad (11.6)$$

From this equation and from the known values of $\Phi_\mu^{(j)}$ and ν_j we can calculate the value of K for the given reaction. Upon introduction of the constant K, Equation (11.5) assumes the form

$$[z^{(1)}]^{\nu_1}[z^{(2)}]^{\nu_2}\ldots[z^{(n)}]^{\nu_n} = K(p, t). \qquad (11.7)$$

Equation (11.7) expresses the law of effective masses. According to this law, in the state of chemical equilibrium the product of the concentrations of the reacting substances, taken through the powers of ν, is a quantity that depends only on the temperature and pressure (or volume).

With increasing equilibrium constant K, the concentration of the initial products increases, while the concentration of the final products of the reaction decreases, that is, the reaction shifts in the direction of the decomposition of the final products and increasing the concentration of the initial products. To the contrary, when the equilibrium constant decreases, the reaction shifts toward the direction of formation of the end products – the "yield" of the reaction increases.

Equation (11.7) for the law of effective masses can be recast in somewhat different form, if it is recognized that in a mixture of ideal gases $z^{(j)} = p^{(j)}|p$. Then

$$[p^{(1)}]^{\nu_1}[p^{(2)}]^{\nu_2}\ldots[p^{(n)}]^{\nu_n} = K', \qquad (11.8)$$

with

$$K' = Kp^{\Sigma \nu_l}.$$

In a state that differs from equilibrium under the given conditions (p and t or V and t), we have $[z^{(1)}]^{\nu}1[z^{(2)}]^{\nu}2 \ldots [z^{(n)}]^{\nu}n \neq K$, and, as follows from the equilibrium conditions, if $[z^{(1)}]^{\nu}1 \ldots [z^{(n)}]^{\nu}n > K$, then the direct reaction occurs, and if $[z^{(1)}]^{\nu}1 \ldots [z^{(n)}]^{\nu}n < K$ the reverse reaction occurs.

Dependence of the Equilibrium Constant on the Pressure and Temperature

For ideal gases the dependence of the equilibrium constants on the pressure and temperature is determined by Equation (11.5). It follows from this that

$$\left(\frac{\partial \ln K}{\partial p}\right)_t = -\frac{1}{A\mu RT}\sum \nu_j \left(\frac{\partial \Phi_\mu^{(j)}}{\partial p}\right)_t. \qquad (11.9)$$

But $\left(\frac{\partial \Phi_\mu^{(j)}}{\partial p}\right)_t = AV_\mu^{(j)}$, and furthermore inasmuch as the value of $\Phi_\mu^{(j)}$ pertains to the pressure p and the temperature t, we have $V_\mu^{(j)} = \mu RT/p$.

Therefore

$$\left(\frac{\partial \ln K}{\partial p}\right)_t = -\frac{1}{p}\sum \nu_l. \qquad (11.10)$$

Analogously

$$\left(\frac{\partial \ln K}{\partial T}\right)_p = \frac{1}{A\mu RT^2}\sum \nu_j \Phi_\mu^{(j)} - \frac{1}{A\mu RT}\sum \nu_j \left(\frac{\partial \Phi_\mu^{(j)}}{\partial T}\right)_p.$$

Since

$$\left(\frac{\partial \Phi_\mu^{(j)}}{\partial T}\right)_p = -S_\mu^{(j)};\ I - TS = \Phi,$$

we have

$$\left(\frac{\partial \ln K}{\partial T}\right)_p = \frac{1}{A\mu RT^2}\sum \nu_j I_\mu^{(j)}.$$

Inasmuch as t and p do not change during the course of the reaction, and there are two substances in the equilibrium state, both the initial ones and those formed as a result of the reaction, then the sum $\sum \nu_j I_\mu^{(j)}$, as can be readily verified, is equal to the difference in the enthalpies of the mixture I_{iskh} and I_{kon} before and after the

reaction, respectively. But $I_{iskh} - I_{kon}$ is the thermal effect Q_p for a given reaction; consequently

$$\left(\frac{\partial \ln K}{\partial T}\right)_p = \frac{Q_p}{A\mu RT^2}. \qquad (11.11)$$

It follows from Equation (11.10) that with increasing pressure at t = const, the chemical equilibrium constant, depending on the sign of the sum $\Sigma\nu_j$, can either increase or decrease or else remain unchanged. If the reaction proceeds with a decrease in the total number of moles (for example, the reaction of combining hydrogen with oxygen to form water vapor $2H_2 + O_2 - 2H_2O = 0$), then $\Sigma\nu_j > 0$ and $(\partial \ln K/\partial p)_t < 0$. This means that with increasing pressure the amount of formed substances or end products of the reaction (H_2O) will increase. If the reaction occurs with an increase in the total number of moles (for example, dissociation of ammonia $2NH_3 - N_2 - 3H_2 = 0$) then $\Sigma\nu_j < 0$ and $(\partial \ln K/\partial p)_t > 0$, that is, an increase in pressure leads to a decrease in the amount of produced reaction products (N_2 and H_2). In a reaction in which the total number of moles does not change (for example, $H_2 + Cl_2 - 2HCl = 0$), we have $\Sigma\nu_j = 0$ and $(\partial \ln K/\partial p)_t = 0$. It follows therefore that the constant of chemical equilibrium does not depend on the pressure.

An increase or a decrease in the temperature at constant pressure leads, according to (11.11), to a change in the value of the constant of chemical equilibrium, the change being the greater, the larger the thermal effect of the reaction. If $Q_p < 0$ (endothermal reaction) then $(\partial \ln K/\partial T)_p < 0$ and the equilibrium constant decreases with increasing temperature. To the contrary, for an exothermal reaction, $(\partial \ln K/\partial T)_p > 0$, that is, the equilibrium constant increases with the temperature.

TABLE 8

Chemical Constants of Different Gases

1 Газ	t	1 Газ	t
CH_4	−1,941	H_2O	−1,860
CO	−0,075	H_2	−1,500
CO_2	+0,850	N_2	−0,153
H_2	−3,690	O_2	+0,547

1) Gas.

A determination of the constants of chemical equilibrium of different reactions and a calculation of the chemical equilibrium constitute a most important problem in chemical thermodynamics. From Equation (11.6), which, recognizing that

$$I_\mu = \int_{0,p}^{T,p} T\left(\frac{\partial S_\mu}{\partial T}\right)_p dT + I_\mu^0,$$

can be rewritten in the form

$$\ln K = \frac{1}{A\mu RT}\left[\sum_i \nu\left(TS_\mu^i - \int_{0,p}^{T,p} T\left(\frac{\partial S_\mu^{(i)}}{\partial T}\right)_p dT\right) - \sum \nu_i I_\mu^{0,(i)}\right],$$

we see that to determine the equilibrium constant K as a function of p and t it is necessary to know the absolute values of the thermodynamic potentials of the reacting substances (initial reagents and end products of the reaction) for the given conditions, or the absolute values of the entropy of the reacting substances and the values of their enthalpy at T = 0. Inasmuch as I^0 enters into the expression for K in the form of the complex $\sum \nu_i I_\mu^{0,(i)} = I_{\text{исх}}^0 - I_{\text{кон}}^0$, which represents the thermal effect of the reaction under consideration at T = 0, it is sufficient to know in place of I^0 at T = 0 only the last quantity, that is, Q_p^0.

We note that sometimes in place of the absolute values of the entropy one uses in the calculation of the chemical equilibrium of ideal gases the so-called chemical constants ξ of the gases:

$$\xi = \frac{1}{A\mu R}(S_\mu^0 - C_{p,\mu}).$$

The values of the chemical constants for different gases are listed in Table 8.

Maximum Useful Work of Reaction

The maximum useful work which can be performed on an external

work object as a result of a reversible chemical reaction occurring at t = const, p = const is equal to the difference between the isobaric potentials of the system in the initial state and in the equilibrium state, while for t = const, V = const it is equal to the loss in the free energy.

If the reaction occurs between ideal gases, then, using the previously presented expression for Φ, we obtain after a few transformations

$$AL'^{\text{макс}}_{p.\,t} = \sum (M_j^{\text{исх}} - M_j^{\text{кон}}) (\Phi'_\mu + A\mu RT \ln z^{(j)}) + \\ + A\mu RT \sum (M_j^{\text{исх}} \ln z^{(j)}_{\text{исх}} - M_j^{\text{кон}} \ln z^{(j)}_{\text{кон}}).$$

But, as was shown above,

$$\Phi^{(j)}_\mu + A\mu RT \ln z^{(j)} = \left(\frac{\partial \Phi^{(j)}_\mu}{\partial M_j}\right)_{pt},$$

and in conformity to the condition for chemical equilibrium (9.3)

$$\sum (M_j^{\text{исх}} - M_j^{\text{кон}}) \left(\frac{\partial \Phi^{(j)}_\mu}{\partial M_j}\right)_{pt} = 0.$$

Therefore,

$$AL'^{\text{макс}}_{p.\,t} = A\mu RT \sum (M_j^{\text{исх}} \ln z^{(j)}_{\text{исх}} - M_j^{\text{исх}} \ln z^{(j)}_{\text{кон}}). \qquad (11.12)$$

If the reacting substances are taken in stoichiometric ratios in the initial state, then upon their complete transformation, when $M_j^{iskh} = \nu_j$, we have

$$AL'^{\text{макс}}_{p.\,t} = A\mu RT \sum \nu_j (z^{(j)}_{\text{исх}} - z^{(j)}_{\text{кон}}). \; *$$

A reaction occurring at t = const, V = const, we have

$$AL'^{\text{макс}}_{t.\,V} = A\mu RT \sum \nu_j (z^{(j)}_{\text{исх}} - z^{(j)}_{\text{кон}}) - AV (p_{\text{исх}} - p_{\text{кон}}). \; * \qquad (11.13)$$

An example of a reversible reaction at constant t and p is a reaction occurring in an electric battery between the electrolyte and the substance of the positive electrode at small current in the closed circuit, when the Joule heat can be neglected. From the maximum voltage of the cell we can determine the maximum work, and conse-

quently the loss of the isobaric potential in the given reaction.

§3. SOLUTIONS

Properties of Binary Solutions

A mixture of several substances or components is called a solution. If the amount of one of the components is much larger than the amount of the others, this component is called the solvent. The composition of a solution is characterized by its concentration (by weight c or molar z), which determines the relation between the components comprising the solution.

The solubilities of different substances vary. Gases at high pressures have unlimited solubility, that is, they mix in all ratios. Liquids are dissolved in each other at all ratios, or, which is encountered most frequently, only in definite ratios. There exist liquids, for example kerosene and water, which are practically insoluble in each other. In the last case each of the components of the mixture is independent, since the presence of the other component does not influence its properties. Solids are dissolved in liquids to a limited degree.

A solution having the maximum concentration of the dissolved substance possible under the given conditions is called a saturated solution. A saturated solution is in equilibrium with the pure dissolved substance. The solubility of one substance in another, and consequently the concentration of the saturated solution, depend on the temperature and pressure of the solution.

The dissolution of matter is accompanied by the release or absorption of a certain amount of heat. Most frequently absorption of heat is observed during the dissolution. The thermal effect of dissolution and the solubility of one substance in another are connected by the following relation, which follows from general thermodynamic

considerations. If the thermal effect of dissolution is positive (when heat is released upon dissolution) the solubility decreases with increasing temperature; if the thermal effect is negative, the solubility, to the contrary, increases with increasing temperature.

Let us consider the properties of binary liquid solutions. The state of a binary solution is determined by three independent variables, which can be chosen to be, for example, the pressure of the solution p, its temperature t, and the concentration of the dissolved substance c. From the general equilibrium conditions it follows that the values of the temperature and chemical potentials of the dissolved substance should be equal in the contiguous phases in the state of equilibrium. Since the state of the binary solution is characterized not by two but by three independent parameters, the quantitative relationship between p and t in phase equilibrium will be different in the presence of a dissolved substance than for the pure solvent. In particular, in the dissolution of a nonvolatile substance, the vapor pressure of the solvent over the solution drops, the boiling temperature of the solution increases, and the freezing temperature of the solution drops.

In weak or dilute binary solutions, the pressure of the saturated vapor of the solvent p on the solution is always smaller than the pressure p^0 of the saturated vapor on the pure solvent, by an amount zp^0 where z is the molar concentration of the solution

$$p^0 - p = zp^0. \tag{11.14}$$

Equation (11.14) expresses Raoult's law, according to which the relative decrease in the saturated-vapor pressure on dissolution is equal to the molar concentration of the solution.

The boiling temperature t_s of a solution is always higher than that of the pure solvent t_s^0. The increase in the boiling temperature

of the solution for a given external pressure is directly proportional to the molar concentration of the dissolved substance

$$t_s - t_s^0 = z\frac{ART^2}{r}. \qquad (11.15)$$

The same equation determines also the decrease in the freezing temperature (we recall that the heat of solidification r is negative).

The tc Diagram

An investigation of the boiling and freezing of solutions of arbitrary concentrations is best carried out with the aid of a diagram of the state of the solution, on which the coordinate axes are the values of the temperature t and the composition (concentration c or z) of the solution for p = const.

For the boiling of a solution of two liquids that mix with each other in all ratios, the diagram of state has the form shown in Fig. 135. The shaded area bounded by the curve ABCD represents the region of the two-phase state, in which the substance is in the form of liquid solution and saturated vapor in equilibrium with it. The region lying above the curve ABC is the vapor region, while the region below the curve ADC is the region of liquid solution. The compositions of the liquid and vapor phases, which are in equilibrium, are represented by different curves – the curve ABC, which pertains to the saturated vapor, and the curve ADC, which pertains to the liquid phase. The points of intersection of the horizontal lines with the curves ABC and ADC, which bound the region of equilibrium existence of the two phases, determine the composition of the phases, into which the solution decomposes at given p and t. Let us take any point inside the region ABCD of the two-phase state, for example, the point E, corresponding to concentration c_1 of the dissolved substance I. At the point E the substance consists of $G^{(1)}$ kg of the

liquid phase, the state of which is characterized by the point E' with concentration $c_1^{(1)}$, and $G^{(2)}$ kg of saturated vapor whose condition is characterized by the point E" and concentration $c_1^{(2)}$. The relation between the quantities of both phases, as is the case in equilibrium between a pure liquid and its saturated vapor, is equal to

$$\frac{G^{(1)}}{G^{(2)}} = \frac{\text{segment } E''E}{\text{segment } E'E}.$$

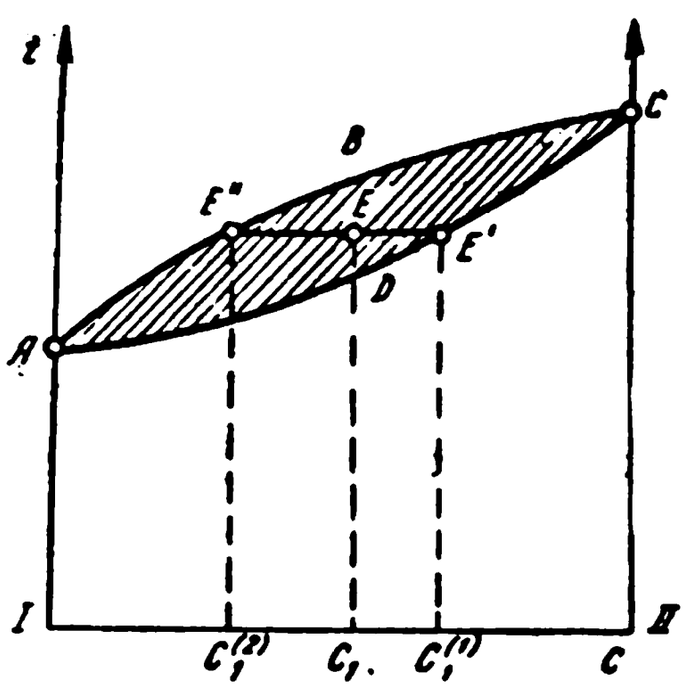

Fig. 135. Diagram of state of boiling of an ideal solution of two liquids.

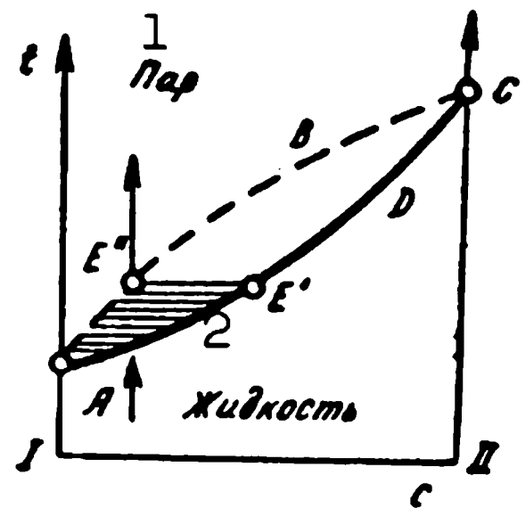

Fig. 136. Boiling of a solution in a closed vessel. 1) Vapor; 2) liquid.

In saturated vapor (for example, at the point E"), the concentration $c_2^{(2)}$ of substance II, which has a higher boiling temperature at a given pressure (less volatile compared with the substance I), is smaller than its concentration $c_2^{(1)}$ in the liquid phase, which is in equilibrium with the vapor (E'). In other words, in liquid solution the vapor is relatively richer in that component, the addition of which to the mixture increases the total vapor pressure (the first rule of Konovalov).

Let us assume now that a solution contained in a closed vessel is heated in such a way that the produced vapor is always above the solution (it is not removed) (Fig. 136). The process of heating the liquid occurs at constant concentration of the solution and is represented by a vertical line. Upon reaching the boiling temperature, at

the point where the vertical line crosses the lower boundary curve ADC, boiling of the liquid will begin, and the vapor produced, as can be seen from the diagram of state, will have a lower concentration of substance II, that is, it will be richer in the volatile component I, than the liquid initially taken. As a result, the concentration of component I in the liquid decreases, while the concentration of the dissolved substance II increases, that is, the content of the higher-boiling-point component in the liquid increases, which causes an increase in the boiling temperature. Thus, unlike the boiling of a pure liquid, the temperature of a liquid solution continuously increases during boiling and the point representing the state of the liquid phase shifts along the lower boundary curve upward to the point E'. Since the boiling process occurs without removal of vapor (at a constant total composition), the ratio of the amount of vapor to the liquid will continuously increase, until, at a certain temperature which is higher than the temperature corresponding to the start of the boiling, the entire liquid turns into vapor and the boiling ends (the point E"). At that instant the concentration of the vapor reaches the concentration of the initial solution, and later on upon heating only an increase in the temperature of the vapor will take place, without a change in its composition. Upon cooling of the vapor under the same conditions, the process will be in reversed order. The lower curve ADC, along which the state of the liquid solution varies during boiling, is called the boiling curve, while the upper curve ABC is called the condensation curve.

If the boiling of the liquid solution occurs in an open vessel (the vapor is continuously removed), then as the liquid continues to boil it will become enriched with the dissolved substance, since the vapor is poorer in the dissolved substance than the liquid. This effect

can be used to separate liquid mixtures. By combining a sequence of evaporation and condensation processes it is possible to separate completely a mixture of liquids. Such a separation of liquid mixtures is called rectification.

Dissolution of Gases in Liquids and Solids

We have considered so far solutions consisting of the same phases of different substances, for example, liquid solutions formed from two liquids, but very frequently it becomes necessary to determine the dissolution of various gases in liquids or in solids.

In solutions of gases in liquids or in solids, the concentration of the gas is directly proportional to the pressure of the gas over the solution. This relation is called the Henry law. The Henry law is valid only for weak solutions. Gases which dissolve well in a liquid do not satisfy this law.

Solid Solutions

Not only liquids can be dissolved in each other, but also in some cases solids. Solutions of solids are called solid solutions.

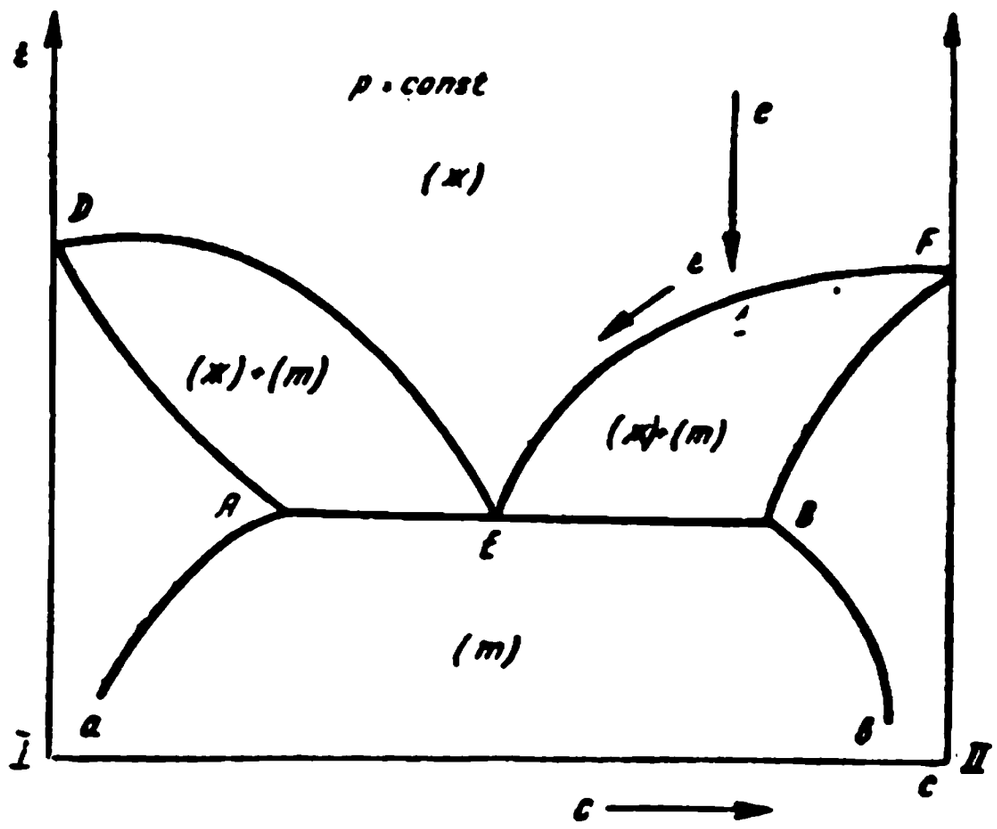

Fig. 137. tc diagram for a solid solution having an immiscibility zone. m = solid; ж = liquid.

Figure 137 shows the tc diagram for a solid solution having an

immiscibility zone. The line DEF represents the solidification curve, and the line DAEBF the melting curve. Above the solidification curve is the region of homogeneous liquid solution, while below the melting curve is the region of solid solution. To the left of the line DAa is the region of homogeneous solid solution with predominance of component I, while to the right of the line FBb is the region of homogeneous solution with predominance of the component II. In the region between the solidification and melting curves the system consists of a liquid and solid phase; on the line AEB the system consists of three phases – a liquid phase (point E) and two solid phases (points A and B). Point E is called the eutectic point, and the solution corresponding to it is called a eutectic solution. Upon cooling the liquid phase (for example, along the vertical line f), crystals of the substance I, precipitated from the solution, begin to form at the point ε. As a result of this, the liquid phase becomes richer in the substance II, and further cooling leads to a decrease in the temperature and a displacement along the solidification line toward the point E. At the same time, crystals are continuously produced of the first substance, until the point E is reached, at which, along with the first crystals, crystals of the second substance begin to be formed. The temperature no longer decreases until the remaining liquid is completely solidified; a mixture of crystals of both types is thus produced.

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[Footnote]

309 When the parameters t and V or t and p are constant, a reversible reaction can be realized by using catalysts or semipermeable partitions, or by carrying out the reaction in an electric cell.

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[Translator's Note]

317 The errata slip attached to the foreign original indicates that the letter symbol z in Equation (11.13) and the display preceding it should be prefaced by the symbol for natural logarithm.

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[List of Transliterated Symbols]

306 исх = iskh = iskhodnyy = initial

306 кон = kon = konechnyy = final

317 макс = maks = maksimum = maximum

323 ж = zh = zhidkost' = liquid

K.D. Voskresenskiy

PART TWO

HEAT TRANSFER

INTRODUCTION

Heat transfer studies processes involved in heat transport. There are three mechanisms for this transport: heat conductivity, convection, and thermal radiation.

Heat conductivity occurs only in an unevenly heated substance. Convection always occurs during motion of a substance which is heated more strongly or more weakly than the surrounding stationary medium. Thermal radiation is characterized by the transformation of the heat of a heated substance into radiant energy, which propagates, encounters the second substance, and is again converted into heat.

In technical structures, as a rule, heat transport is carried out usually by two or three methods simultaneously. Thus, for example, in the coolant stream of an active channel in a reactor, the convection is accompanied by heat conduction, and in some cases also by radiant heat exchange. Such a joint process is called heat transfer. This complicated phenomenon is subdivided into simpler ones and the laws governing each of them are investigated.

The transport of heat is a non-equilibrium thermodynamic process, which proceeds with a finite speed under specific space-time conditions. Therefore heat transfer is a branch of the thermodynamics of irreversible processes, which is presently under vigorous development.

Processes of heat transport play an important role in new technology in general and in nuclear power in particular. Thus, for example, in modern reactors it is necessary to carry away some millions of large calories from a square meter per hour. Under these

conditions the calculation of the heat transfer is a very responsible stage of the design.

Heat transfer as a science in clear-cut form has developed approximately during the last 30 years, although individual branches of it were created much earlier. The formation of heat transfer followed the development of heat engineering and physics. Scientists of many countries participated in the creation of this science. The contribution of Soviet scientists to the development of heat transfer is characterized by such accomplishments as the development of a method for thermal simulation, the creation of the theory of thermal boundary layer, the investigation of heat transfer during boiling and condensation, the creation of a theory of radiant heat exchange between bodies, the development of methods for the regular mode, etc. [20].

In studying the processes of heat transfer in nuclear-power installations, the phenomenological method of investigation is used principally. This method consists of using the fundamental laws of physics and enrolling a few additional considerations (hypotheses) concerning the course of the process, and also an experimental determination of the heat-physics parameters of the substances, so that it becomes unnecessary to study the microscopic mechanism of heat transfer.

The use of this method leads to differential or integral equations of heat transfer. These equations are solved analytically or numerically in simple cases, and in more complicated cases they are used for the determination of criteria of physical similarity, the connection between which is established as a result of an experimental study of the process. Thus, heat transfer is characterized by a combination of theoretical and experimental investigation techniques.

The processes of heat transport in nuclear-power installations

are characterized by a great variety of conditions under which they occur. However, all these processes reduce to the three transport mechanisms that are studied in this course.

The present text is intended for the first study of the principles of heat transfer in higher institutions of learning that offer special courses in nuclear power. It is a supplement to the textbook by M.A. Mikheyev "Fundamentals of Heat Transfer." The text contains information necessary for preparation for study of the special methods of calculating heat transfer in reactors.

An attempt is made here to present heat transfer in a form where the phenomenological method is consistently employed in this branch of knowledge. Such non-traditional exposition leads to a unified logical structure of the entire course, facilitates mastery of the subject, and presents a general point of view on various phenomenological sciences.

Another feature of this text is the joint utilization of the principles "from the general to the particular" and "from the particular to the general." For example, methods of dimensionality and similarity are first demonstrated by means of simple heat-conduction phenomena, and are then treated in more general form, after which they are finally used to study convective heat exchange.

Relatively circumstantial attention is paid in the text to the mathematical formulation of the problems that arise in the study of heat transport. This is caused both by the need for defining the similarity conditions and by the need to expand the possibility of applying computational mathematics for the study of heat transfer.

Some heat-exchange problems have been explained here anew. These include an approximate estimate of the limits of the region of applicability of the equivalent diameter, an account of the tempera-

ture dependence of heat-physical parameters of the liquid phase upon condensation of vapor, etc.

As far as we know, this is the first text of its kind. Indications of shortcomings will be received with gratitude.

The author mentions with gratitude the valuable advice and the interesting statements made concerning the field of heat transfer by M.V. Kirpichev (1879-1955). They exerted an influence on the content of this work.

The text was written while the author was at the Ch'ing-Hua Polytechnic Institute (Peking, China). The administration and the dean's office of the engineering-physics faculty of this institute provided all conditions for the performance of this work within a short time.

The author therefore dedicates this work to the Ch'ing-Hua Polytechnic Institute.

Chapter 1

FORMULATION OF THE FUNDAMENTAL PROBLEMS OF THE THEORY OF HEAT CONDUCTION

§1. THE FOURIER DIFFERENTIAL EQUATION

Phenomenological Method of Investigating Heat-Conduction Processes

The phenomenological method of investigating natural phenomena in general and heat-conduction processes in particular is characterized by the following features.

1. Matter is regarded as a continuous uninterrupted medium (continuum), and therefore its microstructure and the microscopic mechanism of the process drop out of consideration. This simplification greatly facilitates the study of the process.

2. To formulate a mathematical description of the investigated phenomenon, the fundamental laws of physics are used. In particular, to describe the processes of heat conduction use is made of the first and second laws of thermodynamics. However, this yields differential equations in which the number of unknowns exceeds the number of equations. The reason for it is that the fundamental laws of physics do not reflect all the particular features of the microscopic mechanism of the process, inherent in the specific substance.

3. To set up a closed system of differential equations, hypotheses are advanced regarding additional connections between the sought quantities. Such hypotheses occur as a result of direct observations of the phenomenon. This is the reason for the name of the method. In the theory of heat conduction, such a hypothesis was advanced by the French

scientists Biot and Fourier (start of the 19th century). However, these supplementary connections between the sought quantities contain proportionality coefficients, and unless these are determined the system of equations remains unclosed.

4. The values of the proportionality coefficients in the additional equations obtained as a result of involving the hypotheses are determined by experiment. These coefficients are called the physical parameters, if their values depend only on the pressure and temperature for the given substance.* In this case the hypothesis is called a law. In the theory of heat conduction, such a law is the Biot-Fourier law. Thus, the experimental determination of the physical parameters of the substance offsets the failure to consider the microscopic mechanism of the process.**

The obtained closed system of differential equations is supplemented by the uniqueness conditions and is solved either by methods of mathematical physics, or by experimental methods of similarity and analogy.

The phenomenological method can be used only in those cases when the geometrical dimensions of the volume occupied by the process are large compared with the characteristic dimensions of the microstructure (for example, compared with the mean free path of the gas molecules). This condition limits the region of application of the results obtained by the phenomenological method.

Main Concepts and Limitations of Theory of Heat Conduction

Following the first feature of the phenomenological method, we introduce the main concept of the theory of heat conduction without the use of the characteristics of the microstructure of the matter.

Heat conduction is a process of heat transport in a nonuniformly heated substance which can proceed if the heat is not

converted into other forms of energy (the latter can be converted into heat) and if the hotter and the cooler parts of the substance are stationary relative to each other.

Such processes of "pure" heat conduction occur in opaque solid bodies with small coefficients of thermal expansion.

It is customary to describe heat conduction processes with the aid of a temperature field, i.e., a function $t = t(x, y, z, \tau)$.

A temperature field is called stationary if it does not depend on the time. A field is called nonstationary if the temperature depends on the time. Experiment shows that the temperature distribution has the properties of a continuous scalar field. Consequently, the isothermal surfaces do not cross one another and do not become discontinuous inside the body. Experiment has shown also that the process of heat conduction occurs only in nonuniformly heated bodies. To characterize this nonuniformity one introduces the concept of a temperature gradient.

The temperature gradient "grad t" is the rate of increase of the temperature along a normal to the isothermal surface at a given point of the body and at a given instant of time.

The temperature gradient is therefore a vector, directed along the normal to the isothermal surface toward the hotter parts of the body (Fig. 1):

$$\operatorname{grad} t = \frac{\partial t}{\partial x} \cdot i + \frac{\partial t}{\partial y} \cdot j + \frac{\partial t}{\partial z} \cdot k.$$

In addition, the following quantities are used in the theory of heat conduction:

The heat flow $\tilde{Q}$ in kcal is the amount of heat that passes within a certain time τ (hours) through a surface of F m^2; the heat flux $\tilde{Q}$ in kcal/hour is the flow of heat per unit time; the heat flux density $\vec{q}$

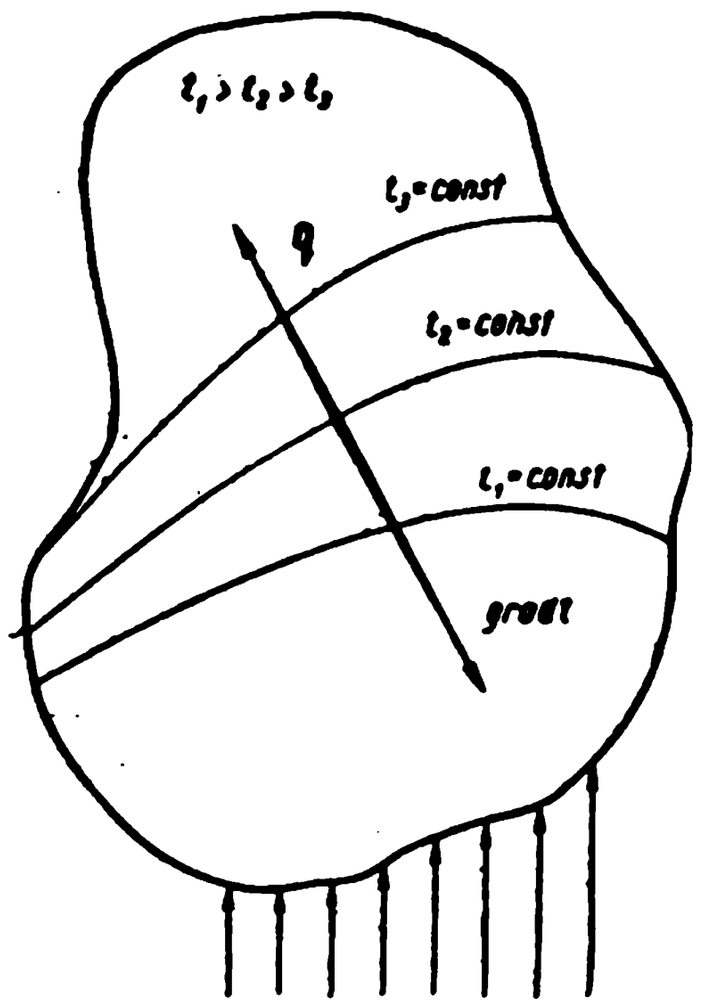

Fig. 1. Location of the vectors $\vec{q}$ and grad $\underline{t}$ in an isotropic body.

<u>in kcal/m^2·hr</u> is the heat flux through a unit surface. The heat flux density is a vector quantity, since it depends on the orientation of the elementary area for which it is determined. The vector $\vec{q}$ is a particular case of the energy flux density vector, which was first introduced (in 1873) by N.A. Umov [51].

The vector $\vec{q}$ is defined by the following equation:

$$dQ = (q, dF).$$

The modulus of this vector $|\vec{q}| \equiv q$ is called <u>the heat load of the surface</u>. In accordance with the second law of thermodynamics, the vector $\vec{q}$ is always directed toward the less hot parts of the body. If the body is isotropic, then the vector $\vec{q}$ is directed along the normal to the isothermal surface.*

<u>The volume density of heat sources q_V in kcal/m^2-hr</u> is the amount of heat which is released ($q_V > 0$) or absorbed ($q_V < 0$) per unit volume of substance and per unit time.

To register heat fluxes we shall henceforth use the Euler reference system: we choose in the volume occupied by the process an elementary stationary and permeable control surface, on which the observer is located. This observer registers the heat fluxes through the control surface.

We shall henceforth use the following limitations:

1) all the characteristics of the heat-conduction process in the specified volume are continuous functions of the coordinates and of the time;

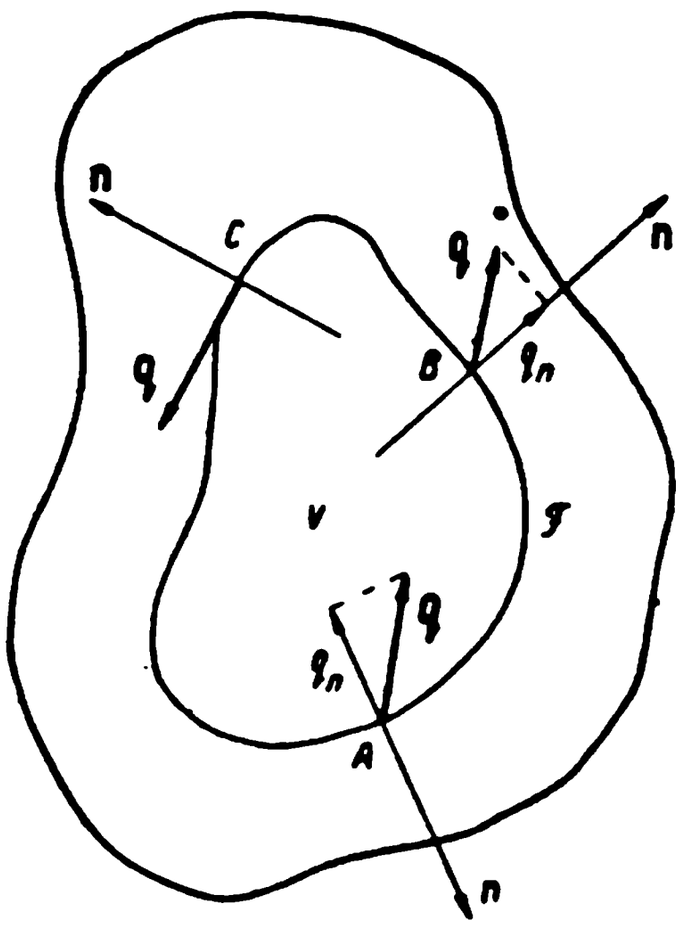

Fig. 2. Scheme of separating an arbitrary volume V with the aid of a control surface F.

2) the energy lost in thermal deformations in the body is negligibly small compared with the changes in the internal energy, i.e., the heat-conduction process occurs with v = const*;

3) there is no heat conduction or radiant heat exchange.

First and Second Laws of Thermodynamics for Heat-Conduction Processes

Following the second feature of the phenomenological method, we write down the expressions of the first and second laws of thermodynamics for the heat-conduction process. For this purpose we consider a nonuniformly heated inhomogeneous and anisotropic body. For the sake of being definite we assume that the temperature at all its points increases in the course of time and that there are active in this body nonnegative sources of heat ($q_V \geq 0$).

With the aid of the control surface F we isolate an arbitrary** volume V. We discard the part of the body that is external with respect to the volume, and we replace its thermal action on the remainder by a certain distribution of the vector $\vec{q}$ on the surface F (Fig. 2). In the general case we have on this surface points of three types: A, B, and C. In points of type A the heat enters the volume and the projection of $\vec{q}$ on the normal $\vec{n}$ is negative. At the points B the heat leaves the body and q_{nb} is positive, while at points C the heat does not pass through the control surface.

Let us set up an equation for the first law of thermodynamics for the volume V over a small time interval $d\tau$:

$$d\tilde{Q}_F + d\tilde{Q}_V = dU + dL. \qquad (1.1)$$

The volume V receives heat $d\tilde{Q}_F$ through the surface F and $d\tilde{Q}_V$ from the internal sources of heat. The heat obtained is consumed in increasing the internal energy dU and in the performance of mechanical work dL by the system.

In accordance with the limitations assumed above, we consider an isochoric heat-transport process. Therefore

$$dL = 0.$$

The quantity $d\tilde{Q}_F$ is equal to the algebraic sum of the heats passing through portions F_a, F_b, and F_c of the control surface F:

$$d\tilde{Q}_F = d\tilde{Q}_{F_a} - d\tilde{Q}_{F_b} - d\tilde{Q}_{F_c}.$$

In turn

$$d\tilde{Q}_{F_a} = d\tau \cdot \int_{F_a} (\mathbf{q}, d\mathbf{F}) = -d\tau \cdot \int_{F_a} q_n \cdot dF;$$

$$d\tilde{Q}_{F_b} = d\tau \cdot \int_{F_b} q_n \cdot dF;$$

$$d\tilde{Q}_{F_c} = d\tau \cdot \int_{F_c} q_n \cdot dF = 0.$$

Therefore

$$d\tilde{Q}_F = -d\tau \cdot \int_{F_a} q_n \cdot dF - d\tau \cdot \int_{F_b} q_n dF - d\tau \cdot \int_{F_c} q_n \cdot dF.$$

Since

$$F_a + F_b + F_c = F,$$

we have

$$d\tilde{Q}_F = -d\tau \cdot \oint_F q_n \cdot dF.$$

In accordance with the Ostrogradskiy theorem

$$\oint_F q_n \cdot dF = \int_V \operatorname{div} \mathbf{q} \cdot dV,$$

therefore

$$d\tilde{Q}_F = -d\tau \cdot \int_V \operatorname{div} \mathbf{q} \cdot dV. \qquad (1.2)$$

From the definition of the volume density of the internal heat sources it follows that

$$d\tilde{Q}_V = d\tau \cdot \int_V q_V \cdot dV. \qquad (1.3)$$

Let us calculate the increment of the internal energy dU in the volume V during the time dτ:

$$dU = \int_V du \cdot dV.$$

where $\underline{u}$ is the internal energy per unit volume of the body and dV is a stationary volume element. It is known from thermodynamics that the internal energy of the body is a function of the parameters of its thermodynamic state

$$u = u(T, v).$$

Consequently,

$$du = \left(\frac{\partial u}{\partial T}\right)_v \cdot dT + \left(\frac{\partial u}{\partial v}\right)_T \cdot dv.$$

We are considering here an isochoric process of heat conduction, therefore $(\partial u/\partial v)_T \cdot dv = 0$. In addition, the quantity $(\partial u/\partial T)_v$ is (by definition) the isochoric specific heat per unit volume:

$$(\partial u/\partial T)_v \equiv C_v \text{ kcal/m}^3\text{-deg}.$$

Therefore

$$dU = C_v \cdot dT = C_v \cdot \frac{\partial T}{\partial \tau} \cdot d\tau \;;\; dU = d\tau \cdot \int_V C_v \cdot \frac{\partial T}{\partial \tau} dV. \qquad (1.4)$$

Substituting the quantities dL, $d\tilde{Q}_F$, and $d\tilde{Q}_V$ in (1.1), we obtain

$$d\tau \cdot \int_V \left(C_v \cdot \frac{\partial T}{\partial \tau} + \operatorname{div} \mathbf{q} - q_V\right) dV = 0.$$

We have assumed above that the volume V is arbitrary, and all the characteristics of the process are continuous functions of the coordinates. Under these conditions the last integral is equal to zero, if the integrand is equal to zero. Consequently,

$$C_v \cdot \frac{\partial T}{\partial \tau} + \operatorname{div} \mathbf{q} - q_V = 0. \qquad (1.5)$$

We thus obtain a differential equation for the first law of ther-

modynamics for an isochoric heat-transport process without its conversion into other forms of energy.

Equation (1.2) is a particular case of the Umov differential equation for energy transport processes [51]. It contains five unknown quantities: the temperature T, the three components of the vector $\vec{q}$, and the specific heat C_v.

The expression of the second law of thermodynamics for heat-conduction processes was found by Planck [43] in the form of the following inequality

$$(q, \operatorname{grad} T) < 0.$$

Planck's inequality indicates that the angle φ between the vectors $\vec{q}$ and grad T lies in the interval $\pi/2 < \varphi < 3\pi/2$. Consequently, the vector $\vec{q}$ always is directed toward the cooler parts of the body.

Thus, the use of the first and second laws of thermodynamics does not enable us to obtain a complete mathematical description of the heat-conduction process: the first law leads to one equation with five unknowns, while the second indicates only the direction of the process and therefore does not yield a supplementary equation.

The Biot-Fourier Law

Following the third feature of the phenomenological method, we use a hypothesis concerning an additional connection between the unknown quantities T and $\vec{q}$. Such a hypothesis was proposed by Biot and Fourier [63]:

the heat flux density vector $\vec{q}$ due to the heat conduction, at a given point of a nonuniformly heated body and at a given instant of time, is directly proportional to the temperature gradient vector grad T at the same point of the body and at the same instant of time

$$q \sim \operatorname{grad} T.$$

Numerous experiments have confirmed the correctness of this hy-

pothesis. It has been established that the proportionality coefficient for each given substance has a unique value (within the limits of measurement accuracy), provided a stable thermodynamic state of this substance is specified. Consequently, it has been established by experiment that the coefficient of proportionality is a physical parameter of the substance. This physical parameter characterizes the ability of the substance to conduct heat. It is therefore called the coefficient of heat conduction and is usually designated λ (in kcal/m-hr-deg).

The hypothesis advanced by Biot and Fourier is at present a law which is written in the form of the equation

$$q = -\lambda \operatorname{grad} T. \qquad (1.6)$$

The physical parameters are customarily considered as positive quantities. In order to obtain this requirement and take into account the opposite directions of the vectors $\vec{q}$ and grad T, a "minus" sign is found in the left side of Eq. (1.6). At the present time no case has been observed where the law of Biot and Fourier was violated for heat-conduction processes that occur under normal conditions. This does not mean, however, that such a case will not be observed in the future.

Equation (1.5) for the first law of thermodynamics and the Biot-Fourier law (1.6) form a system consisting of four equations with six unknowns: $\underline{t}$, q_x, q_y, q_z, λ, C_v.

The Coefficient of Heat Conduction

Following the fourth feature of the phenomenological method, one determines by experiment the heat-conduction coefficients and the specific heats for each substance as a function of the parameters of each thermodynamic state (usually the temperature and pressure). There are several known methods for an experimental determination of the heat-conduction coefficient [55]. Most of them are based on measuring the

absolute values of the vectors $\vec{q}$ and grad T in a given substance, to which a definite geometric form is imparted. Then the heat-conduction coefficient is determined from the ratio of these absolute values:

$$\lambda = \frac{|q|}{|\text{grad } t|} \text{ kcal/m-hr-deg.}$$

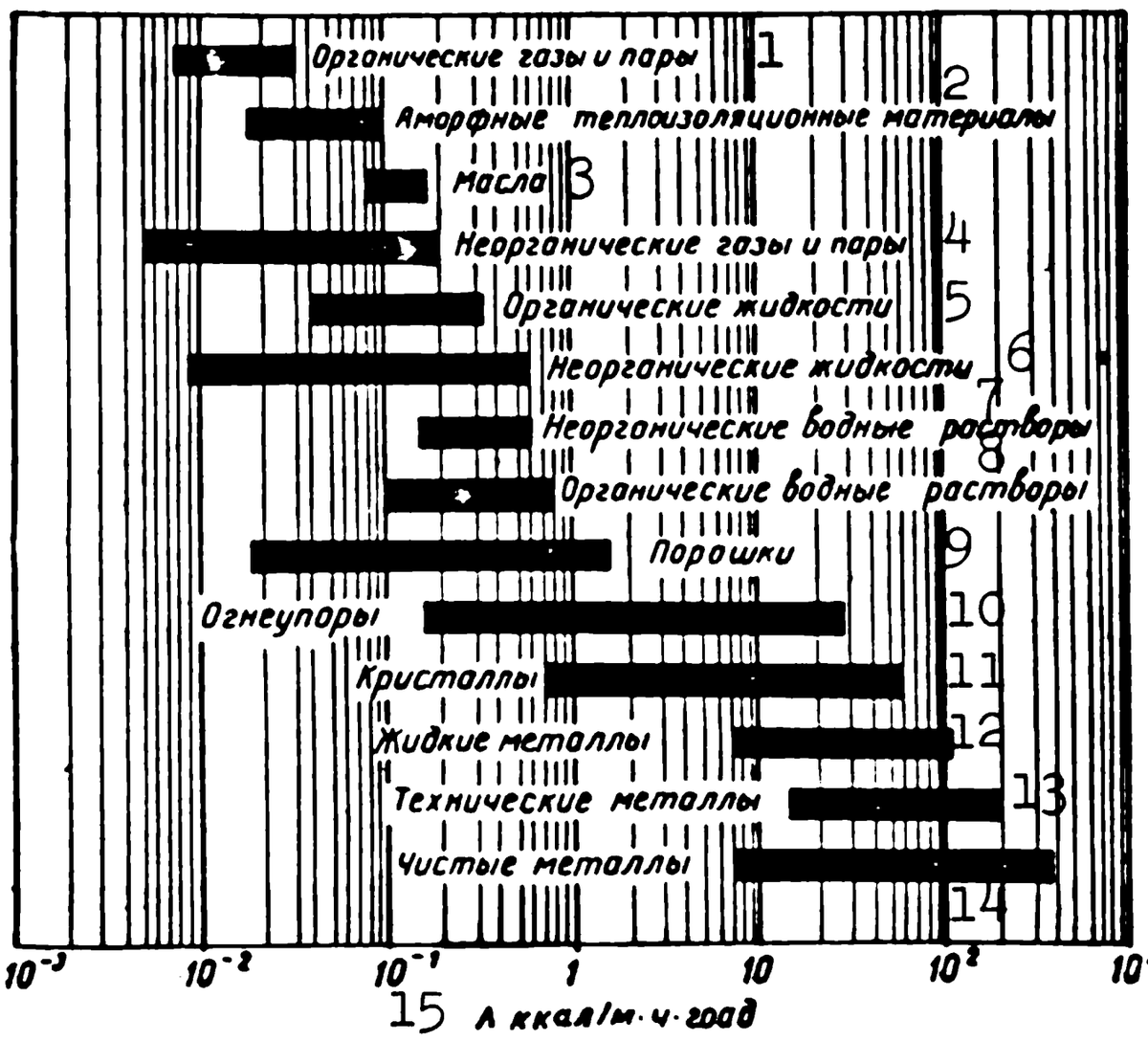

Fig. 3. Order of magnitude of the heat-conduction coefficient of different substances as given by [73]. 1) Organic gases and vapors; 2) amorphous heat-insulating materials; 3) oils; 4) inorganic gases and vapors; 5) organic liquids; 6) inorganic liquids; 7) inorganic aqueous solutions; 8) organic aqueous solutions; 9) powders; 10) refractories; 11) crystals; 12) liquid metals; 13) technical metals; 14) pure metals; 15) λ, kcal/m-hr-deg.

The results of the measurements are summarized in tables [7, 54], which are used in calculations of heat-conduction processes.

The order of magnitude of λ for different substances is shown in Fig. 3 [73].

For gases, the value of λ ranges from $5 \cdot 10^{-3}$ (chloroform vapor) to 0.5 (H_2 at high temperature) and rises with increasing temperature, being practically independent of the pressure over a wide range (from

0.3 to $2 \cdot 10^3$ kg/cm^2).

For liquids this quantity varies from $8 \cdot 10^{-3}$ (oil) to 0.6 (H_2O) and as a rule decreases with increasing temperature, being practically independent of the pressure.

For metals the value of λ ranges from 7 (Bi) to 360 (Ag), slowly decreasing with increasing temperature, being practically independent of the pressure (for Cu $\lambda \approx 340$; for Fe $\lambda \approx 50$; for Al $\lambda \approx 200$).

In inhomogeneous bodies the coefficient of heat conduction depends on the coordinates of the point, while in anisotropic bodies (for example in graphite) it depends on the direction from the given point. In anisotropic bodies there are three directions along which the heat-conduction coefficient reaches extremal values. These directions are called the principal axes of conductivity. If the principal axes of conductivity are taken to be the coordinate axes x_i ($i = 1, 2, 3$), then the Biot-Fourier law for anisotropic bodies assumes the following form [56]:

$$q_i = -\lambda_i \frac{\partial t}{\partial x_i}.$$

In the general case $\lambda_1 \neq \lambda_2 \neq \lambda_3$.

The Fourier Differential Equation

Equations (1.5) and (1.6) form a closed system, which contains four unknown quantities: t, q_x, q_y, q_z. Since the heat-conduction process is customarily described with the aid of a temperature field, one usually eliminates from these equations the vector $\vec{q}$.* Substituting (1.6) in (1.5) we obtain the Fourier differential equation

$$C_v \cdot \frac{\partial t}{\partial \tau} = \operatorname{div} \lambda \operatorname{grad} t + q_v. \qquad (1.7)$$

Equation (1.7) describes isochoric processes of heat conduction in anisotropic and inhomogeneous bodies, the physical properties C_v and λ of which. and also the strength of the internal sources q.. vary

continuously within the volume occupied by the process. In the general case C_v, λ, and q_V depend on the temperature.** Therefore Eq. (1.7) is nonlinear and its integration by analytical means entails great difficulties. In practice, however, one encounters frequently homogeneous and isotropic bodies, in which the specific heat C_v and the heat conductivity λ can be approximately regarded as constant quantities $C_v \approx$ $\approx$ const, λ = const. This means that in the specified temperature interval one can neglect the variation of the quantities C_v and λ as compared with their mean values in this interval (Fig. 4):

$$\Delta C_v \ll C_{v\,cp} \text{ and } \Delta\lambda \ll \lambda_{cp}.$$

In this case the Fourier equation simplifies appreciably

$$\frac{\partial t}{\partial \tau} = a_v \cdot \nabla^2 t + \frac{q_V}{C_v}. \qquad (1.8)$$

The quantity a_v (m^2/hr) is called <u>the isochoric coefficient of temperature conductivity</u>. This physical parameter characterizes the ability of the substance to equalize the temperature. The coefficient of temperature conductivity ranges from $5 \cdot 10^{-4}$ for oils to 0.7 for silver.

Equation (1.8) is a linear partial differential equation of second order, of the parabolic type.

For anisotropic bodies the Fourier equation assumes the following form:

$$C_v \frac{\partial t}{\partial \tau} = \sum_{i=1}^{i=3} \frac{\partial}{\partial x_i} \lambda_i \frac{\partial t}{\partial x_i} + q_V. \qquad (1.9)$$

If the values of λ_i and C_V in the anisotropic body do not depend on the temperature, then Eq. (1.9) can be reduced, by means of the transformation

$$x_i = x_i' \cdot \sqrt{a_v}$$

to the form (1.8). Here x'_i are the new values of the coordinates [52].

In those cases when it is necessary to find the field of the vec-

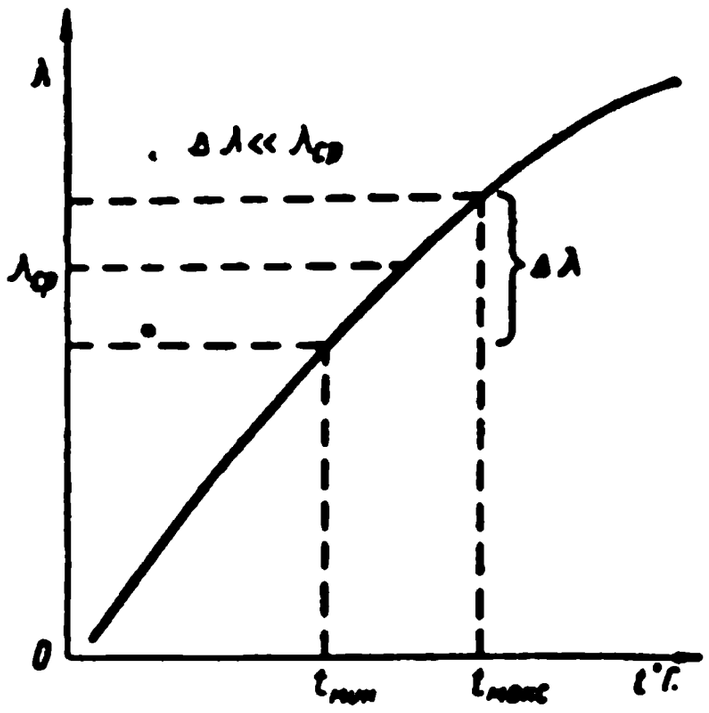

Fig. 4. Approximate condition for the constancy of the heat-conduction coefficient.

tor $\vec{q}$, it is necessary to eliminate the temperature from Eqs. (1.5) and (1.6). This can be readily done for constant values of the physical parameters λ and C_v and under the condition that the strength of the internal sources of heat q_V does not depend on the temperature. Taking the gradient of (1.5), we obtain

$$C_0 \cdot \frac{\partial}{\partial \tau} \operatorname{grad} t = \operatorname{grad} \operatorname{div} \mathbf{q} + \operatorname{grad} q_V.$$

In addition it follows from the Biot-Fourier law that

$$\operatorname{grad} t = -\frac{\mathbf{q}}{\lambda}.$$

Substituting the last equation in the preceding one and transforming the latter, we obtain the vector equation

$$\frac{1}{a_0} \frac{\partial \mathbf{q}}{\partial \tau} = \operatorname{grad} \operatorname{div} \mathbf{q} - \operatorname{grad} q_V, \qquad (1.10)$$

which describes the field of the vector $\vec{q}$ in homogeneous and isotropic bodies.

§2. UNIQUENESS CONDITIONS FOR THE HEAT-CONDUCTION PROCESSES

Purpose and Makeup of the Uniqueness Conditions

Each heat-conduction process occurs in a specific environment, i.e., in specific conditions of space and time, in a definite substance; it is in contact with neighboring processes, which exert definite influence on it. However, the Fourier equations do not take into account these particular features, since the fundamental laws of physics reflect only general features that are characteristic of all heat conduction processes. For this reason, the Fourier equation describes a whole set of processes and has a whole set of solutions. In order to separate from this set the specific process, it is necessary to formulate unique-

ness conditions, which contain the geometrical, physical, time, and boundary conditions [14].

The geometrical conditions in the form of numbers and equations determine the form and dimensions of the body in which the investigated process takes place, and also the orientation of the principal axes of conductivity (for anisotropic substances) relative to the surface of the body.

The physical conditions determine the numerical values of all the physical parameters which exert an influence on the investigated process, i.e., which enter into the Fourier equations and the boundary conditions. In addition, the physical conditions give an indication of the presence of internal sources, their sign, magnitude, and distribution.

The time conditions indicate the features with which the process varies in time (for example, whether it is stationary or nonstationary, the initial temperature distribution, etc.).

The boundary conditions determine the features of the process on the boundaries of the volume occupied by it and on the interaction between the investigated process and the neighboring processes, which exert an influence on it. If for specified uniqueness conditions there exists a unique and stable solution, it is stated that the problem is correctly formulated. It is necessary to note that the generally accepted formulation of problems of the phenomenological theory of heat conduction leads to an arbitrarily large rate of heat propagation [40].

Boundary Conditions for Heat-Conduction Processes

The boundary conditions determine the features of the course of the process on the boundaries of the volume occupied by it. In practice one encounters many such singularities. But if we disregard radiant heat exchange, they are all particular cases of the conditions of

matching of the investigated temperature field 1 and the neighboring temperature field 2. The matching of these fields is carried out on their contact surface (Fig. 5) with the aid of the energy conservation law

$$\lambda_1 \cdot \frac{\partial t_1}{\partial n}\Big|_{n=-0} = \lambda_2 \cdot \frac{\partial t_2}{\partial n}\Big|_{n=+0}. \qquad (2.1)$$

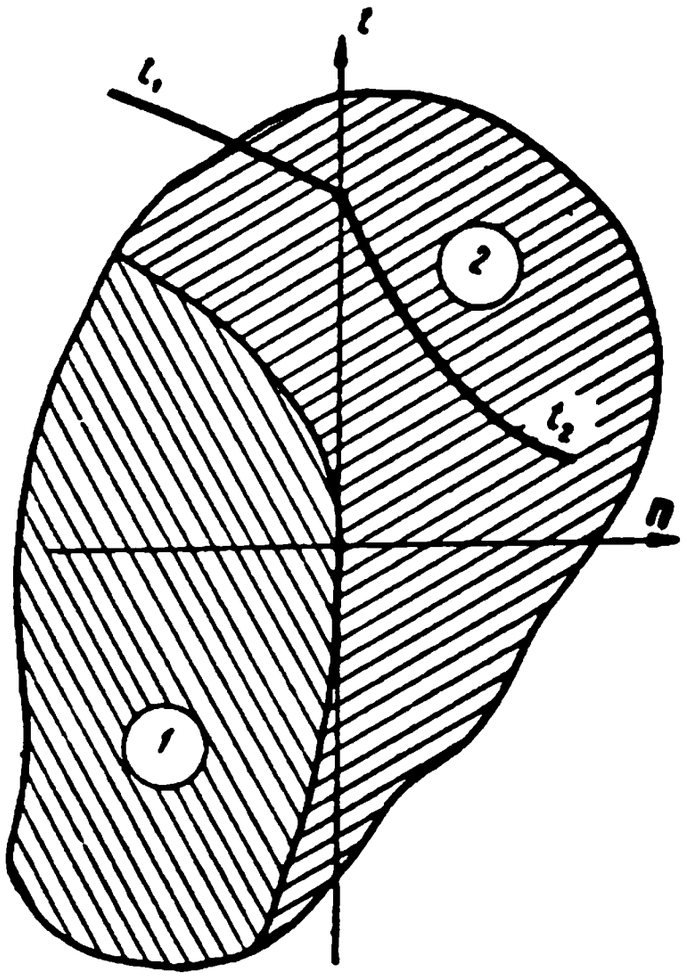

Fig. 5. Diagram showing the matching of two neighboring temperature fields.

The second matching condition for the two temperature fields follows from the continuity of these fields on the surface F

$$t_1|_{n=-0} = t_2|_{n=+0}. \qquad (2.2)$$

The matching conditions (2.1) and (2.2) lead to three kinds of boundary conditions which are most frequently encountered in practice and which have been most completely investigated in mathematical physics.

The first kind of boundary conditions determines the distribution of the temperature on the surface of the body, i.e., in this case one specifies the form of the function

$$t_F = t_F(l_F, \tau). \qquad (2.3)$$

It is necessary here to determine the temperature field inside the body, i.e., to determine the function $t = t(\vec{l}, \tau)$.

The second type of boundary conditions establishes the distribution of the heat load q_F on the surface of the body, i.e., specifies the form of the function

$$-\lambda_F \frac{\partial t}{\partial n}\Big|_{n=-0} = q_F(l_F, \tau). \qquad (2.4)$$

To determine the temperature field in the case under consideration it is necessary to know at least one value of the temperature itself at any point of the body. In addition, the distribution of the

heat load q_F on the surface of the body F should satisfy the equation of the first law of thermodynamics, written in integral form

$$\oint_F q_F \cdot dF = \int_V \left(q_V - C_\sigma \frac{\partial t}{\partial \tau}\right) dV. \qquad (2.5)$$

For a stationary process, Eq. (2.5) simplifies greatly and assumes the following form:

$$\oint_F q_F dF = \int_V q_V dV.$$

The third kind of boundary conditions determine the simplest linear heat-exchange condition between a solid body and a stream of liquid which either heats or cools this body:

$$-\lambda_F \cdot \frac{\partial t}{\partial n}\Big|_{n=0} = \alpha (t|_{n=0} - t_{zh}). \qquad (2.6)$$

The quantity α (kcal/m^2-hr-deg) is called the <u>heat-transfer coefficient</u>. This quantity characterizes the intensity with which heat is transported from the solid surface to the stream of liquid or gas. The value of α in technical installations usually does not exceed 10^5 kcal/m^2-hr-deg.

In Eq. (2.6) the specified quantities are λ_F, the heat conductivity coefficient of the body, α, the heat-transfer coefficient, and t_{zh}, the temperature of the liquid. It is usually assumed that the quantities α and λ_F do not depend on the temperature and that the condition (2.6) is linear.

We note that the boundary condition of the third kind follows from the condition of matching of the temperature fields in the solid body and in the liquid flowing around it [see Eq. (2.1)]

$$-\lambda_F \frac{\partial t}{\partial n}\Big|_{n=0} = -\lambda_{zh} \frac{\partial t_{zh}}{\partial n}\Big|_{n=+0}.$$

Indeed, multiplying and dividing the right half of this relation by the temperature differential $t|_{n=0} - t_{zh}$, we obtain

$$-\lambda_F \frac{\partial t}{\partial n}\Big|_{n=-0} = \frac{-\lambda_{ж} \frac{\partial t_{ж}}{\partial n}\Big|_{n=+0}}{t|_{n=0} - t_{ж}} (t|_{n=0} - t_{ж}).$$

The quantity

$$\frac{-\lambda_{ж} \frac{\partial t_{ж}}{\partial n}}{t|_{n=0} - t_{ж}}$$

does not depend on the temperature if the physical parameters of the single-phase liquid do not depend on the temperature (a proof of this statement is given in Chapter 6). Therefore this quantity has been designated by a special name (the heat-transfer coefficient) and a separate symbol (α). Then the condition of matching assumes the form of the boundary condition of the third kind

$$-\lambda_F \frac{\partial t}{\partial n}\Big|_{n=-0}^{\bullet} = \alpha (t|_{n=0} - t_{ж})$$

or

$$q_F = \alpha (t_F - t_{ж}). \qquad (2.7)$$

Equation (2.7) is called the Newton-Rikhman law: <u>the heat load on the surface of a body in a stream of liquid is directly proportional to the difference in temperature between the surface of the body and the liquid</u>.

The Newton-Rikhman law is satisfied only when the thermal-physical properties of the single-phase liquid do not depend on the temperature. In the opposite case there is no direct proportionality between q_F and $t_F - t_{zh}$.

<u>Logical Scheme of the Theory of Heat Conduction</u>

The scheme of the phenomenological theory of heat conduction is shown in Fig. 6. Four stages of the utilization of the phenomenological method make it possible to obtain the Fourier differential equation. This equation describes a set of heat-conduction processes and therefore has a set of solutions. One adds to the Fourier equation the

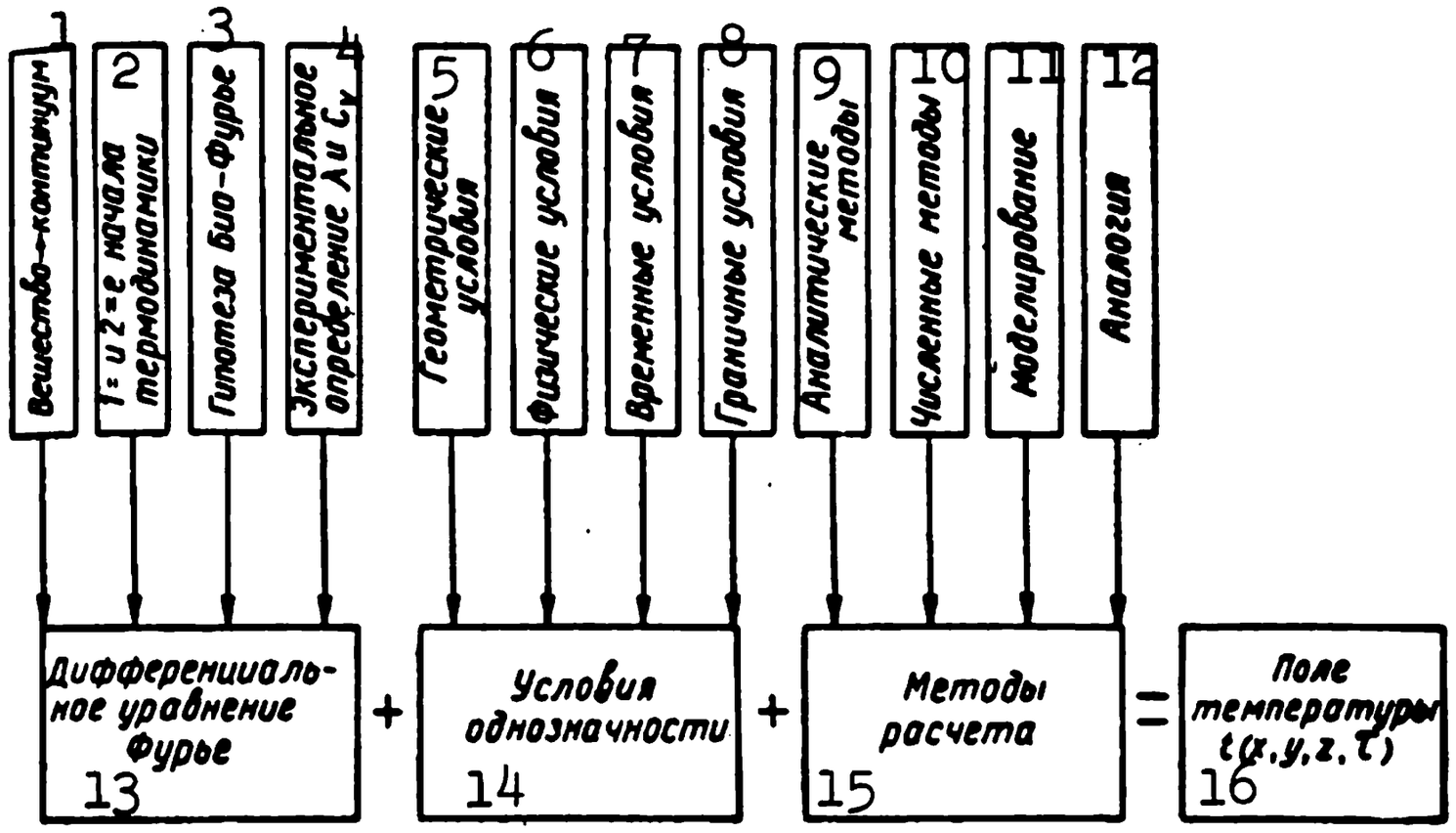

Fig. 6. Logical scheme of the phenomenological theory of heat conduction. 1) Substance → continuum; 2) first and second laws of thermodynamics; 3) the Biot-Fourier hypothesis; 4) experimental determination of λ and C_v; 5) geometrical conditions; 6) physical conditions; 7) time conditions; 8) boundary conditions; 9) analytical methods; 10) numerical methods; 11) simulation; 12) analogy; 13) Fourier differential equation; 14) uniqueness conditions; 15) calculation methods; 16) temperature field $t(x, y, z, \tau)$.

geometrical, physical, time, and boundary uniqueness conditions. The problem thus formulated is solved either analytically, or numerically, or by experimental methods. In the latter case use is made of methods of physical similarity [14; 23; 59] or physical analogies [13; 21]. In the theory of heat conduction analytical [34] and numerical methods of solution [6b] are used relatively more frequently.

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[Footnotes]

332 The physical parameters of anisotropic bodies depend also on the direction from the given point, and those of inhomogeneous bodies on the coordinates of this point.

332 The values of the physical parameters can be determined by calculating the microscopic mechanism of the phenomenon, if this mechanism has been investigated in sufficient detail.

334 In anisotropic bodies the angle φ between the vectors $\vec{q}$ and grad $\underline{t}$ is not equal to two right angles [52].

335 This assumption is well satisfied for the majority of solids.

335 The idea of deriving the equations of mathematical physics for a volume of arbitrary form was proposed by M.V. Ostrogradskiy [39].

341 From Eqs. (1.5) and (1.6) we can eliminate the temperature and obtain a differential equation for the field of the vector $\vec{q}$. For more details see below.

342 The second parameter of the thermodynamic state of matter drops out, since we are considering an isochoric heat-conduction process.

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[List of Transliterated Symbols]

342 ср = sr = sredniy = mean

343 мин = min = minimum = minimum

343 макс = maks = maksimum = maximum

346 ж = zh = zhidkost' = liquid

Chapter 2

SIMPLEST STATIONARY HEAT-CONDUCTION PROCESSES

§3. HEAT CONDUCTION IN A THIN PLATE

Boundary Conditions of the First Kind

Let us consider a stationary temperature field in a thin isotropic plate (Fig. 7), on the surfaces of which temperatures t_{F_1} and $t_{F_2} < t_{F_1}$ are maintained. In the plate there are effective internal sources of heat of strength q_V kcal/m^3-hr. The temperature field in the plate is described by the Fourier equation

$$C_v \frac{\partial t}{\partial \tau} = \operatorname{div} \lambda \operatorname{grad} t + q_V,$$

which encompasses a whole set of heat-conduction processes and has therefore a whole set of solutions.

In order to separate the heat-conduction process in the plate, it is necessary to formulate the uniqueness conditions.

The geometrical conditions determine the thickness of the plate δ and indicate that the plate is thin. This means that its thickness is sufficiently small (compared with the height and width) so that the ends do not exert a noticeable influence on the temperature field and on the flow of heat through the plate. This condition is approximately satisfied if $\delta \leq 0.1H$ and if the temperature on the ends is contained in the interval $t_{F_2} \leq t \leq t_{F_1}$. When these conditions are satisfied the temperature field in the plate can be regarded with sufficient accuracy as one-dimensional. The Fourier equation assumes in this case the following form:

$$C_v \frac{\partial t}{\partial \tau} = \frac{\partial}{\partial x} \lambda \frac{\partial t}{\partial x} + q_V.$$

The physical conditions indicate that the plate has been made of isotropic material, the physical properties of which are specified functions of the temperature

$$C_v = C_v(t) \text{ and } \lambda = \lambda(t).$$

In the plate there are uniformly distributed heat sources with constant strength q_V kcal/m^3-hr.

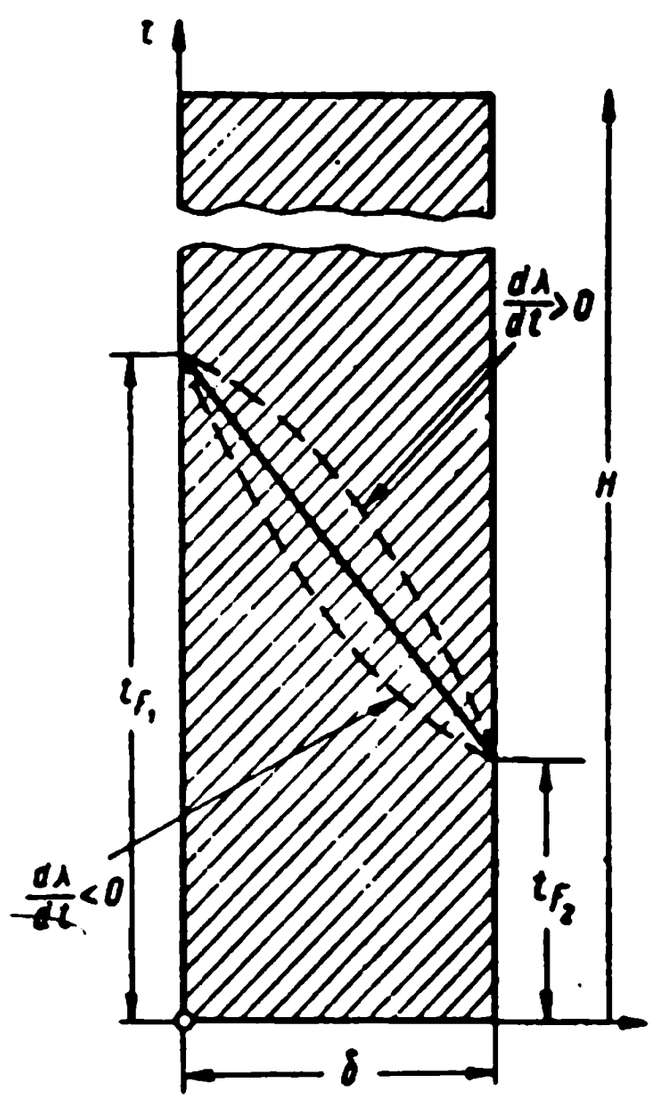

Fig. 7. Temperature distribution in a thin plate for boundary conditions of the first kind.

The time conditions indicate that the temperature field in the plate does not depend on the time. In this case $\partial t/\partial \tau = 0$ and the Fourier equation assumes the simple form

$$\frac{d}{dx}\left[\lambda(t)\frac{dt}{dx}\right] + q_V = 0. \qquad (3.1)$$

The boundary conditions of the first kind specify constant values of the temperature on the surfaces of the plate

$$\left.\begin{aligned} t|_{x=0} &= t_{f_1}; \\ t|_{x=\delta} &= t_{f_2}. \end{aligned}\right\} \qquad (3.2)$$

Thus, the calculation of the temperature field in the plate reduces to a solution of a nonlinear total differential equation (3.1) for Boundary Conditions (3.2).

To simplify the problem we introduce the Kirchhoff variable [64]

$$\vartheta \equiv \int_{t_{f_2}}^{t(x)} \lambda(t)\, dt. \qquad (3.3)$$

Differentiating (3.3) with respect to x we obtain

$$\frac{d\vartheta}{dx} = \lambda(t)\frac{dt}{dx}.$$

Substituting the obtained relation into (3.1) and introducing the

Kirchhoff variable into the boundary conditions (3.2), we obtain the linear problem

$$\left.\begin{aligned} &\frac{d^2\vartheta}{dx^2}+q_V=0;\\ &\vartheta|_{x=0}=\int_{t_{F_1}}^{t_{F_2}}\lambda(t)\,dt=\vartheta_{F_1};\\ &\vartheta|_{x=\delta}=0. \end{aligned}\right\} \qquad (3.4)$$

The solution of the problem (3.4) has the following form:

$$\vartheta=\vartheta_{F_1}-\frac{q_V x^2}{2}+x\left(\frac{q_V\delta}{2}-\frac{\vartheta_{F_1}}{\delta}\right).$$

Substituting the values of ϑ and ϑ_{F_1} in the last equation we obtain

$$\int_{t_{F_2}}^{t(x)}\lambda(t)\,dt=\int_{t_{F_2}}^{t_{F_1}}\lambda(t)\,dt-\frac{q_V x^2}{2}+x\left(\frac{q_V\delta}{2}-\right.$$
$$\left.-\frac{1}{\delta}\int_{t_{F_2}}^{t_{F_1}}\lambda(t)\,dt\right). \qquad (3.5)$$

Equation (3.5) describes the temperature field in a plate with variable heat-conduction coefficient, internal heat sources, and boundary conditions of the first kind.

Let us consider particular cases of this equation for $q_V = 0$. The simplest case is observed at a constant heat-conductivity coefficient $\lambda = \text{const}$. Substituting the conditions $q_V = 0$ and $\lambda = \text{const}$ into Eq. (3.5), we obtain

$$t(x)=t_{F_1}-\Delta t_F\frac{x}{\delta}. \qquad (3.6)$$

Consequently, the temperature field in a homogeneous and isotropic plate without internal heat sources obeys a straight-line law.

Let us determine the thermal load of the plate

$$q_F=-\lambda\frac{dt}{dx}=\frac{\lambda}{\delta}\Delta t=\frac{\Delta t}{\frac{\delta}{\lambda}}\ \text{kcal/m}^3\text{-hr}. \qquad (3.7)$$

The quantity δ/λ is called the internal thermal resistance of a thin

plate or the thermal resistance to heat conduction in it.

If we put $(t - t_{F_2})/(t_{F_1} - t_{F_2}) \equiv \Theta$ and $x/\delta \equiv X$, then the equation for the temperature distribution in the plate can be represented in the following form (Fig. 8):

$$\Theta = 1 - X. \tag{3.8}$$

Here Θ is the dimensionless temperature and X is the dimensionless abscissa. Equation (3.8) contains only two dimensionless quantities, and is therefore much more convenient than the dimensional equation (3.6), both for experimental verification and for plotting. These properties of dimensionless equations are extensively used in heat transfer.

When $q_V = 0$ and the heat-conductivity coefficient $\lambda = \lambda(t)$ is variable we obtain in place of Eq. (3.5)

$$\int_{t_{F_2}}^{t(x)} \lambda(t)\,dt = \lambda_{cp}\Delta t_F\left(1 - \frac{x}{\delta}\right). \tag{3.9}$$

The heat load of the plate is then

$$q = \frac{\lambda_{cp}}{\delta}\Delta t_F, \tag{3.10}$$

where

$$\lambda_{cp} \equiv \frac{1}{\Delta t_F}\int_{t_{F_2}}^{t_{F_1}} \lambda(t)\,dt$$

is the mean-integral value of the heat-conductivity coefficient.

Consequently, for a variable heat-conductivity coefficient the temperature distribution in the plate no longer obeys the straight-line law. In this case if $d\lambda/dt > 0$, then the temperature field is convex upward, and if $d\lambda/dt < 0$, then $t(x)$ is convex downward (see Fig. 7).

The heat load of a plate made of a material with variable heat-conductivity coefficient is given by the same equation (3.10) as for λ = const. In this case the heat-conductivity coefficient must be chosen

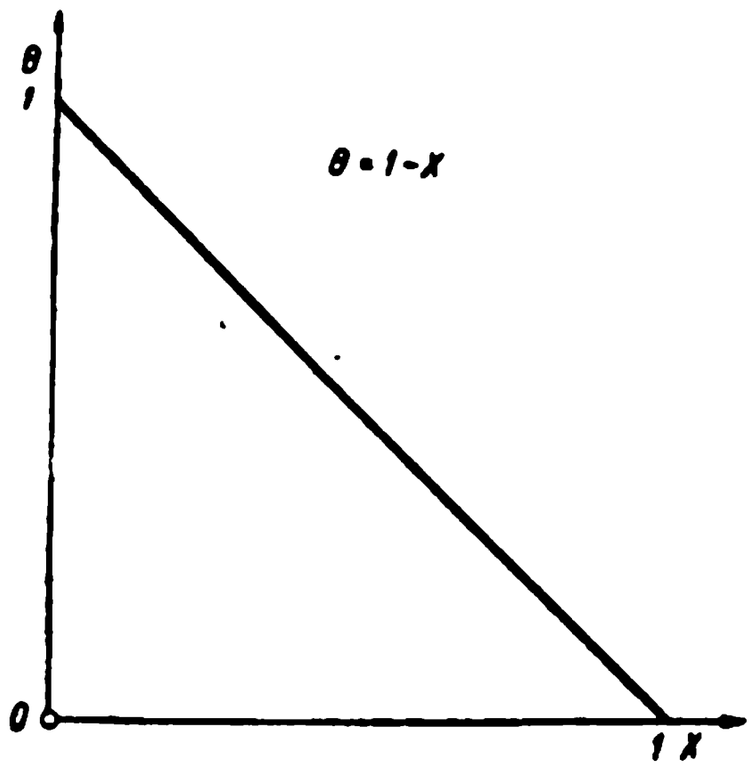

Fig. 8. Dimensionless temperature field in a thin plate.

to be its mean-integral value in the temperature interval from t_{F_2} to t_{F_1}. This rule can be employed only for boundary conditions of the first kind, which contain two temperatures: the maximum t_{F_1} and the minimum t_{F_2}.

Let us consider particular cases of Eqs. (3.5) for $\lambda = \text{const}$ and $q_V \neq 0$. For this purpose we substitute the condition $\lambda = \text{const}$ into Eq. (3.5) and reduce it to the dimensionless form. After transformations we obtain

$$\frac{t - t_{F_2}}{\frac{q_V \delta^2}{2\lambda}} = \frac{t_{F_1} - t_{F_2}}{\frac{q_V \delta^2}{2\lambda}} - \left(\frac{x}{\delta}\right)^2 + \left(1 - \frac{t_{F_1} - t_{F_2}}{\frac{q_V \delta^2}{2\lambda}}\right)\frac{x}{\delta} \qquad (3.11)$$

or

$$\Theta = \Theta_{F_1} - X^2 + (1 - \Theta_{F_1})X,$$

where Θ, Θ_{F_1}, and X are the dimensionless temperatures and the abscissa.

In the case of symmetrical cooling of the plate, when $t_{F_1} = t_{F_2} = t_F$, $\Theta_{F_1} = 0$, Eq. (3.5) assumes the simple form

$$\Theta = X - X^2. \qquad (3.12)$$

The dimensionless temperature has its maximum value ($\Theta_{maks} = 1/4$) at the center of the plate (Fig. 9).

The second case corresponds to the condition

$$\Theta_{F_1} = 1.$$

Equation (3.5) likewise assumes the simple form

$$\Theta = 1 - X^2. \qquad (3.13)$$

The dimensionless temperature has a maximum value ($\Theta_{maks} = 1$) on the edge of the plate.

From a comparison of the two last variants we can conclude that in the case of symmetrical cooling the greatest dimensionless tempera-

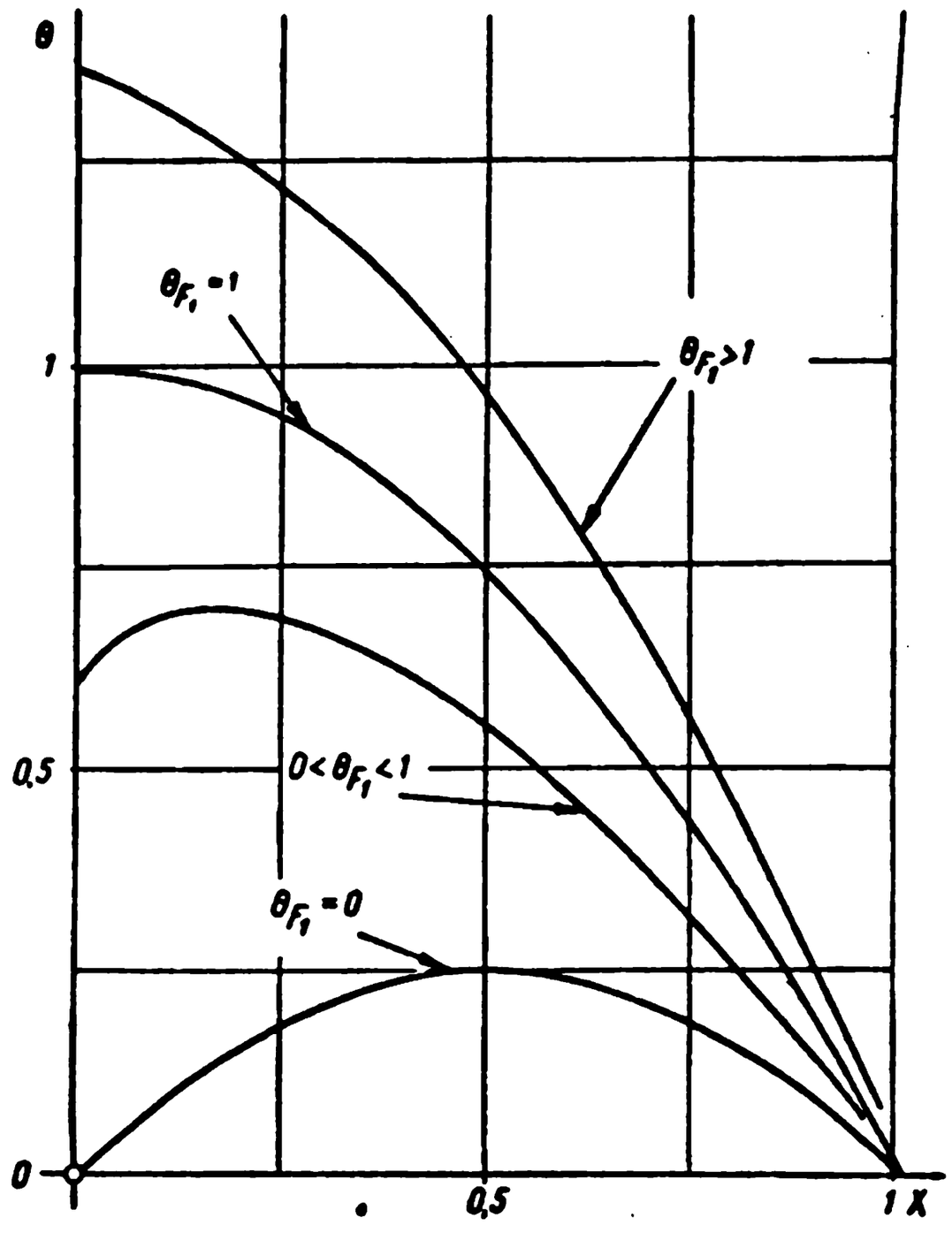

Fig. 9. Possible variants of the distribution of the dimensionless temperature in a thin plate with internal heat sources.

ture of the plate is one quarter as large as in the case of unilateral cooling. The quantity $q_V\delta^2/2\lambda$ has the meaning of the total temperature drop $t_{F_1} - t_{F_2}$ in a plate with internal heat sources under unilateral cooling. Finally, one can indicate two other cases which are of intermediate character and correspond to the conditions $0 < \Theta_{F_1} < 1$ and $\Theta_{F_1} > 1$. The temperature distribution in the plate obeys in this case also Eq. (3.5).

Boundary Conditions of the Second Kind

Let us consider the simplest case when boundary conditions of the second kind are specified on the surface of a thin homogeneous and isotropic plate, and there are no internal sources of heat. Assume that a constant heat load q_F = const is maintained on the surface x = 0 (Fig. 10) along with a constant temperature t_{F_1}. Then the temperature field

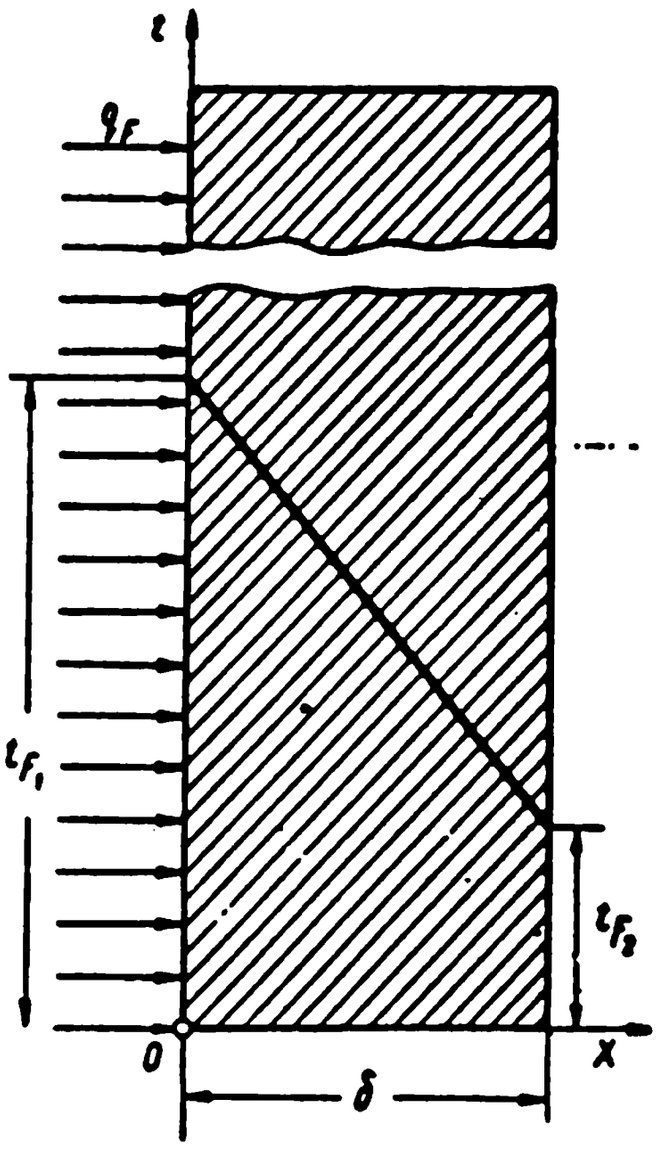

Fig. 10. Distribution of temperature in a thin plate for boundary conditions of the second kind.

obeys the following differential equation and boundary conditions:

$$\left.\begin{aligned} \frac{d^2t}{dx^2} &= 0; \\ -\lambda \frac{dt}{dx}\Big|_{x=+0} &= q_F; \\ t|_{x=0} &= t_{F_1}. \end{aligned}\right\} \qquad (3.14)$$

Integrating Eqs. (3.14) we obtain

$$t = t_{F_1} - \frac{q_{F_1}}{\lambda} x.$$

It is necessary to note here once more that boundary conditions of the second kind specify not only the heat load q_F on the surface of the body, but also the temperature at any point in it. Otherwise such a problem becomes indeterminate.

Boundary Conditions of the Third Kind

Let us consider the stationary temperature field in a thin homogeneous and isotropic plate in which heat is released as a result of the action of internal sources. This heat is carried away by two cooling liquids flowing past the plate on both sides and having unequal temperatures $t_{zh1} > t_{zh2}$ (Fig. 11).

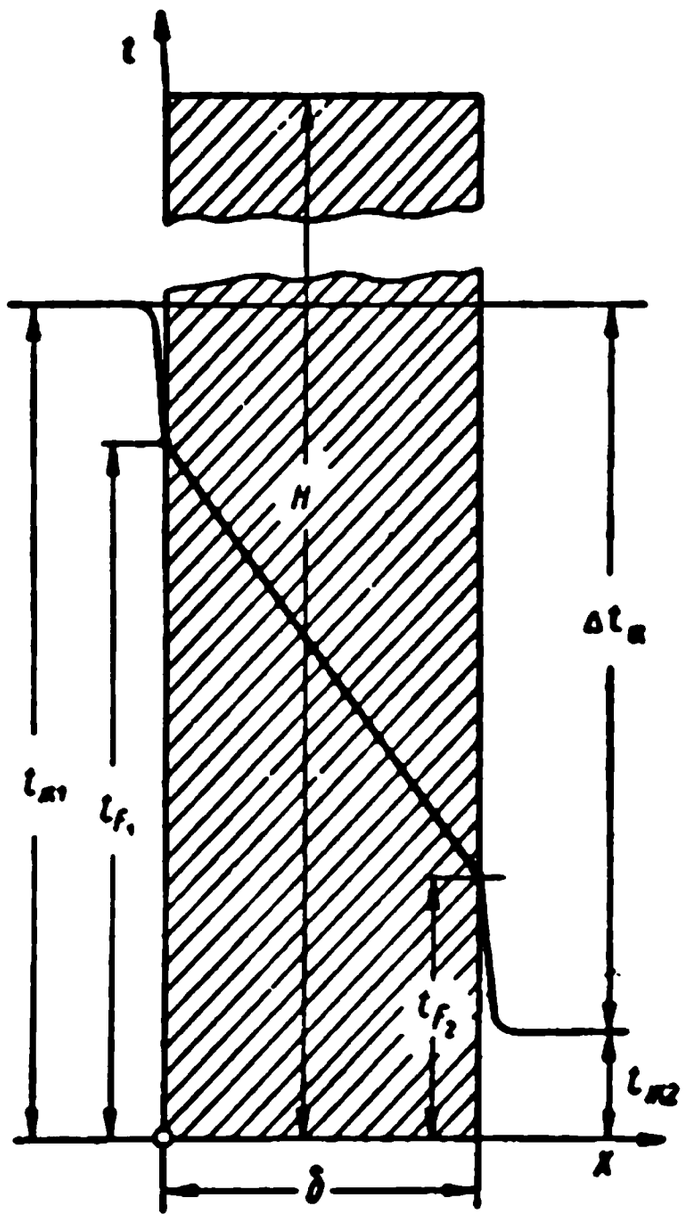

Fig. 11. Distribution of temperature in a thin plate for boundary conditions of the third kind.

The temperature distribution in the plate obeys the Fourier equation

$$\frac{d^2t}{dx^2} + \frac{q_V}{\lambda} = 0 \qquad (3.15)$$

and the boundary conditions of the third kind

$$\left.\begin{aligned} -\lambda \frac{dt}{dx}\Big|_{x=+0} &= \alpha_1 (t|_{x=0} - t_{zh1}); \\ -\lambda \frac{dt}{dx}\Big|_{x=+\delta-0} &= \alpha_2 (t|_{x=\delta} - t_{zh2}), \end{aligned}\right\} \qquad (3.16)$$

where the quantities λ, α_1, α_2, δ, t_{zh1}, and t_{zh2} are specified.

If there is no heat released in the plate, $q_V = 0$, heat is transferred from the hot liquid to the cold one via the plate, and the temperature distribution in the plate obeys the straight-line law:

$$t = t_{zh1} - q\left(\frac{1}{\alpha_1} + \frac{x}{\lambda}\right) \qquad (3.17)$$

The thermal load of the plate q is equal to

$$q = k\Delta t_{zh}. \qquad (3.18)$$

where $k \equiv 1/[(1/\alpha_1) + (\delta/\lambda) + (1/\alpha_2)]$ is the heat-transfer coefficient, which characterizes the intensity of the heat transfer through the plate and has the same unit of measurement as the heat-delivery coefficient α (kcal/m^2-hr-deg).

The quantities $1/\alpha_1$ and $1/\alpha_2$ are called the external thermal resistance of the plate or the thermal resistance to heat transfer, while the quantity Δt_{zh} is called the total temperature differential. The heat-transfer coefficient is determined by the value of the largest thermal resistance $1/\alpha_1$, $1/\alpha_2$, or δ/λ. If, for example, $1/\alpha_2 \gg (\delta/\lambda) + (1/\alpha_2)$, then $k \approx \alpha_2$.

In this case the largest thermal resistance must be reduced to such a value that all three thermal resistances have approximately the same values.

This can be attained by different methods, and in particular by locating fins on that side of the plate, where the external thermal resistance $1/\alpha$ is large. The heat rate of the smooth side of a finned plate can be estimated by using the following approximate relation

$$q_{F_1} < \frac{\Delta t_{zh}}{\frac{1}{} + \frac{\delta}{} + \frac{1}{} \frac{F}{}}. \qquad (3.19)$$

The right half of (3.19) gives too large a value of q_F, for in the derivation of Formula (3.19) the internal thermal resistance of the fins was not taken into account, and the temperature on their external surface t_{F_2} was assumed constant.

Let us return to the initial problem (3.15) and (3.16) and let us write down the general integral of Eq. (3.15)

$$t = -\frac{q_V x^2}{2\lambda} + c_1 x + c_2.$$

Let us determine c_1 and c_2 with the aid of the boundary conditions (3.16) and let us substitute the obtained values in the expressions for the general integral. After transformation we obtain

$$\frac{t - t_{\text{ж}2}}{\frac{q_V \delta^2}{2\lambda}} = \frac{t_{\text{ж}1} - t_{\text{ж}2}}{\frac{q_V \delta^2}{2\lambda}}\left(1 - \frac{k}{\alpha_1}\right) - \left(\frac{x}{\delta}\right)^2 + \frac{k\delta}{\lambda}\left(\frac{2}{\frac{\alpha_2 \delta}{\lambda}} + 1 - \right.$$

$$\left. - \frac{t_{\text{ж}1} - t_{\text{ж}2}}{\frac{q_V \delta^2}{2\lambda}}\right)\frac{x}{\delta} + \frac{k}{\alpha_1}\left(\frac{2}{\frac{\alpha_2 \delta}{\lambda}} + 1\right)$$

or

$$\Theta = \Theta_{\text{ж}1}\left(1 - \frac{k}{\alpha_1}\right) - X^2 + Bi_k\left(\frac{2}{Bi_2} + 1 - \Theta_{\text{ж}1}\right)X + \\ + \frac{k}{\alpha_1}\left(\frac{2}{Bi_2} + 1\right). \qquad (3.20)$$

where $\Theta \equiv (t - t_{zh2})/(q_V\delta^2/2\lambda)$ is the dimensionless temperature at any point of the plate; $\Theta_{zh1} \equiv (t_{zh1} - t_{zh2})/(q_V\delta^2/2\lambda)$ is the dimensionless temperature of the hotter liquid; $X \equiv x/\delta$ is the dimensionless abscissa; $k \equiv 1/[(1/\alpha_1) + (\delta/\lambda) + (1/\alpha_2)]$ is the coefficient of heat transfer through the plate; $Bi_2 \equiv \alpha_2\delta/\lambda$ is the partial Biot number, equal to the ratio of internal thermal resistance of the plate to its external thermal resistance from the right side; $Bi_k \equiv k\delta/\lambda$ is the total Biot number, equal to the ratio of the internal thermal resistance of the plate to its total thermal resistance.

Let us consider heat conduction in a plate when the boundary conditions of the third kind are specified. In symmetrical cooling, when

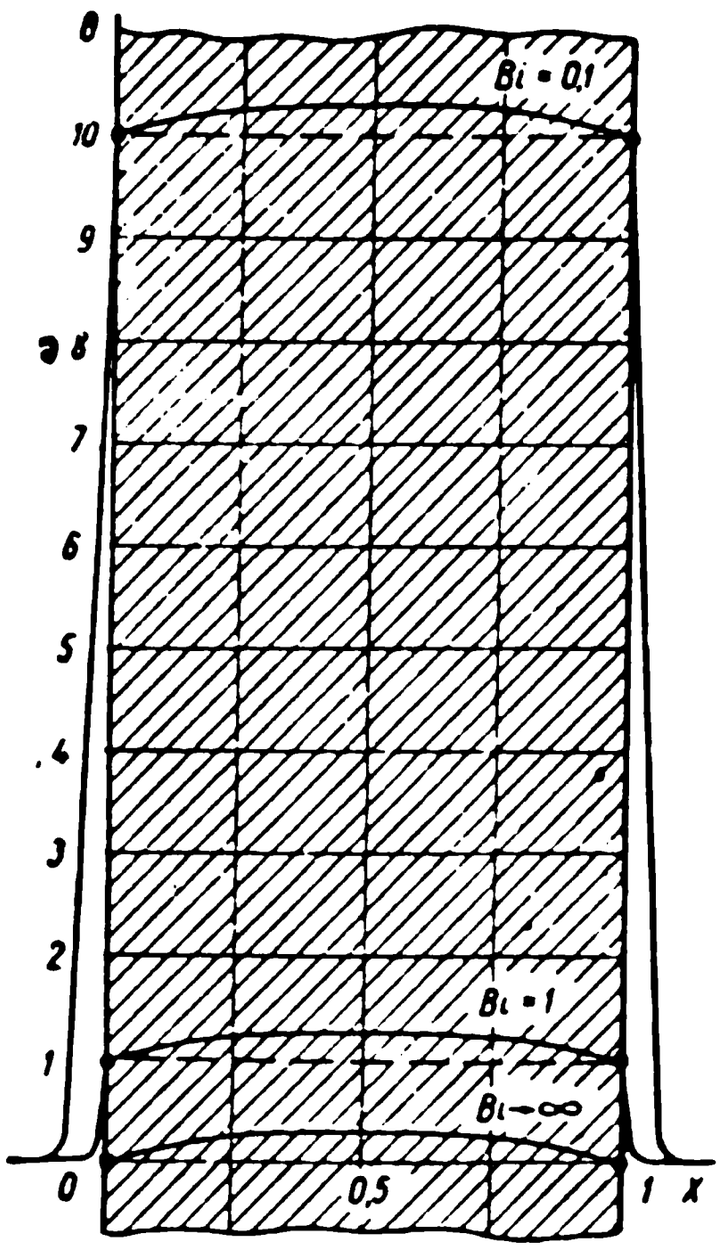

Fig. 12. Influence of the Biot number on the distribution of the dimensionless temperature in a plate with internal sources of heat for symmetrical bilateral cooling.

$t_{zh1} = t_{zh2} = t_{zh}$ and $\alpha_1 = \alpha_2 = \alpha$, $\Theta_{zh1} = 0$, Eq. (3.20) assumes the form

$$\theta = X - X^2 + \frac{1}{Bi}. \quad (3.21)$$

The dimensionless temperature has its maximum value [$\Theta_{maks} = (1/4) + (1/Bi)$] at the center of the plate where $X = 0.5$ (Fig. 12). If the Biot number is small ($Bi \ll 1$) and $\delta/\lambda \ll 1/\alpha$, then the decrease in temperature occurs in the cooling-liquid layers adjacent directly to the plate, while the decrease in temperature inside the plate is quite insignificant. To the contrary, for $Bi \gg 1$, when $\delta/\lambda \gg 1/\alpha$, almost the entire change in temperature occurs inside the plate. Consequently, the value of Bi greatly influences the character of the distribution of the temperature in a plate cooled by a liquid stream.

The installation of fins makes it possible to improve the cooling of the plate. Then the formula for the maximum temperature assumes the form

$$\theta_{макс\ р} = \frac{1}{4} + \frac{1}{Bi_р}. \quad (3.22)$$

where $Bi_r \equiv \alpha\delta F_r/\lambda F$. This formula has been obtained without account of the internal thermal resistance of the fins, so that it is approximate. Inasmuch as $F_r/F > 1$, we have $\Theta_{maks\ r} < \Theta_{maks}$. For $\Theta_{zh1} = (2/Bi) + 1$, when $\alpha_1 = \alpha_2 = \alpha$, Eq. (3.20) assumes the form

$$\theta = \frac{2}{Bi} + 1 - X^2. \quad (3.23)$$

The dimensionless temperature Θ has its largest value [$\Theta_{maks} = 1 +$

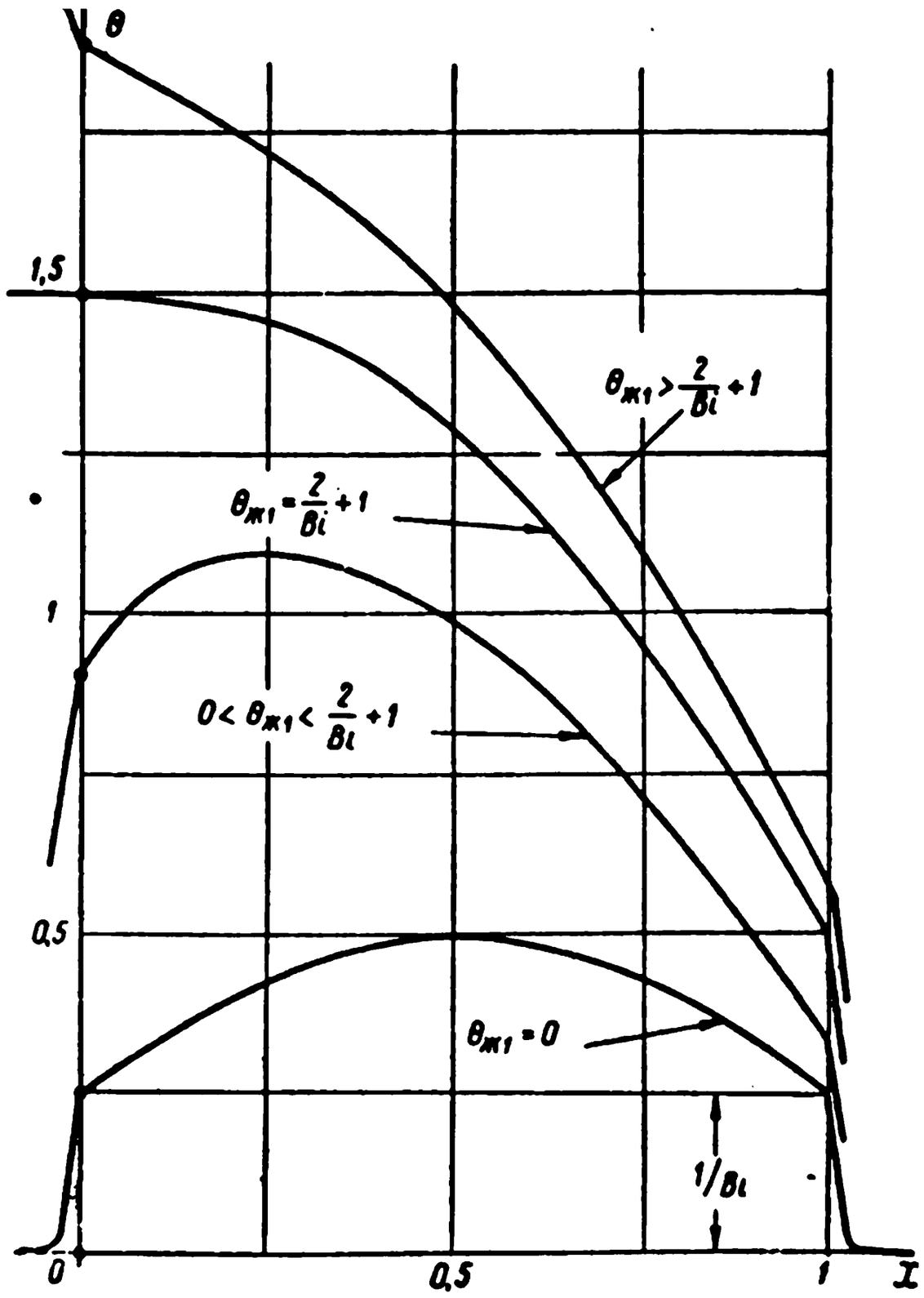

Fig. 13. Possible variants of the distribution of the dimensionless temperature in a plate with internal sources of heat for bilateral cooling.

$+ (2/Bi)]$ at $X = 0$ (Fig. 13).

Let us indicate, finally, two other cases which are intermediate in character and correspond to the conditions $0 < \Theta_{zh2} < 1 + (2/Bi)$ and $\Theta_{zh2} > 1 + (2/Bi)$. In this case the temperature distribution in the plate also obeys Eq. (3.20).

If the plate is made of a material with variable heat-conductivity coefficient $\lambda = \lambda(t)$, then the calculation of the heat-conduction process in the plate with $q_V \neq 0$ and boundary conditions of the third kind is in the general case difficult, since the boundary conditions are nonlinear in this case

$$\left.\begin{aligned} -\lambda(t)\cdot\frac{dt}{dx}\Big|_{x=+0} &= -\alpha_1\cdot(t|_{x=0}-t_{ж1});\\ -\lambda(t)\cdot\frac{dt}{dx}\Big|_{x=+\delta-0} &= +\alpha_2\cdot(t|_{x=+\delta}-t_{ж2}). \end{aligned}\right\} \qquad (3.24)$$

However, we can obtain an approximate solution of this problem. For this purpose we introduce the Kirchhoff variable

$$\vartheta = \int_{t_{ж2}}^{t}\lambda(t)\,dt. \qquad (3.25)$$

Then the boundary conditions (3.24) assume the form

$$\left.\begin{aligned} \frac{d\vartheta}{dx}\Big|_{x=+0} &= \alpha_1\cdot(t(\vartheta)|_{x=0}-t_{ж1});\\ -\frac{d\vartheta}{dx}\Big|_{x=+\delta-0} &= \alpha_2\cdot(t(\vartheta)|_{x=+\delta}-t_{ж2}). \end{aligned}\right\} \qquad (3.26)$$

If the dependence of the heat-conductivity coefficient on the temperature $\lambda = \lambda(t)$ is such that in a specified interval $t_{zh2} < t < t_{zh1}$ the function $t(\vartheta)$ in Eq. (3.26) can be approximated with sufficient accuracy by means of a straight line, i.e., if we assume for the right halves of the boundary conditions $\lambda \approx \lambda_0 = \text{const}$,

$$t = t_{ж2} + \frac{\vartheta}{\lambda_0}, \qquad (3.27)$$

then these boundary conditions become approximate and linear:

$$\left.\begin{aligned} \frac{d\vartheta}{dx}\Big|_{x=+0} &\approx \alpha_1\cdot\left(t_{ж2}-t_{ж1}+\frac{1}{\lambda_0}\cdot\vartheta\Big|_{x=0}\right);\\ -\frac{d\vartheta}{dx}\Big|_{x=+\delta-0} &\approx \alpha_2\cdot\left(\frac{\vartheta}{\lambda_0}\right)_{x=+\delta}. \end{aligned}\right\} \qquad (3.28)$$

In this case it becomes possible to find a solution of the linear differential equation (3.4), satisfying the linear boundary conditions (3.27). The results of such a calculation are approximate in character.

§4. HEAT CONDUCTION IN A WALL OF A LONG TUBE

Boundary Conditions of the First Kind

Let us consider the stationary temperature field in the wall of a long tube made of isotropic material with variable heat-conduction coefficient. On the surfaces of this tube are maintained temperatures $t_п$ and $t_п < t_п$ (Fig. 14). In the wall there are effective external

heat sources of strength $q_V > 0$. Under these conditions the Fourier equation assumes the form

$$\frac{1}{r}\frac{d}{dr}\left[\lambda(t)\cdot r\cdot\frac{dt}{dr}\right]+q_V=0, \tag{4.1}$$

and the boundary conditions of the first kind are

$$\left.\begin{array}{l}\text{for } r=r_1 \quad t=t_{F_1}\\ \text{for } r=r_2 \quad t_{F_2}<t_{F_1}.\end{array}\right\} \tag{4.2}$$

Solving this problem with the aid of the Kirchhoff transformation, we obtain

$$\frac{4\int_{t_{F_2}}^{t(r)}\lambda(t)\,dt}{q_V\cdot r_2^2}=1-R^2-\frac{\ln R}{\ln R_1}\left(1-R_1^2-\frac{4\int_{t_{F_2}}^{t_{F_1}}\lambda(t)\,dt}{q_V\cdot r_2^2}\right) \tag{4.3}$$

where $R \equiv r/r_2$ and $R_1 \equiv r_1/r_2$.

Let us consider particular cases of this equation, when there are no internal heat sources in the tube: $q_V = 0$.

The simplest case is observed at a constant heat-conduction coefficient. Substituting the conditions $q_V = 0$ and $\lambda = \text{const}$ into Eq. (4.3), we obtain after transformation

$$t=t_{F_2}+\Delta t_F\frac{\ln\left(\frac{r_2}{r}\right)}{\ln\left(\frac{r_2}{r_1}\right)}. \tag{4.4}$$

In dimensionless form this equation can be written:

$$\Theta=\frac{\ln R}{\ln R_1}. \tag{4.5}$$

Consequently, the temperature distribution in a homogeneous isotropic wall of a long tube without internal heat sources obeys a logarithmic law. The heat flow through the wall of the tube is

$$Q=-2\pi\cdot r_1 l\left.\frac{dt}{dr}\right|_{r=r_1+0}=\frac{\pi\cdot l\cdot\Delta t_F}{\frac{1}{2\lambda}\cdot\ln\left(\frac{d_2}{d_1}\right)}. \tag{4.6}$$

The quantity $(1/2\lambda)\cdot\ln(d_2/d_1)$ is called the internal thermal resistance of the wall of a long tube.

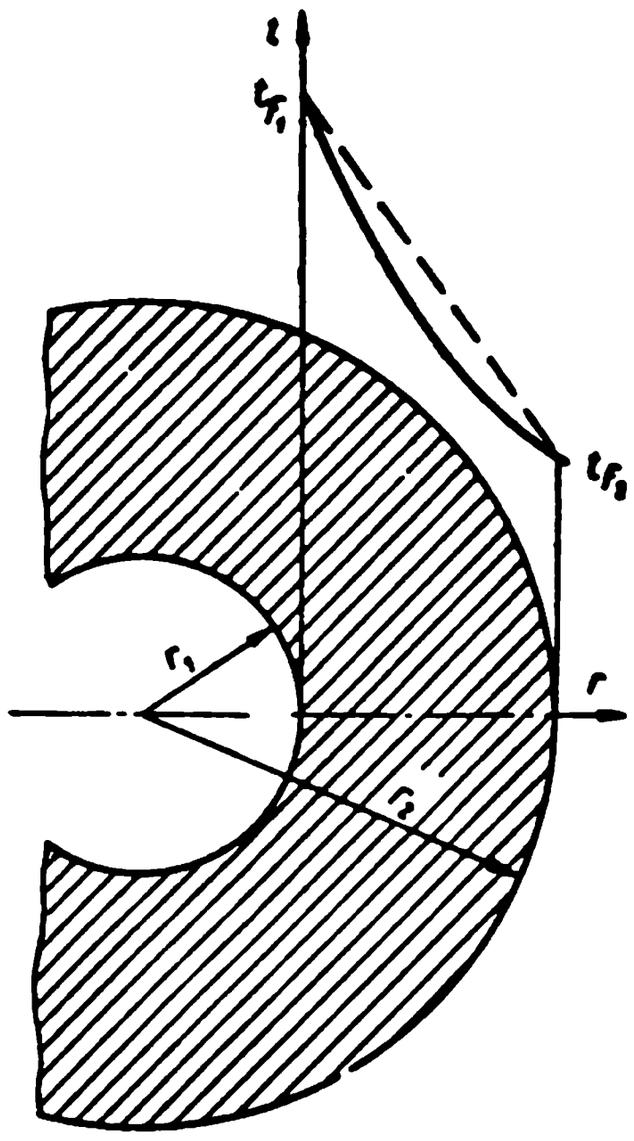

Fig. 14. Temperature distribution in the wall of a long tube for boundary conditions of the first kind.

A second variant is observed in the case of a variable heat-conduction coefficient, when $\lambda = \lambda(t)$. Substituting $q_V = 0$ in Eq. (4.3) we obtain

$$\int_{t_{F_2}}^{t(r)} \lambda(t)\,dt = \lambda_{cp} \cdot \Delta t_F \cdot \frac{\ln\left(\frac{r_2}{r}\right)}{\ln\left(\frac{r_2}{r_1}\right)} \qquad (4.7)$$

The flow of heat through the wall of the tube will now be

$$Q = \lambda_{cp} \frac{2\pi \cdot l \cdot \Delta t_F}{\ln\left(\frac{d_2}{d_1}\right)}. \qquad (4.8)$$

Here, as before

$$\lambda_{cp} = \frac{1}{\Delta t_F} \int_{t_{F_2}}^{t_{F_1}} \lambda(t)\,dt.$$

Let us consider the heat-conduction processes in the wall of a tube made of a material with constant heat-conduction coefficient in the presence of heat release inside the wall ($q_V > 0$). Substituting the condition $\lambda = \text{const}$ into Eq. (4.3) we obtain after transformations

$$\frac{t - t_{F_2}}{\frac{q_V \cdot r_2^2}{4\lambda}} = 1 - \left(\frac{r}{r_2}\right)^2 - \frac{\ln\left(\frac{r}{r_2}\right)}{\ln\left(\frac{r_1}{r_2}\right)}\left(1 - \frac{r_1^2}{r_2^2} - \frac{4 \cdot \lambda \cdot \Delta t_F}{q_V \cdot r_2^2}\right) \qquad (4.9)$$

or

$$\Theta = 1 - R^2 - \frac{\ln R}{\ln R_1}(1 - R_1^2 - \Theta_{F_1}).$$

In the case of symmetrical cooling of the tube walls $t_{F_1} = t_{F_2} = t_F$ and $\Theta_{F_1} = 0$. Then Eq. (4.3) assumes the form

$$\Theta = 1 - R^2 - \frac{1 - R_1^2}{\ln R_1} \ln R. \qquad (4.10)$$

The dimensionless temperature has a maximum value at the follow-

ing value of the dimensionless radius:

$$R_{\text{макс}} = \sqrt{\frac{1 - R_1^2}{2 \ln\left(\frac{1}{R_1}\right)}}. \qquad (4.11)$$

Let us consider the case corresponding to the condition

$$\Theta_{F_1} = 1 - R_1^2. \qquad (4.12)$$

Now Eq. (4.3) assumes the simplest form

$$\Theta = 1 - R^2, \qquad (4.13)$$

and in this case the dimensionless temperature has no maximum, since $0 < R_1 < R \leq 1$, and the largest value $\Theta = 1 - R_1^2$ is assumed by the dimensionless temperature when $R = R_1$.

Finally, a case is possible when no heat passes through the internal surface of the tube. Then

$$\left.\frac{d\Theta}{dR}\right|_{R = R_1 + 0} = 0. \qquad (4.14)$$

Substituting Relation (4.9) into Eq. (4.14) we obtain after subtraction

$$\Theta_{F_1} = (2 \ln R_1 - 1) R_1^2 + 1. \qquad (4.15)$$

Consequently, if the quantity Θ_{F_1} is contained in the interval $0 \leq \Theta_{F_1} \leq (2 \ln R_1 - 1)R_1^2 + 1$, then the distribution of the dimensionless temperature in the tube wall has a maximum. On the other hand if $\Theta_{F_1} > (2 \ln R_1 - 1)R_1^2 + 1$, this is not observed. The foregoing holds true for $t_{F_2} < t_{F_1}$. Analogous conclusions can be obtained also when the temperature of the external tube surface t_{F_2} exceeds the temperature t_{F_1} of its internal surface.

Boundary Conditions of the Third Kind and Critical Diameter of the Tube

Let us consider the condition for the transfer of heat through a homogeneous and isotropic wall of a long tube without internal heat sources. In the tube flows a hotter liquid, the temperature of which varies weakly along the stream and can therefore be regarded as con-

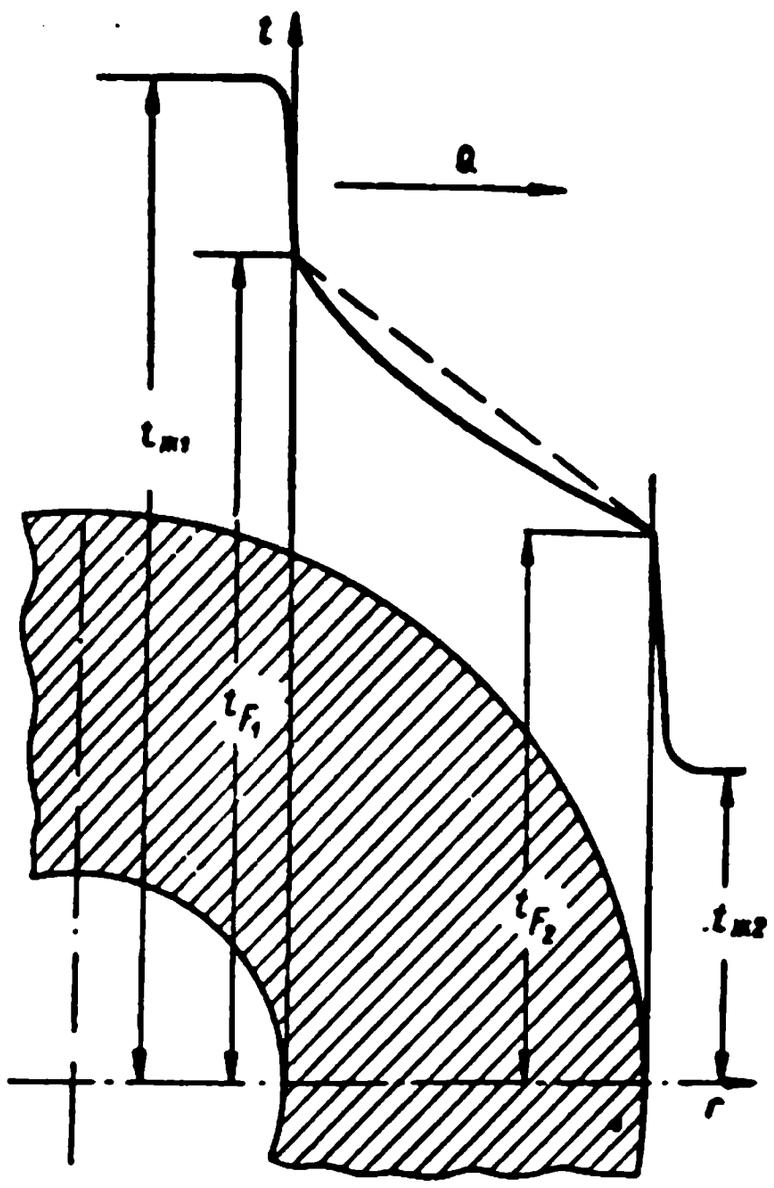

Fig. 15. Temperature distribution in the wall of a long tube for boundary conditions of the third kind.

stant. On the outside of the tube there flows a stream of cooler liquid, which also has a constant temperature $t_{zh2} < t_{zh1}$ (Fig. 15). The tube dimensions r_1, r_2, and l are specified along with the heat-delivery coefficients α_1 and α_2 and also the heat-conduction coefficient λ of the material of which the tube is made.

In accordance with the Newton-Rikhman law the heat flux Q (kcal/hr) through the wall of the tube is

$$\left.\begin{aligned} Q &= \pi d_1 \cdot l \cdot \alpha_1 \cdot (t_{w1} - t_{F_1}); \\ Q &= \pi d_2 \cdot l \cdot \alpha_2 \cdot (t_{F_2} - t_{w2}) \end{aligned}\right\} \quad (4.16)$$

These equations contain tnree unknowns Q, t_{F_1}, and t_{F_2}. As the third relation we use Eq. (4.6)

$$Q = \frac{\pi l \cdot (t_{F_1} - t_{F_2})}{\frac{1}{2\lambda} \cdot \ln\left(\frac{d_2}{d_1}\right)}.$$

If we eliminate from these equations the temperature of the internal and external surfaces of the wall, i.e., t_{F_1} and t_{F_2}, then we obtain an equation for the heat transfer through the tube wall

$$Q = \frac{\pi \cdot l \cdot \Delta t_{zh}}{\frac{1}{\alpha_1 \cdot d_1} + \frac{1}{2\lambda} \cdot \ln\frac{d_2}{d_1} + \frac{1}{\alpha_2 \cdot d_2}}, \quad (4.17)$$

where $\Delta t_{zh} \equiv t_{zh1} - t_{zh2}$ is the total temperature drop, i.e., the difference between the temperatures of the hotter and the colder liquids.

The heat transfer through the tube wall can have a maximum when the outside diameter d_2 of the tube is varied. Actually, by fixing the values of Δt_{zh}, l, d_1, α_1, λ, α_2 and increasing d_2 we find that the internal thermal resistance of the tube wall $(1/2\lambda) \ln (d_2/d_1)$ will

increase, and its internal resistance $1/(\alpha_2 \cdot d_2)$ will decrease. Therefore the dependence $Q = Q(d_2)$ can have an extremum. From the condition $\partial Q/\partial d_2 = 0$ we obtain an expression for the critical diameter of the tube*

$$d_{кр} = \frac{2\lambda}{\alpha_2}. \qquad (4.18)$$

It is easy to show that in this case

$$\left.\frac{\partial^2 Q}{\partial (d_2)^2}\right|_{d_2 = d_{кр}} < 0.$$

Consequently, the dependence $Q = Q(d_2)$ has a maximum. The value of the critical tube diameter d_{kr} depends on the previously specified values of λ and α_2. For the same tube dimensions d_1 and d_2, the value of d_{kr} can vary.

Let α_2 and λ be chosen such that $d_{kr} < d_1$. In this case the function $Q(d_2)$ is monotonic and has no maximum (Fig. 16). If $d_{kr} = d_1$, then the dependence $Q = Q(d_2)$ has a maximum when $d_2 = d_1$. If finally $d_{kr} > d_1$, then the dependence $Q(d_2)$ has a clearly pronounced maximum. In the latter case an increase in the thickness of the wall tube first leads to an increase in the heat flow Q through the wall, due to the increase in the cooled outer surface.** However, with further increase of the wall thickness the heat transferred through it begins to decrease, owing to the appreciable increase of its internal thermal resistance.

This property of the tube wall is utilized in various fields of engineering. For example, the dimensions, the cooling conditions, and the heat-conduction coefficient of insulation for electric wiring are chosen such that the outside diameter is critical or nearly critical $d_2 \approx d_{kr}$. In this case an insulated electric wire is easier to cool than a bare one. In designing thermal insulation for pipes, the properties of the insulation are chosen in such a way as to satisfy the

condition $d_{kr} < d_1$, where d_1 is the inside diameter of the thermal insulation or the outside diameter of the insulated pipe.

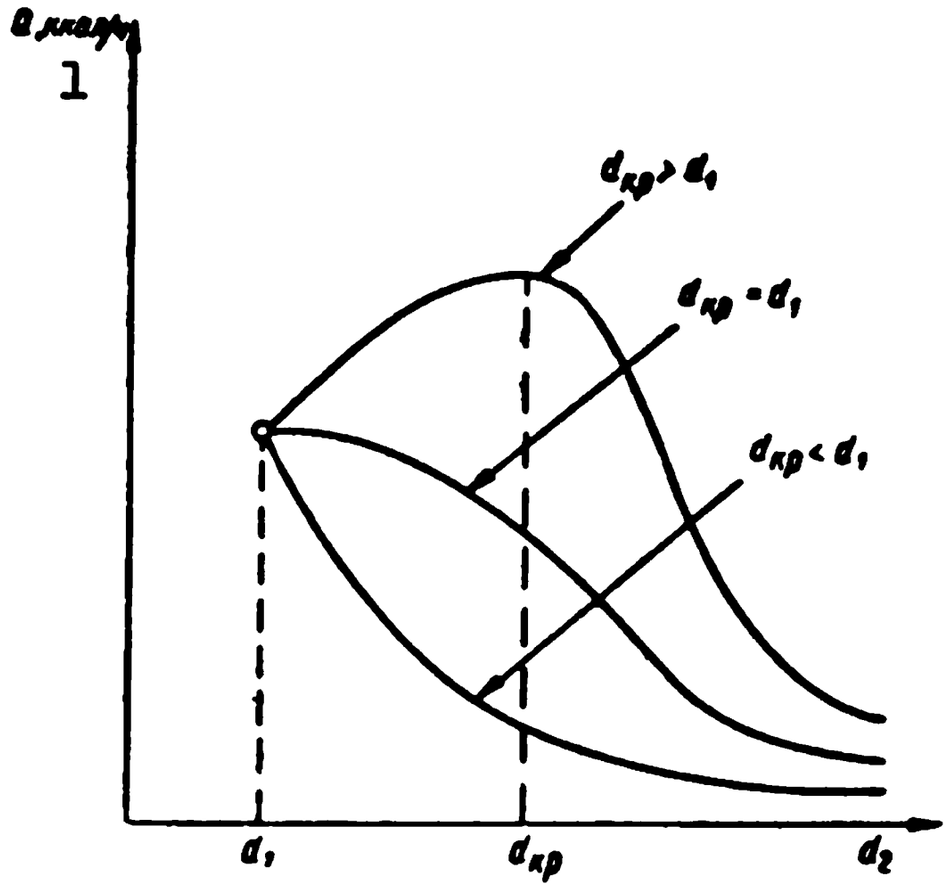

Fig. 16. Character of dependence of heat flux through the wall of a tube on its outside diameter. 1) kcal/hr.

§5. HEAT CONDUCTION IN RODS WITH SMALL "TRANSVERSE" BIOT NUMBER

Formulation of the Problem of Heat Conduction in a Thin Rod

Let us consider the stationary process of heat conduction in a cylindrical rod, for which the following uniqueness conditions are specified:

The geometrical conditions indicate that the straight cylindrical rod has a transverse cross section f, a perimeter P, and a length l (Fig. 17).

The physical conditions indicate that the material of the rod is isotropic but not homogeneous. The coefficient of heat conduction of the material depends on the temperature $\lambda = \lambda(t)$. In the rod there are internal heat sources with variable strength $q_V(x)$ kcal/m^3-hr. The internal thermal resistance transversely to the rod at any section of the rod is small compared with the outside resistance. Consequently, $Bi \equiv \alpha \cdot d_e/\lambda \ll 1$. Here $d_e = 4f/P$ is the equivalent diameter of the transverse section of the rod. Consequently, the temperature of the

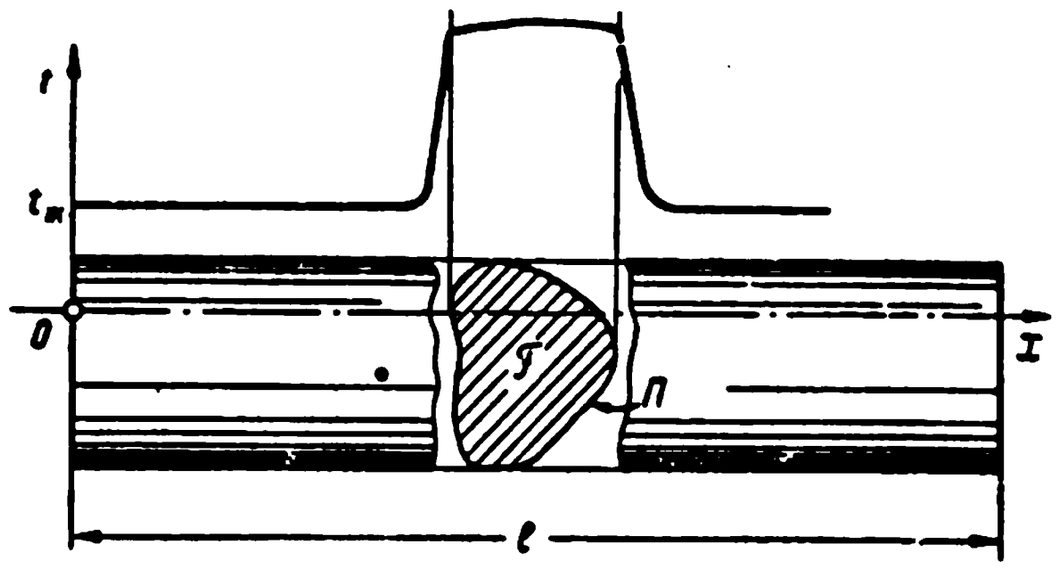

Fig. 17. Distribution of temperature in transverse cross section of a rod having a small "transverse" Biot number.

rod is distributed almost uniformly over any cross section of the rod.

The time conditions indicate that the process of heat conduction is stationary, i.e., $\partial t/\partial \tau = 0$.

The boundary conditions of the third kind specify the variable value of the heat-delivery coefficient on the surface of the rod $\alpha = \alpha(x)$ kcal/m^2-hr-deg, and also the temperature of the liquid away from the rod

$$\left[-\lambda(t)\frac{\partial t}{\partial n}\right]_{n=-0} = \alpha(x)\cdot(t|_{n=0} - t_ж).$$

Here n is the normal at any point of the perimeter P of the rod section. In addition, some fully defined heat-exchange conditions on both ends of the rod are specified.

Differential Equations for Heat Conduction in a Thin Rod

The stationary distribution of the temperature in the rod obeys the Fourier equation

$$\operatorname{div}\lambda(t)\operatorname{grad}t + q_V = 0. \qquad (5.1)$$

This equation can be simplified by recognizing that the "transverse" Biot number is small and that the temperature distribution t at any section f practically does not differ from the average temperature $^f\!-t$ in this section:

$$t \approx {}^{f}\bar{t} \equiv \frac{1}{f}\int_{f} t\, df. \tag{5.2}$$

Let us integrate Eq. (5.1) over the section $\underline{f}$. Assuming that the $\underline{x}$ axis is directed along the rod axis, we obtain

$$\int_{f}\left[\frac{\partial}{\partial y}\lambda(t)\frac{\partial t}{\partial y}+\frac{\partial}{\partial z}\lambda(t)\frac{\partial t}{\partial z}\right]dz\,dy+\int_{f}\frac{\partial}{\partial x}\lambda(t)\frac{\partial t}{\partial x}\,df+$$
$$+\int_{f} q_V(x)\,df=0. \tag{5.3}$$

In accordance with Green's theorem [5] we have

$$\int_{f}\left[\frac{\partial}{\partial y}\lambda(t)\frac{\partial t}{\partial y}+\frac{\partial}{\partial z}\lambda(t)\frac{\partial t}{\partial z}\right]df=\int_{\Pi}\lambda(t)\frac{\partial t}{\partial n}\Big|_{n=-0}d\Pi,$$

where $\underline{n}$ is the normal to any point of the perimeter P of the cross section of the rod $\underline{f}$.

Taking into account the boundary conditions of the third kind, we obtain

$$\int_{\Pi}\lambda(t)\frac{\partial t}{\partial n}\Big|_{n=-0}d\Pi=-\int_{\Pi}\alpha(x)\cdot(t|_{n=0}-t_{\infty})\,d\Pi.$$

Inasmuch as for each section the value of the heat-transfer coefficient is constant (α = const) and $t|_{n=0} \approx {}^{f}\bar{t}$, we have

$$\int_{\Pi}\lambda(t)\frac{\partial t}{\partial n}\Big|_{n=-0}d\Pi=\alpha(x)\left({}^{f}\bar{t}-t_{\infty}\right)\cdot\Pi. \tag{5.4}$$

Further, on the basis of Eq. (5.2) we have

$$\int_{f}\frac{\partial}{\partial x}\lambda(t)\frac{\partial t}{\partial x}\,df\approx f\frac{d}{dx}\lambda\left({}^{f}\bar{t}\right)\frac{d\bar{t}}{dx}. \tag{5.5}$$

Substituting Relations (5.4) and (5.5) into Eq. (5.3) and transforming the latter, we obtain

$$\frac{d}{dx}\left[\lambda\left({}^{f}\bar{t}\right)\frac{d{}^{f}\bar{t}}{dx}\right]-\frac{\alpha(x)\cdot\Pi}{f}\left({}^{f}\bar{t}-t_{\infty}\right)+{}^{f}\bar{q}_V(x)=0, \tag{5.6}$$

where ${}^{f}\bar{q}_V$ is the strength of the internal heat sources averaged over the section of the rod.

The differential equation (5.6) describes the distribution of the temperature $^{f}\bar{t}$, averaged over the cross section f, along the axis in a straight cylindrical rod having variable values of the coefficient of heat conduction, internal heat sources, and coefficient of heat transfer.

Many technical calculations reduce to solving the problem of the heat conduction in a thin rod. By way of examples we point out the calculation of the temperature field along control rods of a nuclear reactor or the calculation of the errors in the readings of a thermometer placed in a sleeve.

Thermal Calculation of a Thermometer Sleeve

The thermometer sleeve (Fig. 18) can be represented in the form of a rod with transverse section $f = \pi \cdot d_{sr} \cdot \delta$ and perimeter $P \approx \pi d_{sr}$.

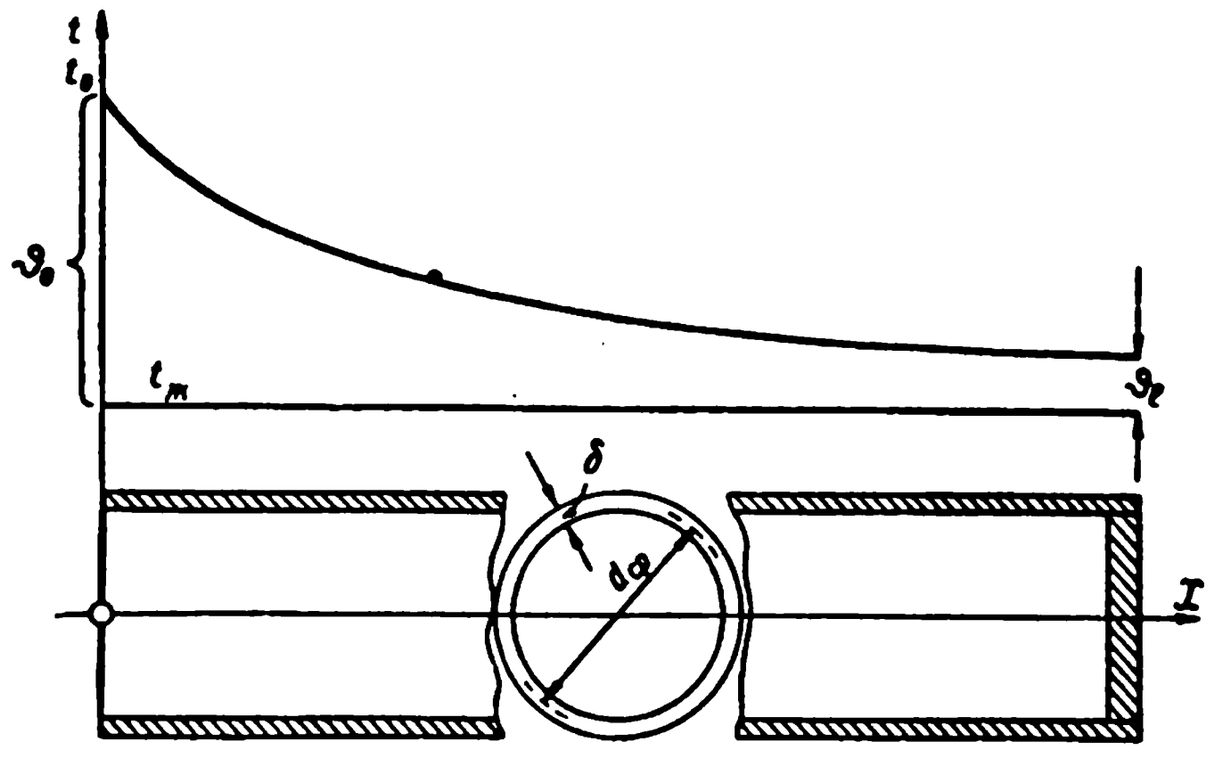

Fig. 18. Temperature distribution along a thermometer sleeve.

If λ = const and there are no internal sources of heat, then Eq. (5.6) assumes the form

$$\frac{d^2v}{dx^2} - m^2 v = 0 \qquad (5.7)$$

where

$$\vartheta \equiv {}^{f}\bar{t} - t_m \; ; \; m^2 \equiv \frac{\alpha \cdot \Pi}{\lambda \cdot f} \approx \frac{\alpha}{\lambda \cdot \delta}.$$

To determine the error in the readings of the thermometer due to

the heat conduction along the sleeve, let us determine ϑ_l for $x = l$ for the boundary conditions

$$\vartheta|_{x=0} = \vartheta_0 \text{ and } \left.\frac{d\vartheta}{dx}\right|_{x=l} \approx 0. \qquad (5.8)$$

The solution of Eq. (5.7) satisfying the boundary conditions (5.8) has the following form:

$$\vartheta = \vartheta_0 \frac{\operatorname{ch} m(x-l)}{\operatorname{ch} ml}.$$

In this case the error in the readings of a thermometer placed in a sleeve is

$$\vartheta|_{x=l} = \frac{\vartheta_0}{\operatorname{ch}(ml)}. \qquad (5.9)$$

Consequently, to reduce this error it is necessary to decrease ϑ_0 and to increase l and $m \equiv \sqrt{\alpha/\lambda \cdot \delta}$.

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[Footnotes]

366 Equation (4.18) is approximate, since it has been obtained under the assumption that the heat-delivery coefficient α_2 on the outer surface of the tube does not depend on its diameter. In actual conditions α_2 depends on d_2. See [9a and 30c].

366 We observe here the same effect as produced by the installation of fins on a flat wall.

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[List of Transliterated Symbols]

354 макс = maks = maksimum = maximum

356 ж = zh = zhidkost' = liquid

359 р = r = rebra = fin

366 кр = kr = kriticheskiy = critical

367 П = P = perimeter = perimeter

367 э = e = ekvivalentnyy = equivalent

371 ch = cosh

Chapter 3

SIMPLEST NONSTATIONARY HEAT-CONDUCTION PROCESSES

§6. NONSTATIONARY HEAT-CONDUCTION PROCESSES IN A THIN PLATE

Formulation of the Problem for Boundary Conditions of the Third Kind

Let us consider a nonstationary temperature field, which is produced in a plate immersed in a stream of heated liquid.

The geometrical conditions specify a thin plate of thickness 2δ (Fig. 19).

The physical conditions indicate that the plate is made of a homogeneous and isotropic material; there are no internal sources of heat. Consequently, $\lambda = \text{const}$, $C_v = \text{const}$, and $q_V = 0$.

The time conditions indicate that before the start of the heating the plate has a temperature t_0, and after prolonged heating the temperature at all points of the plate tends to the temperature of the heating liquid t_{zh}. Consequently, the time conditions specify the initial and final temperature distributions in the plate

$$t|_{\tau=0;\ -\delta<x<+\delta} = t_0;$$

$$t|_{\tau\to\infty;\ -\delta<x<+\delta} \to t_{\infty}.$$

The boundary conditions of the third kind specify the conditions of heat exchange between the plate and the heated liquid. For the right and left surfaces of the plate we have, respectively,

$$-\lambda \frac{\partial t}{\partial x}\Big|_{x=+\delta-0;\ \tau>0} = -\alpha(t_{\infty} - t|_{x=+\delta})_{\tau>0};$$

$$-\lambda \frac{\partial t}{\partial x}\Big|_{x=-\delta+0;\ \tau>0} = +\alpha(t_{\infty} - t|_{x=-\delta})_{\tau>0}.$$

The latter equation can be replaced by a simpler condition of symmetry

of the temperature field

$$\left.\frac{\partial t}{\partial x}\right|_{x=0;\ \tau>0}=0.$$

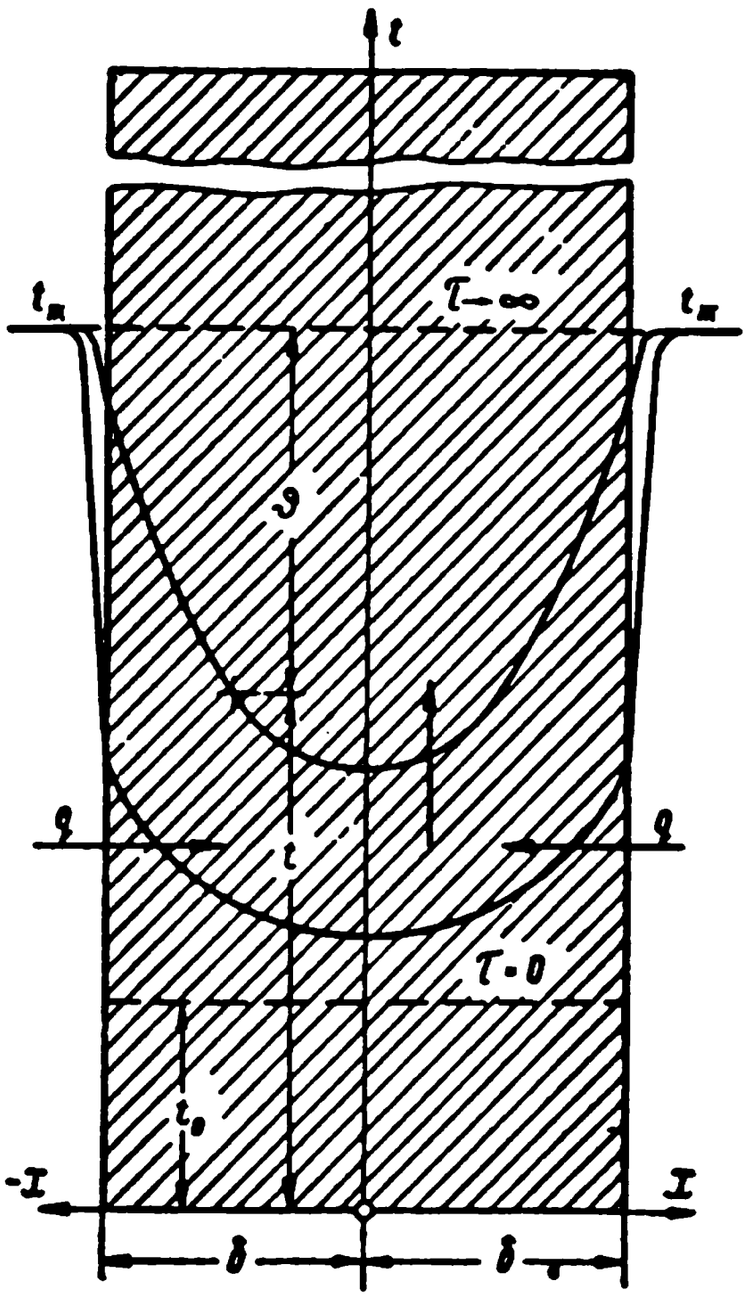

Fig. 19. Nonstationary temperature field in a plate heated on both sides in a liquid stream.

For the specified uniqueness conditions, the Fourier differential equation simplifies and assumes the form

$$\frac{\partial t}{\partial \tau}=a_0\frac{\partial^2 t}{\partial x^2}.$$

In these equations the unknown is the temperature distribution in the plate, i.e., the function $t = t(x, \tau)$. In this case the quantities δ, $\underline{a}$, λ, α, t_{zh}, and t_0 should be regarded as specified parameters. The mathematical formulation of the problem can be simplified somewhat by introducing the excess temperature $\vartheta \equiv t_{zh} - t$. Then the Fourier equation, the initial conditions, and the boundary conditions assume the following form:

$$\frac{\partial\vartheta}{\partial\tau}=a_0\frac{\partial^2\vartheta}{\partial x^2}; \tag{6.1}$$

$$\vartheta|_{\tau=0;\ 0<x<+\delta}=t_ж-t_0\equiv\Delta t; \tag{6.2}$$

$$\vartheta|_{\tau\to\infty;\ 0<x<+\delta}\to 0; \tag{6.3}$$

$$\lambda\left.\frac{\partial\vartheta}{\partial x}\right|_{x=+\delta-0;\ \tau>0}=-\alpha\vartheta|_{x=+\delta;\ \tau>0}; \tag{6.4}$$

$$\left.\frac{\partial\vartheta}{\partial x}\right|_{x=0;\ \tau>0}=0. \tag{6.5}$$

Calculation of the Heating of a Plate by the Method of Separation of Variables

In accordance with the method of separation of variables, the field of the excess temperature $\vartheta(x, \tau)$ must be represented in the

form of a product of two functions X(x) and T(τ), each of which depends only on one argument

$$\vartheta = T(\tau) \cdot X(x). \qquad (6.6)$$

Substituting Relation (6.6) into Eq. (6.1) and separating variables, we obtain

$$\frac{1}{T}\frac{dT}{d\tau} = \frac{a_0}{X}\frac{d^2X}{dx^2}.$$

The left half of this equation does not depend on $\underline{x}$:

$$\frac{1}{T}\frac{dT}{d\tau} = f(x),$$

while the right hand does not depend on τ:

$$\frac{a}{X}\cdot\frac{d^2X}{dx^2} = f(\tau).$$

Consequently, the common value of the right and left parts of this equation should not depend either on $\underline{x}$ or on τ; therefore

$$\frac{1}{T}\frac{dT}{d\tau} = \frac{a_0}{X}\frac{d^2X}{dx^2} = \text{const}.$$

We have thus obtained two equations

$$\frac{1}{T}\frac{dT}{d\tau} = \text{const}; \qquad (6.7)$$

$$\frac{a_0}{X}\cdot\frac{d^2X}{dx^2} = \text{const}. \qquad (6.8)$$

From Eq. (6.7) we find that

$$T = c_1 \cdot e^{\text{const}\cdot\tau} = c_1 \cdot \exp(\text{const}\cdot\tau),$$

where c_1 is an integration constant. From Condition (6.3) it follows that when the plate is heated $\partial\vartheta/\partial\tau < 0$. Consequently, we also have $dT/d\tau < 0$. The last inequality will be satisfied if

$$\text{const} = -c_2^2 < 0.$$

Then

$$T = c_1 \cdot \exp(-c_2^2 \cdot \tau) \qquad (6.9)$$

and in lieu of Eq. (6.8) we obtain

$$\frac{d^2X}{dx^2} + \frac{c_2^2}{a_0}X = 0.$$

The general integral of the left equation has the following form [5]:

$$X = c_4 \cdot \sin\left(\frac{c_3}{\sqrt{a_v}} x\right) + c_5 \cdot \cos\left(\frac{c_3}{\sqrt{a_v}} x\right).$$

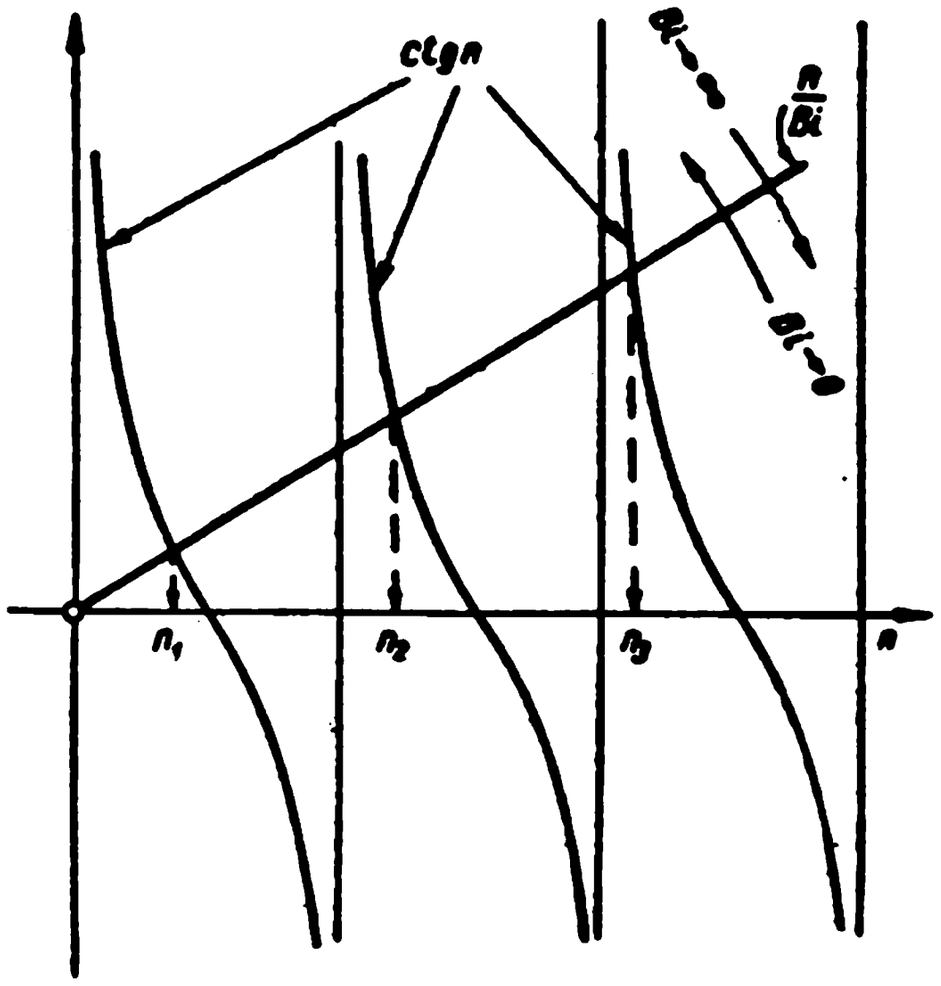

Fig. 20. Scheme for graphical solution of the transcendental equation cot n = n/Bi.

Using the symmetry conditions (6.5) we obtain $c_3 = 0$. Consequently,

$$X = c_4 \cdot \cos\left(\frac{c_3}{\sqrt{a_v}} x\right). \qquad (6.10)$$

Substituting Eqs. (6.10) and (6.9) into Eq. (6.6), we obtain

$$\vartheta = c_4 \cdot \exp\left(-c_1^2 \cdot \tau\right) \cdot \cos\left(\frac{c_3}{\sqrt{a_v}} x\right). \qquad (6.11)$$

To determine the constant c_2 and c_5 we have the initial condition (6.2) and the boundary condition (6.4). In order to find c_2, the solution (6.11) must be substituted in Eq. (6.4). After transformation we obtain a transcendental equation for c_2

$$\frac{\lambda}{\alpha} \frac{c_2}{\sqrt{a_v}} = \operatorname{ctg}\left(\frac{c_2 \cdot \delta}{\sqrt{a_v}}\right).$$

If we put $(c_2 \cdot \delta)/\sqrt{a_v} \equiv n$ and $\alpha\delta/\lambda \equiv Bi$, then the equation for the determination of c_2 assumes the following form:

$$\frac{n}{Bi} = \operatorname{ctg} n. \qquad (6.12)$$

The scheme for graphical solution of this equation is shown in

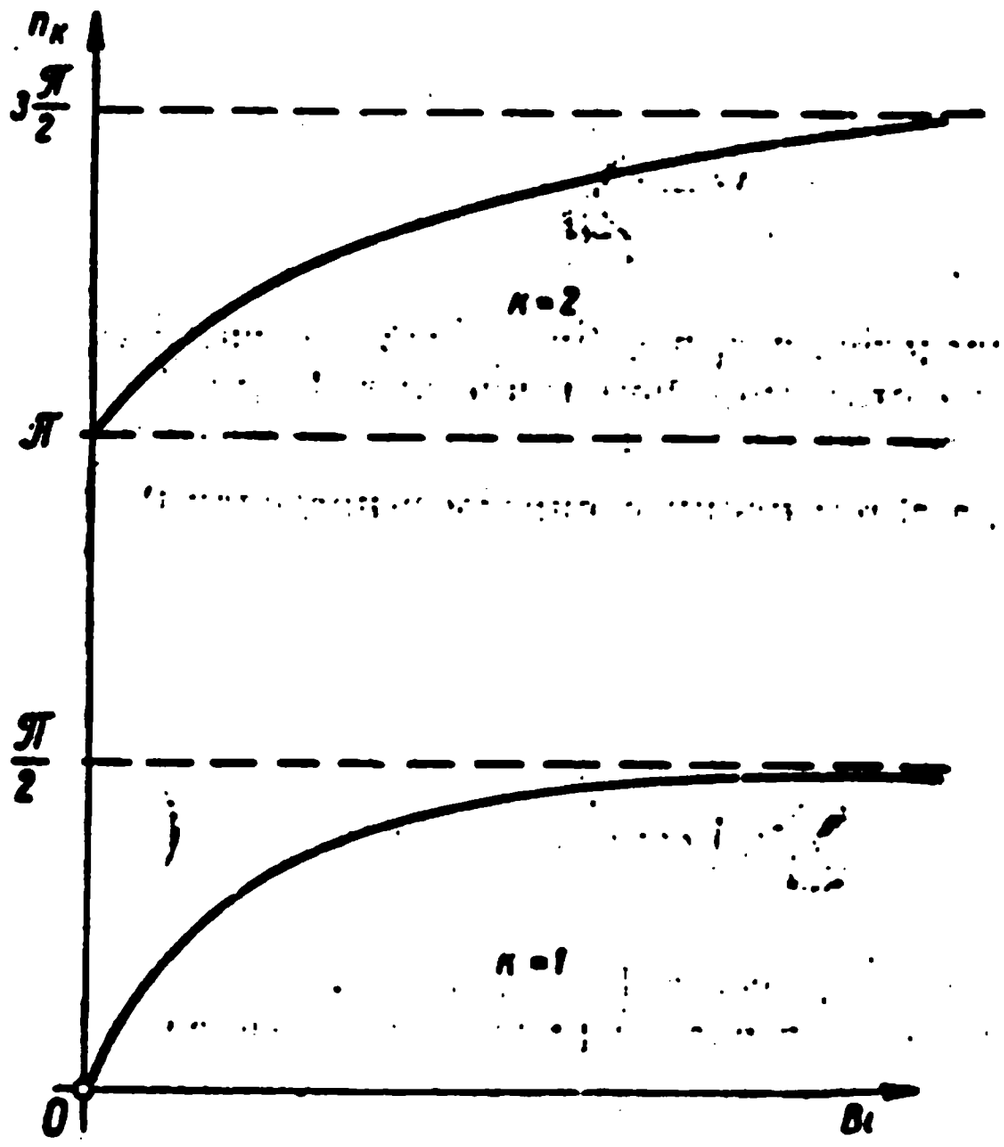

Fig. 21. Character of the dependence of the eigenvalue on its serial number and on the Biot number.

Fig. 20. It is seen from this figure that Eq. (6.12) has an infinite number of solutions n_k ($k = 1, 2, 3, \ldots$).

The quantities n_k are called the eigenvalues of the problem. From Eqs. (6.12) and from Fig. 20 it also follows that the eigenvalue depends on its ordinal number $\underline{k}$ and on the Biot number

$$n_k = n_k(k, Bi).$$

This dependence has been tabulated [34] and is schematically represented in Fig. 21. Thus, the constant c_2 has an infinite number of values

$$c_2 = n_k \cdot \frac{\sqrt{a_0}}{\delta} \quad (k = 1, 2, 3, \ldots).$$

Substituting the value of c_2 into Eq. (6.11) we obtain a particular solution of the problem

$$\vartheta_k = c_k \cdot \exp\left(-n_k^2 \frac{a\tau}{\delta}\right) \cdot \cos\left(n_k \frac{x}{\delta}\right). \qquad (6.13)$$

However, this equation cannot be used as a calculation formula for two

reasons: it leads to a whole set of solutions ϑ_k ($k = 1, 2, 3, \ldots$) and does not make it possible to determine c_5 from the initial condition (6.2).

To eliminate these difficulties, the general solution of the problem under consideration is represented in the form of a sum of all the partial solutions*

$$\vartheta = \sum_{k=1}^{\infty} \vartheta_k = \sum_{k=1}^{\infty} c_k \cdot \exp\left(-n_k^2 \frac{a\tau}{\delta^2}\right) \cdot \cos\left(n_k \cdot \frac{x}{\delta}\right). \qquad (6.14)$$

Substituting Eq. (6.14) into the initial condition (6.2), we obtain

$$\Delta t = \sum_{k=1}^{\infty} c_k \cdot \cos\left(n_k \cdot \frac{x}{\delta}\right).$$

To determine c_5 we use the so-called "orthogonality" property of trigonometric functions [5]. For this purpose we multiply the left and right halves of the last equation by $\cos(n_l x/\delta)$, and then integrate within the limits $0 \leq x/\delta \leq 1$

$$\Delta t \int_0^1 \cos\left(n_l \frac{x}{\delta}\right) d\left(\frac{x}{\delta}\right) =$$

$$= \sum_{k=1}^{\infty} c_k \int_0^1 \cos\left(n_k \frac{x}{\delta}\right) \cos\left(n_l \frac{x}{\delta}\right) d\left(\frac{x}{\delta}\right). \qquad (6.15)$$

Here l is an arbitrary but fixed value of $\underline{k}$. The "orthogonality" property is written in this case in the form

$$\int_0^1 \cos\left(n_k \frac{x}{\delta}\right) \cdot \cos\left(n_l \frac{x}{\delta}\right) d\left(\frac{x}{\delta}\right) = \begin{cases} 0, & \text{if } k \neq l, \\ \int_0^1 \cos^2\left(n_l \frac{x}{\delta}\right) d\left(\frac{x}{\delta}\right), & \text{if } k = l. \end{cases} \qquad (6.16)$$

Substituting Eq. (6.16) into Eq. (6.15) we obtain

$$c_5 = \Delta t \frac{\int_0^1 \cos\left(n_l \frac{x}{\delta}\right) d\left(\frac{x}{\delta}\right)}{\int_0^1 \cos^2\left(n_l \frac{x}{\delta}\right) d\left(\frac{x}{\delta}\right)}.$$

Substituting the obtained expression for c_5 into Eq. (6.14) we obtain after calculations the following calculation formula

$$\frac{t_{\infty}-t}{t_{\infty}-t_0}=2\sum_{k=1}^{\infty}\frac{\sin n_k\cdot\cos\left(n_k\frac{x}{\delta}\right)}{n_k+\sin n_k\cdot\cos n_k}\cdot\exp\left(-n_k^2\frac{a\tau}{\delta^2}\right).$$

where $n_k = n_k(k, Bi)$ is determined from a table [34] and l is replaced by $\underline{k}$, since l is an arbitrary value of $\underline{k}$.

The quantity $a\tau/\delta^2$ is called the Fourier number and is designated Fo, while the quantity $(t_{zh} - t)/(t_{zh} - t_0) \equiv \Theta$ is called the dimensionless temperature. Now the calculation formula assumes the form

$$\Theta=2\sum_{k=1}^{\infty}\frac{\sin n_k\cdot\cos(n_k\cdot X)}{n_k+\sin n_k\cdot\cos n_k}\exp(-n_k^2\cdot Fo). \qquad (6.17)$$

Computation Charts

It is inconvenient to use Eq. (6.17) for practical calculations, since this entails the summation of a series and the determination of eigenvalues. To simplify the problem, Eq. (6.17) is represented in the form of auxiliary computation charts as shown in Figs. 22 and 23. For this purpose we make use of the fact that the computation equation (6.17) relates four dimensionless quantities Θ, X, Bi, and Fo and has therefore the following structure:

$$\Theta = \Theta(X, Bi, Fo).$$

In addition, in many practical calculations it is necessary to determine either the temperature $\Theta_1 = \Theta$ (X = 1, Bi, Fo) on the surface of the plate, where X = 1, or the temperature $\Theta_0 = \Theta$ (X = 0, Bi, Fo) in its central plane, where X = 0. The equations $\Theta_1 = \Theta_1$(Bi, Fo) and $\Theta_0 = \Theta_0$(Bi, Fo) are plotted in semilogarithmic coordinates (see Figs. 22 and 23), where the ordinates represent the logarithms of Θ_1 or Θ_0, while the abscissas represent the Fourier numbers; the Biot number is used in this case as a parameter. By making use of these charts we can

carry out calculations of three types.

– For a specified duration of heating (cooling) of a plate, i.e., the number Fo and for specified intensity of heat delivery on its surface, i.e., the Biot number, determine the temperatures Θ_1 and Θ_0.

– For specified Θ_1 (or Θ_0) and Bi, determine Fo, i.e., determine the duration of the heating (cooling).

– For specified Fo and Θ_1 (or Θ_0) determine the required heat-transfer intensity on the surfaces of the plate, i.e., the Bi number.

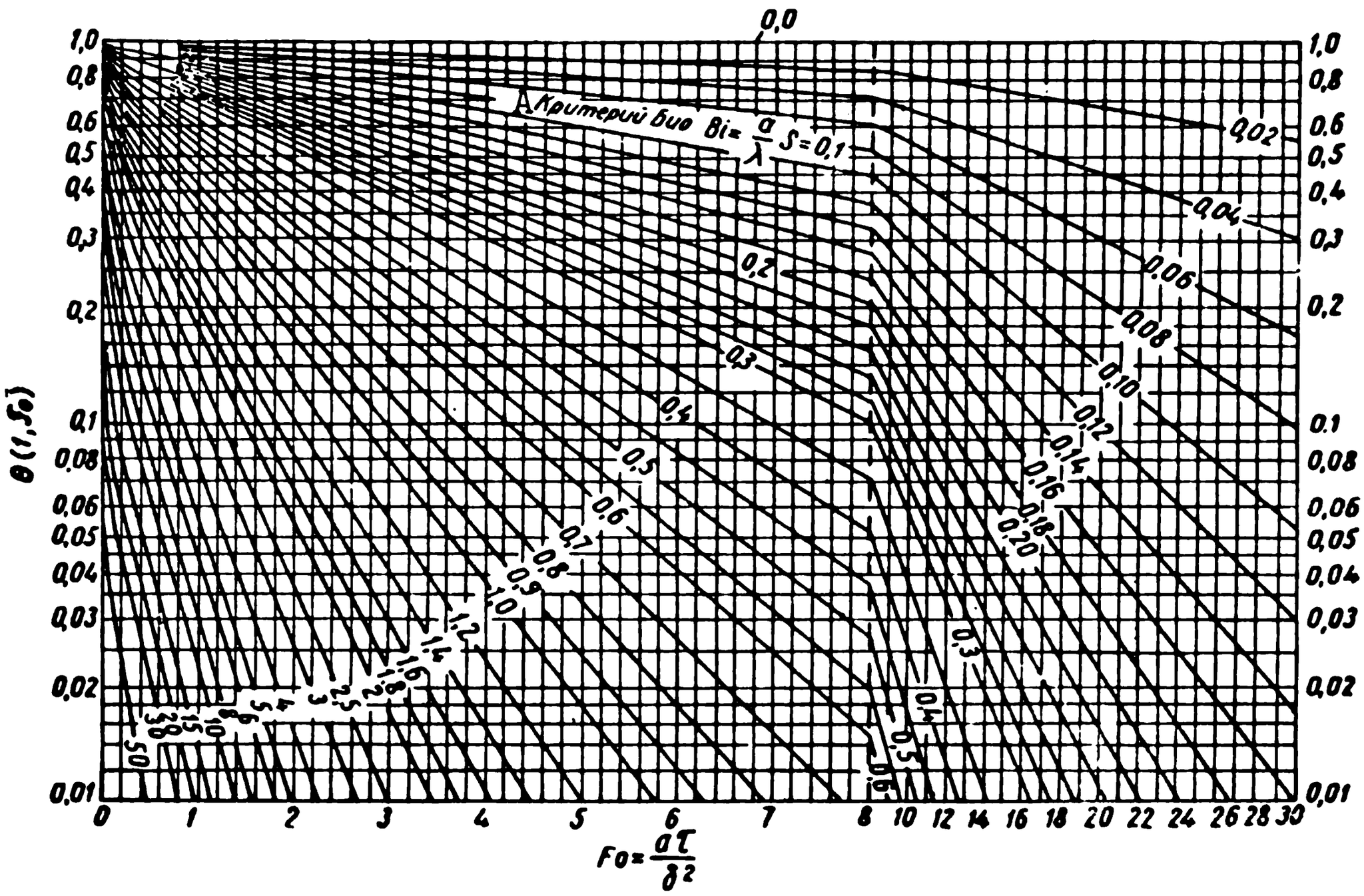

Fig. 22. Chart for the calculation of the surface temperature of a plate that is being heated or cooled. A) Biot number.

Figures 22 and 23 show the advantages of the dimensionless calculation equation. Indeed, the solution of this problem, represented in the form of a connection between the dimensional quantities, has the following form:

$$\vartheta = \vartheta(x, \tau, a_v, \delta, \lambda, \alpha, \Delta t)$$

and contains eight dimensional quantities, whereas in dimensionless form

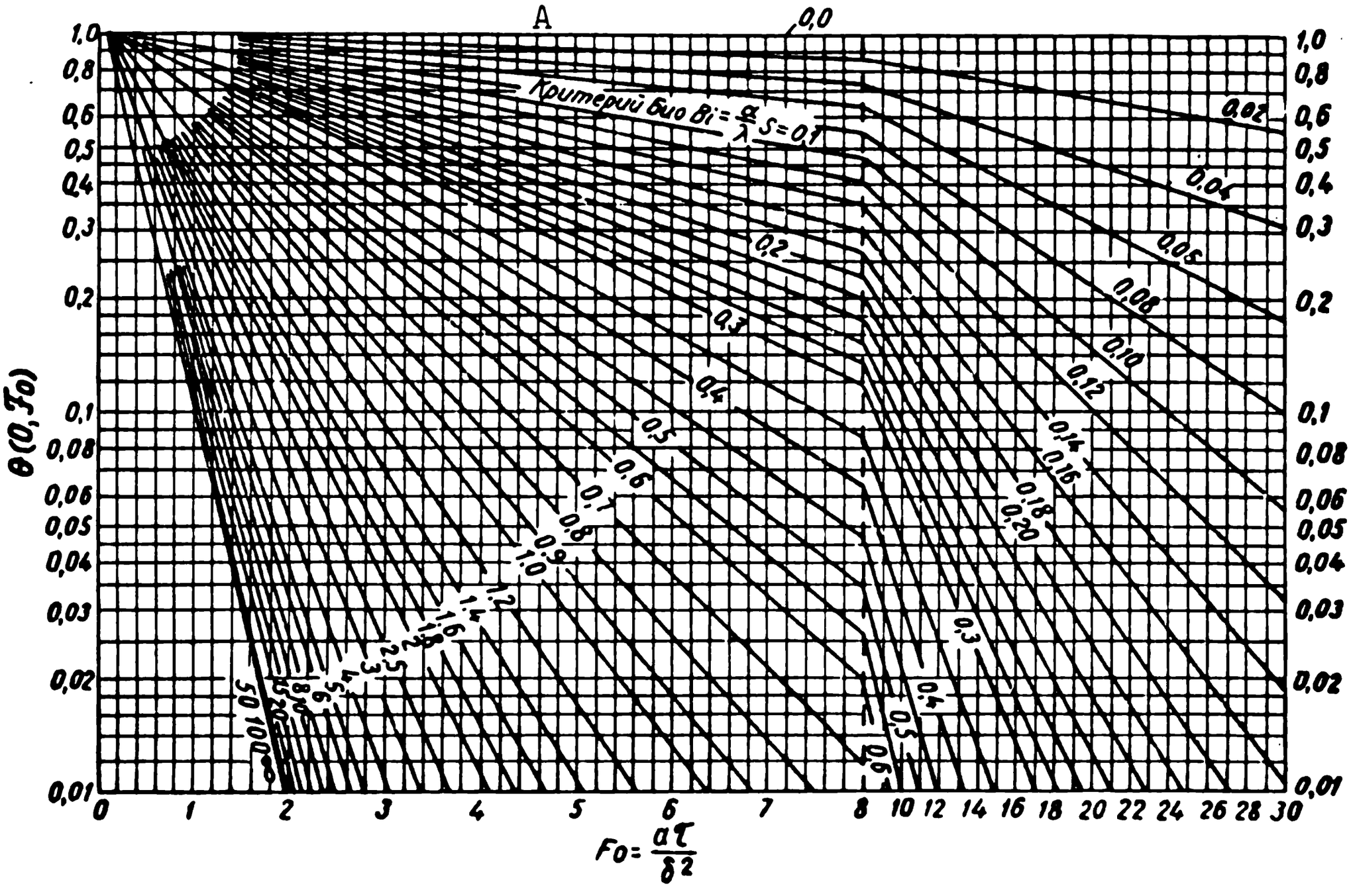

Fig. 23. Chart for calculation of the temperature in the midplane of a plate that is being heated or cooled. A) Biot number.

$$\Theta = \Theta(X, Fo, Bi)$$

this solution contains only four quantities, which simplifies the construction of the computation charts.

Heat Conduction in Certain Bodies that Results from the Intersection of Thin Plates

The plots of the functions $\Theta_1(\mathrm{Fo}, \mathrm{Bi})$ and $\Theta_0(\mathrm{Fo}, \mathrm{Bi})$ make it possible to calculate the heating or cooling of rectangular bars and parallelepipeds. Indeed, each of these bodies can be regarded as a result of the intersection of two or three mutually perpendicular thin plates, having the same uniqueness conditions as the bar or the parallelepiped. It turns out here that the dimensionless temperature field, say in a bar, is equal to the product of the dimensionless temperature fields Θ_x and Θ_y in the thin plates whose intersection at right angle forms the bar under consideration

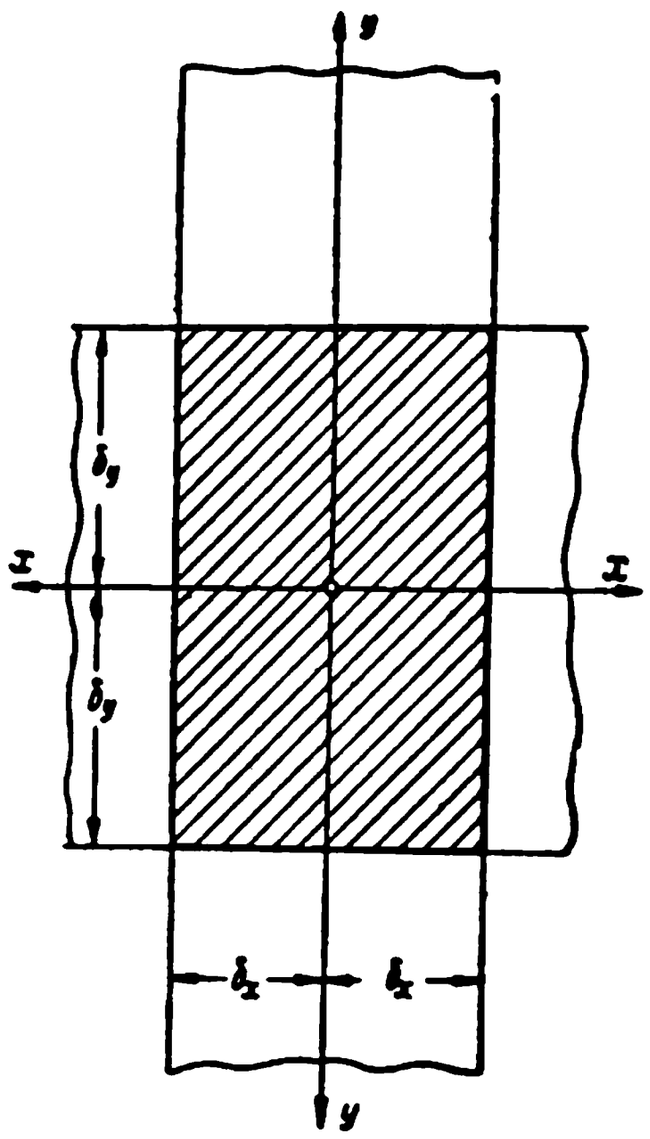

Fig. 24. Rectangular bar, which is the result of the intersection of two thin plates at right angles.

$$\Theta = \Theta_x \cdot \Theta_y \qquad (6.18)$$

where

$$\Theta_y = \Theta_y(X, Fo_x, Bi_x) \text{ and } \Theta_x = \Theta_x(Y, Fo_y, Bi_y).$$

In addition

$$X \equiv \frac{x}{\delta_x},\ Y \equiv \frac{y}{\delta_y},\ Fo_x \equiv \frac{a_v \cdot \tau}{\delta_x^2},$$

$$Fo_y \equiv \frac{a_v \cdot \tau}{\delta_y^2},\ Bi_x \equiv \frac{\alpha \cdot \delta_x}{\lambda_F} \text{ and } Bi_y \equiv \frac{\alpha \cdot \delta_y}{\lambda_F}.$$

This rule holds also for other bodies, which can be regarded as a result of the intersection of two or three bodies of simple shape, having only one characteristic dimension.

For example, a short cylinder can be regarded as a result of the intersection of a long cylinder and a thin plate at right angles to each other, etc.

Let us prove the correctness of this rule using as a particular example a rectangular bar as shown in Fig. 24. The bar is made of homogeneous isotropic material and has no internal heat sources. The nonstationary field of the excess temperature $\vartheta \equiv t_{zh} - t$, produced in the bar when the latter is heated in a stream of liquid, obeys the Fourier equation

$$\frac{\partial \vartheta}{\partial \tau} = a_v \cdot \left(\frac{\partial^2 \vartheta}{\partial x^2} + \frac{\partial^2 \vartheta}{\partial y^2} \right). \qquad (6.19)$$

and also the initial and boundary conditions

$$\vartheta|_{\tau=0;\ 0<x<\delta_x;\ 0<y<\delta_y} = \Delta t; \qquad (6.20)$$

$$\vartheta|_{\tau\to\infty;\ 0<x<\delta_x;\ 0<y<\delta_y} \to 0; \qquad (6.21)$$

$$\left.\begin{aligned} \left| \lambda_F \frac{\partial \vartheta}{\partial x} + \alpha \vartheta \right|_{x=+\delta_x-0;\ \tau>0} = 0; \\ \left| \lambda_F \frac{\partial \vartheta}{\partial y} + \alpha \vartheta \right|_{y=+\delta_y-0;\ \tau>0} = 0; \end{aligned}\right\} \qquad (6.22)$$

$$\left.\begin{aligned}\frac{\partial\vartheta}{\partial x}\Big|_{x=0;\ \tau>0;\ 0<y<\delta_y} &= 0;\\ \frac{\partial\vartheta}{\partial y}\Big|_{y=0;\ \tau>0;\ 0<x<\delta_x} &= 0.\end{aligned}\right\} \tag{6.23}$$

Let us show that Eq. (6.18) satisfies all the conditions (6.19)-(6.23). For this purpose we rewrite it in a different form

$$\vartheta = \frac{\vartheta_x\cdot\vartheta_y}{\Delta t}. \tag{6.24}$$

where $\vartheta_y = \vartheta_y(x, \tau)$ and $\vartheta_x = \vartheta_x(y, \tau)$ are the nonstationary fields of the excess temperature in the thin plates whose intersection forms the bar under consideration. It is assumed here that the processes of heat conduction have in these plates the same physical, time, and boundary conditions as in the bar.

Substituting Eq. (6.24) into Eqs. (6.19)-(6.23) and taking into account the last remark, we obtain

$$\Delta t\cdot\vartheta_x\cdot\left(\frac{\partial\vartheta_y}{\partial\tau} - a_0\frac{\partial^2\vartheta_y}{\partial x^2}\right) + \Delta t\cdot\vartheta_y\cdot\left(\frac{\partial\vartheta_x}{\partial\tau} - a_0\frac{\partial^2\vartheta_x}{\partial y^2}\right) = 0;$$

$$\frac{\Delta t\cdot\Delta t}{\Delta t} = \Delta t;\quad \frac{\vartheta_x\cdot\vartheta_y}{\Delta t}\Big|_{\tau\to\infty} \to 0;$$

$$\left|\lambda_F\frac{\vartheta_x}{\Delta t}\cdot\frac{\partial\vartheta_y}{\partial x} + \alpha\,\frac{\vartheta_y\cdot\vartheta_x}{\Delta t}\right|_{x=+\delta_x-0;\ \tau>0} = 0;$$

$$\left|\lambda_F\frac{\vartheta_y}{\Delta t}\cdot\frac{\partial\vartheta_x}{\partial y} + \alpha\,\frac{\vartheta_x\cdot\vartheta_y}{\Delta t}\right|_{y=+\delta_y-0;\ \tau>0} = 0;$$

$$\frac{\vartheta_x}{\Delta t}\cdot\frac{\partial\vartheta_y}{\partial x}\Big|_{x=0;\ \tau>0;} = 0;$$

$$\frac{\vartheta_y}{\Delta t}\cdot\frac{\vartheta_x}{\partial y}\Big|_{y=0;\ \tau>0;} = 0.$$

Consequently, Eq. (6.24), meaning also Eq. (6.18), satisfies all the conditions of the problem involving the heating of a bar in a liquid stream. Consequently, Eq. (6.18) can be used to calculate the heating of the bar.

The method developed here greatly broadens the region of applicability of the charts (see Figs. 22 and 23). Analogous charts [34] for long cylinders in conjunction with the charts given here for the plate enable us to calculate the heating (cooling) of short cylinders, which

can be regarded as a result of the intersection of a long cylinder and a thin plate.

Influence of the Biot and Fourier Numbers on the Temperature Field in a Plate

With increasing Fourier number, the convergence of the series in Eq. (6.17) improves, since each succeeding eigenvalue n_{k+1} is larger than the preceding one n_k. Therefore, starting with a certain value of the Fourier number this series can be replaced with sufficient degree of accuracy by its first term

$$\Theta \approx 2 \frac{\sin n_1 \cdot \cos (n_1 \cdot X)}{n_1 + \sin n_1 \cdot \cos n_1} \exp \left(- n_1^2 \cdot Fo\right) = A_1 \cdot \exp \left(- n_1^2 \cdot Fo\right) \qquad (6.25)$$

Taking logarithms of the last equation, we obtain

$$\ln \Theta \approx \ln A_1 - n_1^2 \cdot Fo.$$

Consequently, in semilogarithmic coordinates (ln Θ, Fo) the calculation formula can be represented, starting with a certain value of Fo, in the form of a straight line. Figures 22 and 23 confirm this consideration.

The region where Formula (6.17) degenerates into the first term of the series is customarily called the regular thermal mode [25]. The regular field of excess temperature

$$\vartheta = A_1 (X) \cdot \Delta t \cdot \exp \left(- n_1^2 \cdot Fo\right)$$

has the remarkable property that it remains similar to itself at all succeeding instants of time. Indeed, the ratio of the excess temperatures ϑ_A and ϑ_B at points with abscissas X_A and X_B does not depend on the number Fo, i.e., on the time

$$\frac{\vartheta_A}{\vartheta_B} = \frac{A_1 (X_A) \cdot \Delta t \cdot \exp \left(- n_1^2 \cdot Fo\right)}{A_1 (X_B) \cdot \Delta t \cdot \exp \left(- n_1^2 \cdot Fo\right)} = \frac{A_1 (X_A)}{A_1 (X_B)}.$$

Such processes are called self-similar in time. This property of regular thermal mode can be conveniently used to define the concept of the regular mode (see §7).

At small values of the Fourier number (Fo << 1) the series in the computation equation converges quite weakly. To calculate the heat conduction in this region it is more convenient to use the operator method [34].

We shall start the analysis of the influence of the Biot number on the temperature field in a plate by considering the case when Bi → ∞. When Bi → ∞ (see Figs. 20 and 21) we have

$$\begin{aligned} n_1 &\to \pi/2; \\ n_2 &\to 3\pi/2; \\ &\cdots\cdots\cdots \\ n_k &\to (2k-1)\,\pi/2. \end{aligned}$$

with

$$\begin{aligned} \sin n_k &\to \sin (2k-1)\,\pi/2 = (-1)^{k-1}; \\ \cos n_k &\to \cos (2k-1)\,\pi/2 = 0, \end{aligned}$$

and Eq. (6.17) assumes the form

$$\Theta\Big|_{Bi\to\infty} \to \frac{4}{\pi} \sum_{k=1}^{k\to\infty} \frac{(-1)^{k-1} \cdot \cos\left[(2k-1)\frac{\pi}{2} \cdot X\right]}{2k-1} \times$$

$$\times \exp\left[-(2k-1)^2 \frac{\pi^2}{4} \cdot Fo\right].$$

In this case the temperature Θ_1 on the cooled surface of the plate, where X = 1, is equal to zero:

$$\Theta_1\Big|_{Bi\to\infty} \to \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \cdot \cos\left|(2k-1)\frac{\pi}{2}\right| \cdot \exp\left[-(2k-1)^2 \frac{\pi^2}{4} \cdot Fo\right]}{2k-1} = 0,$$

since cos (2k − 1) π/2 = 0.

Consequently, as Bi → ∞, when the internal thermal resistance greatly exceeds the external one $\delta/\lambda >> 1/\alpha$, the temperature of the cooled surface of the plate $t|_{x=\delta}$ immediately becomes equal to the temperature of the heating liquid t_x. In other words, when Bi → ∞ the third kind of boundary conditions goes over into the first kind. The temperature distribution in a plate for Bi → ∞ is shown in Fig. 25.

It follows from more detailed calculations [34] that the temperature field has such properties when Bi ≥ 100.

Let us consider another limiting case, when the internal thermal resistance δ/λ is small compared with the external one $1/\alpha$, i.e., when $Bi \to 0$. In this case

$$\begin{aligned} &n_1 \to 0;\\ &n_2 \to \pi;\\ &\cdots\cdots\\ &n_k \to (k-1)\pi. \end{aligned}$$

Consequently,

$$A_1 = \lim_{n_1 \to 0} \frac{2\frac{\sin n_1}{n_1}}{1+\frac{\sin n_1}{n_1}\cdot\cos n_1} = 1;$$

$$A_2 = \frac{2\frac{\sin \pi}{\pi}}{1+\frac{\sin \pi}{\pi}\cdot\cos \pi} = 0;$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$A_k = \frac{2\frac{\sin (k-1)\pi}{(k-1)\pi}}{1+\frac{\sin (k-1)\pi}{(k-1)\pi}\cdot\cos[(k-1)\pi]} = 0.$$

Thus, for $k \geq 2$ all $A_k \to 0$. Consequently, when $Bi \to 0$ we have

$$\Theta|_{Bi\to 0} \to \exp(-n_1^2 \cdot Fo)|_{n_1\to 0} \to 1.$$

Further, let us consider a case when $Bi \neq 0$ ($Bi \ll 1$) (see Fig. 20). In this case, $\tan n_1 \approx n_1$ and from the transcendental equation

$$\frac{n_1}{Bi} = \frac{1}{\operatorname{tg} n_1}$$

it follows that $n_1 \approx \sqrt{Bi}$.

Therefore when $Bi \ll 1$ we have

$$\Theta \approx \cos(X\cdot\sqrt{Bi})\cdot\exp(-Bi\cdot Fo).$$

The dependence of Θ on the abscissa X for $Bi \ll 1$ is quite weak. Therefore the change in temperature occurs in this case principally in time. The character of variation of the temperature field in the plate for $Bi \ll 1$ is shown in Fig. 26.

More detailed calculations show that the condition $Bi \ll 1$ is satisfied when $Bi \leq 0.1$ [34]. Thus, the limiting cases $Bi \gg 1$ and $Bi \ll 1$ correspond to the conditions $Bi \geq 100$ and $Bi \leq 0.1$, and the

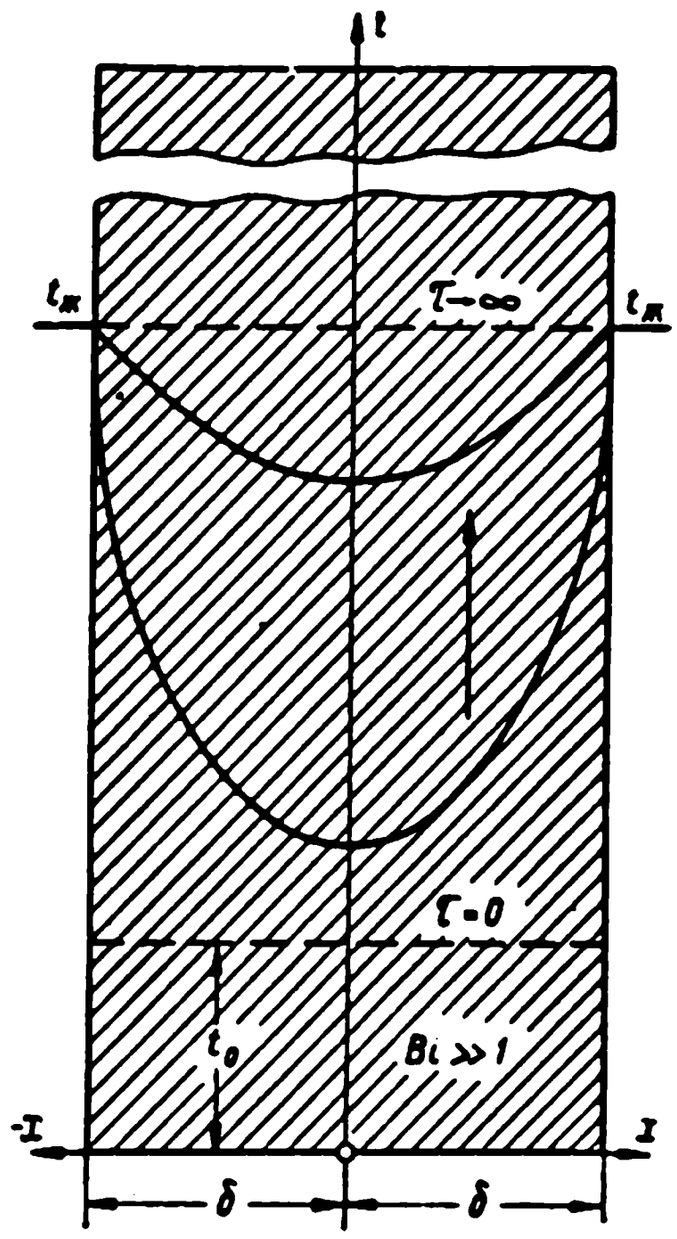

Fig. 25. Nonstationary temperature field in a plate with a large Biot number.

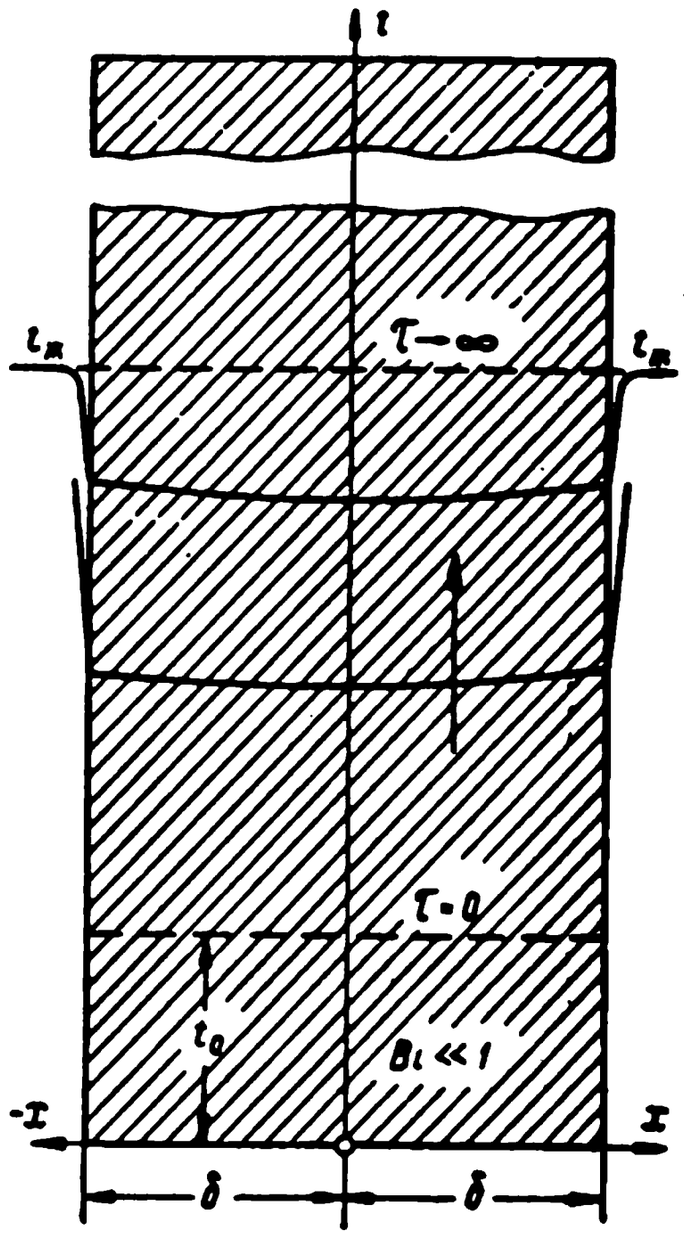

Fig. 26. Nonstationary temperature field in a plate with a small Biot number.

range of variation of the Biot number from 0.1 to 100 is intermediate. In this region the calculation of heat conduction in the plate must be carried out by using Eq. (6.17) or the corresponding charts.

Plate with Internal Heat Sources

Let us consider a thin homogeneous and isotropic plate situated in a stream of liquid with temperature t_{zh}. At the initial instant of time heat starts to be released in the plate because of the action of internal sources, which have a constant strength q_V kcal/m^3-hr.

The nonstationary temperature field in the plate obeys the Fourier equation

$$\frac{\partial t}{\partial \tau} = a_0 \frac{\partial^2 t}{\partial x^2} + \frac{qV}{C_0}. \qquad (6.26)$$

the initial and boundary conditions

$$t\big|_{\tau=0} = t_ж; \quad -\lambda_F \frac{\partial t}{\partial x}\Big|_{x=+0-0} = \alpha (t|_{x=0} - t_ж), \qquad (6.27)$$

and also the symmetry condition

$$\left.\frac{\partial t}{\partial x}\right|_{x=0;\ \tau>0} = 0. \qquad (6.28)$$

After the lapse of a prolonged time interval a stationary temperature distribution is established in the plate

$$t_\infty \equiv \lim_{\tau\to\infty} t = t_\infty + \frac{q_V\cdot\delta^2}{2\lambda_F}\left(1+\frac{2}{Bi}-\frac{x^2}{\delta^2}\right). \qquad (6.29)$$

In Eqs. (6.26)-(6.29) the quantities a_v, C_v, q_v, λ, α, δ are specified parameters. The character of the temperature field in the plate is shown in Fig. 27. Calculation of the heat conduction is materially simplified in this case if one introduces the excess temperature $\vartheta \equiv \equiv t_\infty - t$, reckoned from the stationary field $t_\infty(x)$. We note that the stationary field $t_\infty(x)$ obeys the following equation and boundary conditions

$$\left.\begin{aligned} -\lambda_F\left.\frac{\partial t_\infty}{\partial x}\right|_{x=+\delta-0} &= \alpha\left(t_\infty|_{x=\delta}-t_\infty\right);\\ \frac{\partial t_\infty}{\partial\tau} &= a_v\frac{\partial^2 t_\infty}{\partial x^2}+\frac{q_v}{C_v};\ \left.\frac{\partial t_\infty}{\partial x}\right|_{x=0}=0. \end{aligned}\right\} \qquad (6.30)$$

Subtracting Eqs. (6.26)-(6.28) from Eqs. (6.30) we obtain

$$\frac{\partial\vartheta}{\partial\tau} = a_v\frac{\partial^2\vartheta}{\partial x^2}; \qquad (6.31)$$

$$\left.\begin{aligned} -\lambda_F\cdot\left.\frac{\partial\vartheta}{\partial x}\right|_{x=+\delta-0} &= \alpha\cdot\vartheta|_{x=\delta};\\ \vartheta|_{\tau=0;\ x} &= \frac{q_V\cdot\delta^2}{2\lambda_F}\left(1+\frac{2}{Bi}-\frac{x^2}{\delta^2}\right);\\ \left.\frac{\partial\vartheta}{\partial x}\right|_{x=0;\ \tau>0} &= 0. \end{aligned}\right\} \qquad (6.32)$$

The character of variation of the field of excess temperature ϑ is shown in Fig. 28. The problem (6.32) is solved by separation of variables, as was done above. Using this method, we obtain the general computation equation

$$\Theta \equiv \frac{t_\infty - t}{\frac{q_V\cdot\delta^2}{2\lambda}} = \sum_{k=1}^{\infty} A_k\cdot\cos\left(n_k\cdot X\right)\cdot\exp\left(-n_k^2\cdot Fo\right),$$

where the eigenvalues n_K satisfy the transcendental equation (6.12),

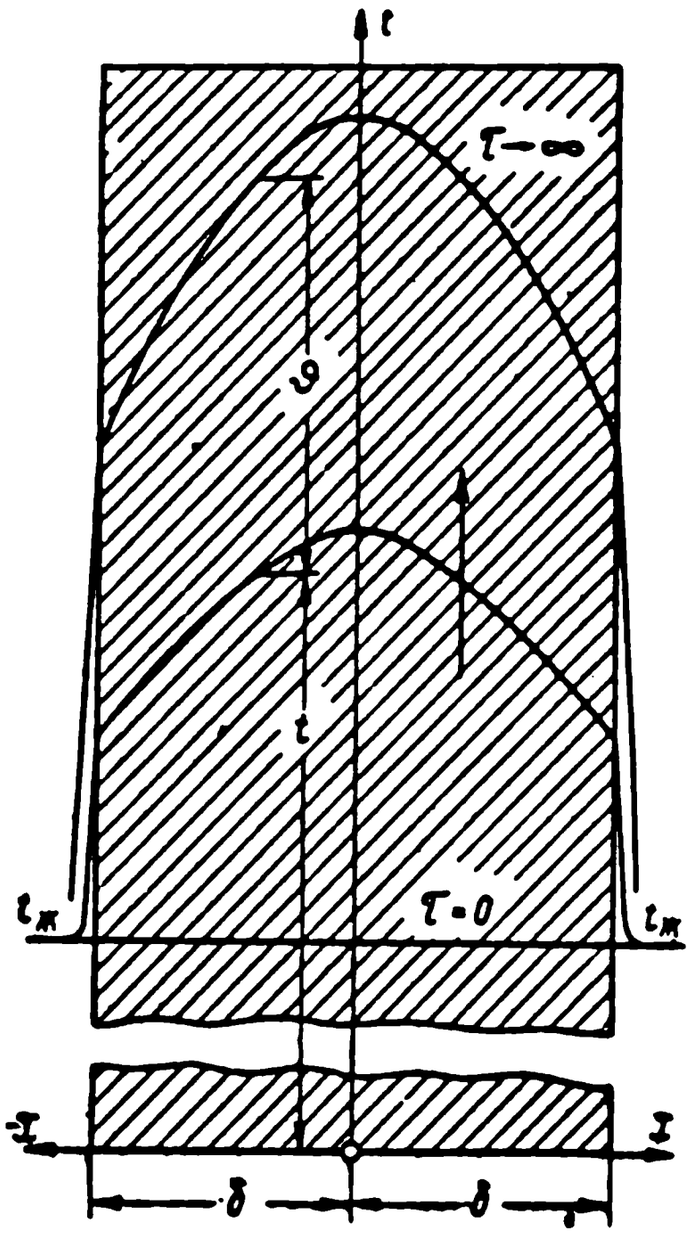

Fig. 27. Nonstationary temperature field in a plate with internal heat sources.

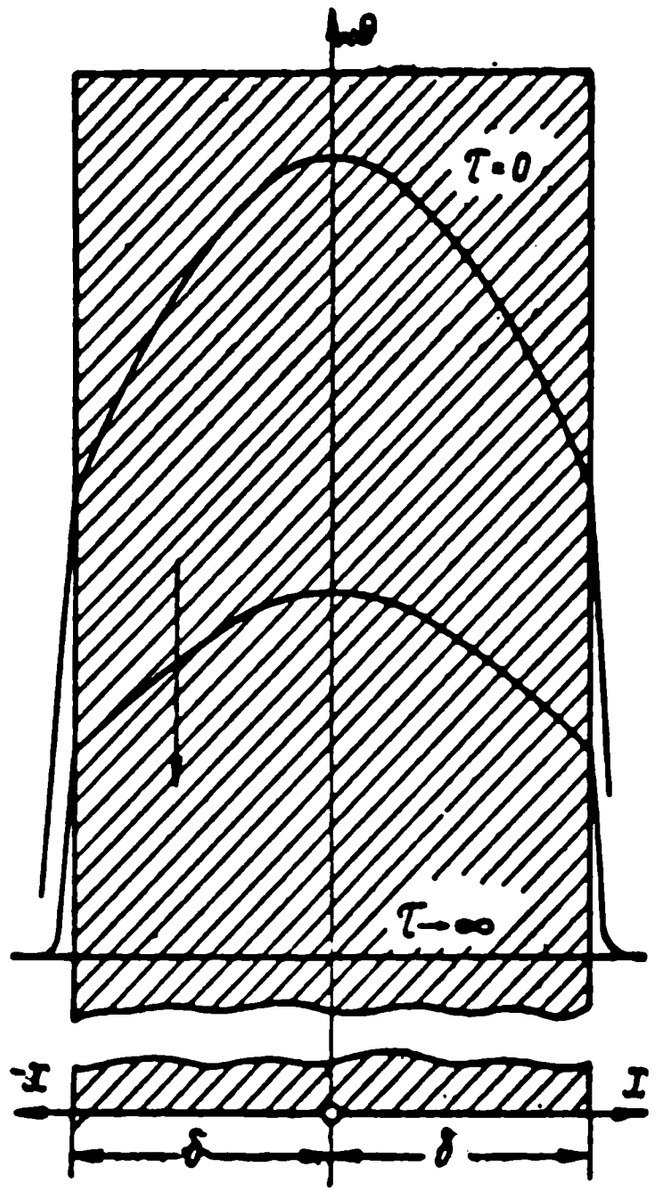

Fig. 28. Nonstationary field of excess temperature in a plate with internal heat sources.

and the values of A_k are

$$A_k = \frac{\int_0^1 \left(1 + \frac{2}{Bi} - X^2\right) \cdot \cos(n_k \cdot X) \cdot dX}{\int_0^1 \cos(n_k \cdot X)\, dX}.$$

Account of the Dependence of the Specific Heat and Heat Conduction on the Temperature

Let us consider the heating of a plate situated in a stream of liquid. The plate is made of a material whose specific heat and heat conduction depend on the temperature $C_v = C_v(t)$ and $\lambda = \lambda(t)$. The nonstationary temperature field in this plate is described by the following equations:

$$\left.\begin{aligned} C_v(t)\frac{\partial t}{\partial \tau} &= \frac{\partial}{\partial x}\left[\lambda(t)\frac{\partial t}{\partial x}\right]; \\ t|_{\tau=0;\ 0<x<\delta} &= t_0; \\ t|_{\tau\to\infty;\ 0<x<\delta} &\to t_ж > t_0; \end{aligned}\right\} \qquad (6.33)$$

$$\left.\begin{aligned} \frac{\partial t}{\partial x}\Big|_{x=0;\ \tau>0} &= 0; \\ -\lambda_F(t)\,\frac{\partial t}{\partial x}\Big|_{x=+\delta-0} &= \alpha(t|_{x=\delta} - t_{\infty}). \end{aligned}\right\} \qquad (6.33)$$

The quantities δ, α, t_{zh}, and t_0, and also the functions $\lambda_F(t)$ and $C_v(t)$ are assumed specified. The problem (6.33) is nonlinear. It is therefore difficult to solve it analytically in general form. The calculation of the heating of the plate must be carried out by numerical methods (see, for example, [6]). However, when the dependence of the heat conduction and the specific heat on the temperature has the same character

$$C_v(t) = \frac{1}{a_v}\cdot\lambda_F(t),$$

i.e., when the temperature conductivity of the material of the plate is constant,* the nonlinear problem (6.33) can be reduced approximately to the linear one and then solved by separation of variables. For this purpose we introduce the Kirchhoff variable

$$\vartheta \equiv \int_{t_0}^{t} \lambda(t)\,dt. \qquad (6.34)$$

Differentiating (6.34) with respect to τ and $\underline{x}$, we obtain

$$\frac{\partial \vartheta}{\partial \tau} = \lambda(t)\cdot\frac{\partial t}{\partial \tau};\quad \frac{\partial \vartheta}{\partial x} = \lambda(t)\cdot\frac{\partial t}{\partial x} \qquad (6.35)$$

Substituting Eqs. (6.35) into Eqs. (6.33) we obtain after transformations

$$\left.\begin{aligned} &\frac{\partial \vartheta}{\partial \tau} = a_v\frac{\partial^2 \vartheta}{\partial x^2};\ \vartheta|_{\tau=0;\ 0<x<\delta} = 0; \\ &\vartheta|_{\tau\to\infty;\ 0<x<\delta} \int_{t_0}^{t_{\infty}} \lambda(t)\,dt \equiv \vartheta_{\infty}; \\ &-\frac{\partial \vartheta}{\partial x}\Big|_{x=+\delta-0} = \alpha[t(\vartheta)|_{x=\delta} = t_{\infty}]. \end{aligned}\right\} \qquad (6.36)$$

The problem (6.36) contains a nonlinear term $t(\vartheta)$ in the boundary condition of the third kind. Further simplification of the calculation is possible if the dependence $\lambda = \lambda(t)$ is such that in the specified temperature interval in the plate, $t_0 < t < t_{zh}$, the dependence $t(\vartheta)$

can be approximated with sufficient accuracy by a straight line. We then obtain for the right half of the boundary condition of the third kind

$$t(\vartheta)|_{x=\delta} \approx c_1 \cdot \vartheta|_{x=\delta} + c_2. \qquad (6.37)$$

The system of equations (6.36) and (6.37) is linear and can be solved by separation of variables. It must be noted that Condition (6.37) is equivalent to the approximate condition

$$\lambda \approx \text{const}. \qquad (6.38)$$

Indeed, it follows from Eq. (6.37) that

$$dt/d\vartheta \approx c_1 \text{ and } d\vartheta/dt \approx 1/c_1.$$

From (6.34) we obtain in turn $d\vartheta/dt = \lambda(t)$. Consequently, $\lambda \approx 1/c_1$. However, the approximate condition (6.38) is used only to simplify the right half of the boundary condition of the third kind. All the remaining equations in (6.36) remain unchanged in this case. It must be noted that the equations in (6.36) become linear also when $Bi_{min} \equiv \equiv \alpha \cdot \delta/\lambda_{maks} \gg 1$. Then the boundary condition of the third kind assumes the form of the boundary condition of the first kind

$$\vartheta|_{x=\delta;\ \tau>0} = \vartheta_c$$

and the nonlinearity is eliminated.

Let us consider, finally, a third version, when the problem (6.33) can be solved in analytic form. This version corresponds to the condition $Bi_{maks} \equiv \alpha \cdot \delta/\lambda_{min} \ll 1$. When the latter condition is satisfied the temperature varies insignificantly transversely to the plate. Consequently, the following condition is approximately satisfied in this case:

$$t(x, \tau) \approx {}^{V}\bar{t}(\tau) \equiv \frac{1}{V}\int_V t\,dV \approx t|_{x=\delta}. \qquad (6.39)$$

where ${}^{V}\bar{t}$ is the plate temperature averaged over the volume. In order to use Relation (6.39), let us integrate the Fourier equation over the

volume of the plates

$$\int_V C_0(t)\frac{\partial t}{\partial \tau}dV=\int_V \frac{\partial}{\partial x}\lambda(t)\frac{\partial t}{\partial x}dV. \qquad (6.40)$$

On the basis of Condition (6.39) we have

$$\int_V C_0(t)\frac{\partial t}{\partial \tau}dV\approx C_0(\overline{{}^{V}t})\cdot\frac{d\overline{{}^{V}t}}{d\tau}\cdot V. \qquad (6.41)$$

From the Ostrogradskiy theorem, the boundary condition of the third kind, and Condition (6.40) it follows that

$$\int_V \frac{\partial}{\partial x}\left[\lambda(t)\frac{\partial t}{\partial x}\right]dV=\int_F\left[\lambda(t)\frac{\partial t}{\partial x}\right]_{x=+0-0}dF=$$
$$=\int_F \alpha\left(\overline{{}^{V}t}-t_{\infty}\right)dF=-\alpha\cdot F\left(\overline{{}^{V}t}-t_{\infty}\right). \qquad (6.42)$$

where F is the lateral heat-delivering surface of the plate whose volume is V. Substituting (6.41) and (6.42) into (6.40) we obtain

$$V\cdot C_0\left(\overline{{}^{V}t}\right)\frac{d\overline{{}^{V}t}}{d\tau}=-\alpha\cdot F\left(\overline{{}^{V}t}-t_{\infty}\right). \qquad (6.43)$$

Separating variables and integrating Eq. (6.43) we obtain

$$\int_{t_0}^{V_t}\frac{C_0(\overline{{}^{V}t})\cdot d\overline{{}^{V}t}}{\overline{{}^{V}t}-t_{\infty}}=-\frac{\alpha\cdot F}{V}\int_0^{\tau}d\tau=-\frac{\alpha\cdot F\cdot\tau}{V}. \qquad (6.44)$$

Equation (6.44) is a solution of the problem for $Bi_{maks} \ll 1$. This equation is also valid for a body of arbitrary shape having a volume V and a heat-delivery surface F.

§7. REGULAR MODE OF HEAT-CONDUCTION PROCESSES

Concept of the Regular Thermal Mode

The nonstationary process of heat conduction is called the regular thermal mode if the field of the excess temperature ϑ is self-similar in time, i.e., if it remains similar to itself in the course of time. The temperature ϑ is called the excess temperature if it is reckoned from any temperature t_0 characterizing the given heat conduction process.*

At the present time the following modifications of the regular

mode are known and are used in practice.

The exponential regular mode [25]

$$\vartheta = \Phi_1(l) \cdot e^{-m\tau},$$

which is observed in bodies that have boundary conditions of the third kind (see §6).

The linear regular mode [19b]

$$\vartheta = \Phi_2(l) + m \cdot \tau,$$

which is observed in the presence of boundary conditions of the second kind.

The periodic regular mode [47] is observed in the case of periodic variation of the temperature in some point of the body.

The indicated types of regular modes are used in experimental determinations of the thermal diffusivity a and the thermal conductivity λ of various substances, and also in the determination of the heat-transfer coefficient α. In addition to the foregoing, there exist also other types of regular modes.

By way of an example let us indicate the stationary regular mode of the heat-conduction process in long rods or the regular thermal mode of convective heat exchange in long tubes, when thermal stabilization occurs along the liquid stream.

Thus, both the types of regular modes and the regions of their application vary greatly in the investigations of the heat-transfer processes. They are all characterized, however, by the presence of self-similarity in the temperature field, either with respect to time or with respect to the coordinates, or else with respect to the time and the coordinates simultaneously.

Therefore the notion of the self-similarity of the temperature field serves as the basis of the definition of the concept of the thermal regular mode. Such a definition enables us to use one and the

same self-similarity condition to set up equations for various types of regular modes.

Some Properties of the Regular Mode of Nonstationary Heat-Conduction Processes

The properties of the regular mode of nonstationary heat-conduction processes are best illustrated using as a particular example the heating of a thin plate. At values of Fourier number of the order of unity and more, Fo $\geq$ 1, the series in the computation formula (6.17) can be replaced with sufficient accuracy by its first term

$$\vartheta \approx A_1 \cdot \cos(n_1 \cdot X) \cdot \exp\left(-\frac{n_1^2 \cdot a_0}{\delta^2} \cdot \tau\right).$$

The factor preceding τ is called the rate of the regular mode and denoted

$$\dot{m} \equiv \frac{n_1^2 \cdot a_0}{\delta^2}. \qquad (7.1)$$

Therefore the equation of the exponential regular mode in a plate has the form

$$\vartheta \approx F(x) \cdot e^{-m\tau}. \qquad (7.2)$$

The main property of the field of the excess temperature ϑ is self-similarity with respect to time. This property has served as the basis for the definition of the concept of the regular mode. In addition, the following properties of this mode are the results of Formula (7.2):

1) the rate m of the regular mode does not depend on either the coordinates or the time;

2) the magnitude of the rate m is determined by the geometrical form and dimensions of the body, its physical properties (the thermal diffusivity), and also by the conditions of heat transfer on the surface of the body;

3) the character of the time dependence [exp ($-m\tau$)] of the field

of the excess temperature ϑ is determined by the boundary conditions and is independent of the initial condition (accurate to a proportionality coefficient);

4) if a system of bodies in close contact is in the regular mode, then all the bodies of this system have an identical rate m. Indeed, on the boundary between two bodies k and k + 1 of the system we have

$$F_k(l_{rp})\cdot e^{-m_k\cdot\tau} = F_{k+1}(l_{rp})\cdot e^{-m_{k+1}\cdot\tau}.$$

This condition for the matching of two neighboring temperature fields is satisfied at any instant of time if

$$m_k = m_{k+1} = \ldots = m.$$

Use of the Regular Mode in an Experimental Determination of the Temperature Conductivity of Substances

The measurement of temperature conductivity a of a substance reduces to the measurement of the rate m of the regular mode in some body which is made of the specified substance and has a convenient form (for example a sphere).

If the body has the form of a plate, then by determining m and knowing the thickness of the plate 2δ, it is easy to calculate with the aid of the identity (7.1) the thermal diffusivity a_v, provided the first eigenvalue n_1 is known:

$$a_v = m\left(\frac{\delta}{n_1}\right)^2. \qquad (7.3)$$

However, the value of n_1 depends on the Biot number, i.e., on the heat-transfer coefficient and on the heat conduction, the determination of which is an independent problem. To eliminate this difficulty, the regular mode of cooling or heating of the body (for example, a plate) is realized at large values of the Biot number ($Bi \geq 100$). Then $n_1 \approx \pi/2$ (see Fig. 21), and Eq. (7.3) simplifies to

$$a_v = m\cdot\left(\frac{2\cdot\delta}{\pi}\right)^2. \qquad (7.4)$$

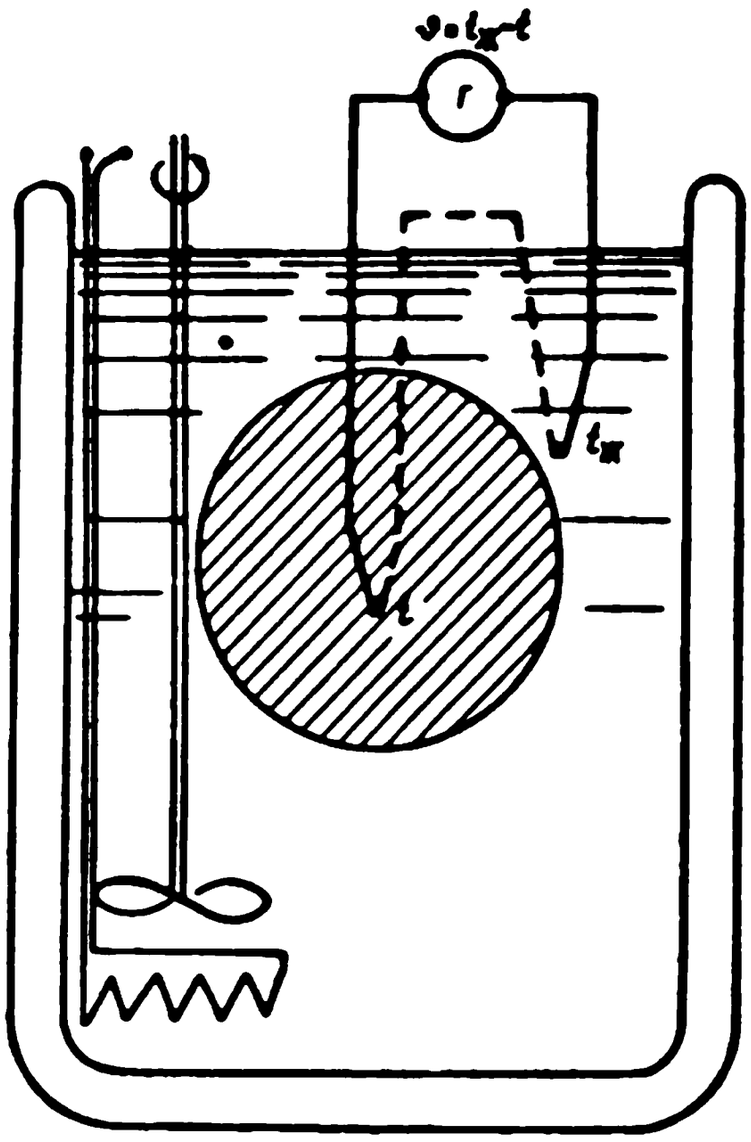

Fig. 29. Diagram of experimental setup for the measurement of the thermal diffusivity by the regular-mode method.

The rate of the regular mode is usually measured in the following sequence.

A body made of the investigated material is heated in a thermostat, in which liquid is maintained at a constant temperature (Fig. 29). Two thermocouples connected to "buck" each other and a galvanometer are used to measure the excess temperature $\vartheta \equiv t - t_{zh}$ at any point of the body at different instants of the time τ. The results of the measurements are plotted (Fig. 30) in semilogarithmic coordinates ($\ln \vartheta$; τ). The value of the rate $\underline{m}$ is determined by calculating the slope of the

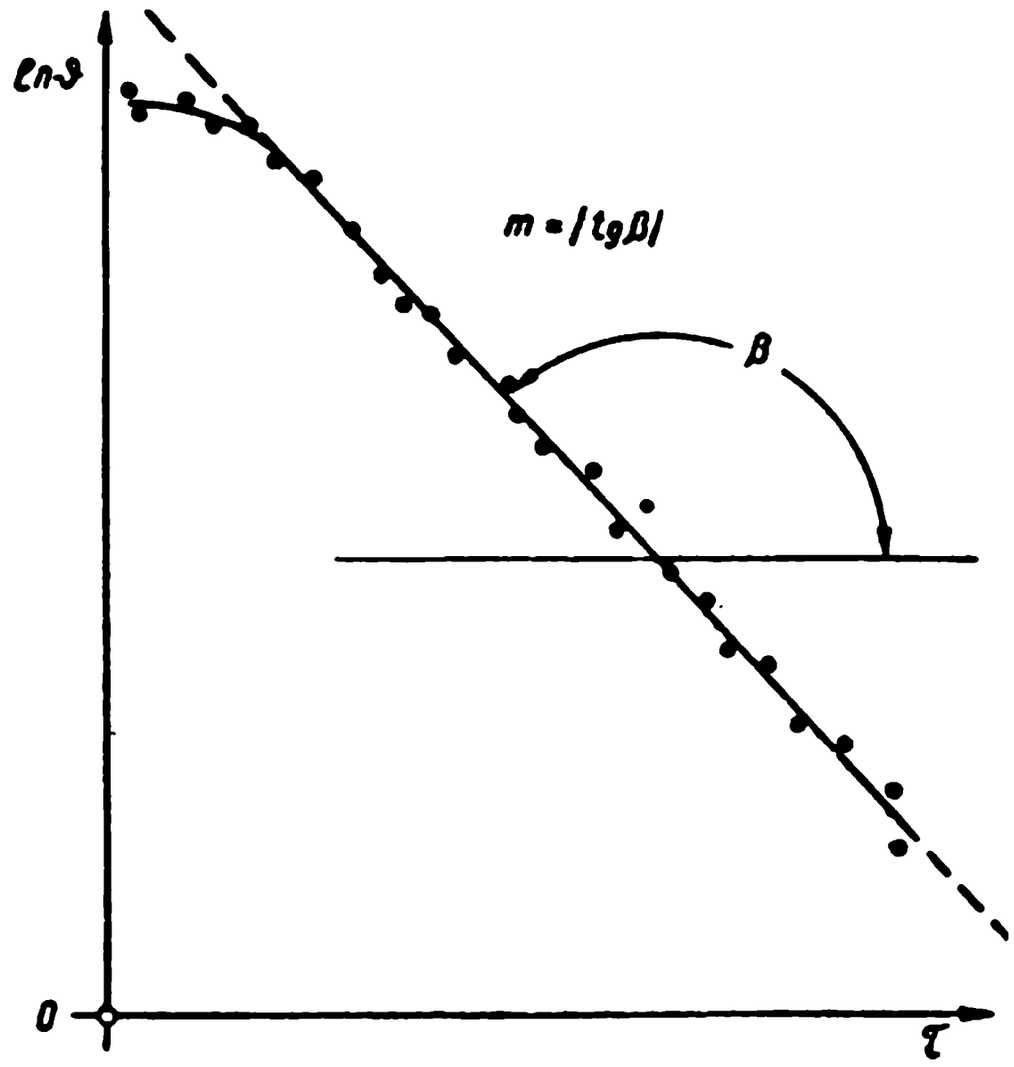

Fig. 30. Scheme for graphical determination of the rate of a regular mode.

straight line which is formed after the regular mode sets in.

However, this method can be conveniently used under laboratory

conditions for the determination of a_v only for substances with low heat conductivity ($\lambda \leq 0.5$). Actually, when the body is heated it is necessary to have $Bi \geq 100$. Consequently, $\lambda \leq \alpha \cdot \delta/100$. The value of α in thermostats of the usual type does not exceed 2000 kcal/m^3-hr-deg, while the dimensions of the body must not exceed $2\delta = 0.05$ m. Consequently, $\lambda \leq 2000 \cdot 0.025/100 = 0.5$. This circumstance limits the use of the regular mode for the measurement of thermal diffusivity under laboratory conditions.

Use of Regular Mode for Experimental Determination of the Heat-Transmission Coefficient on the Surface of a Body Situated in a Stream of Liquid

Assume that some body with volume V and surface F is cooled in a stream of liquid and has a small Biot number ($Bi \ll 1$). The average coefficient of heat transfer $\bar{\alpha}^F$ on the surface of this body can be determined with the aid of the regular thermal mode [25a]. Let us consider first the theory of this problem.

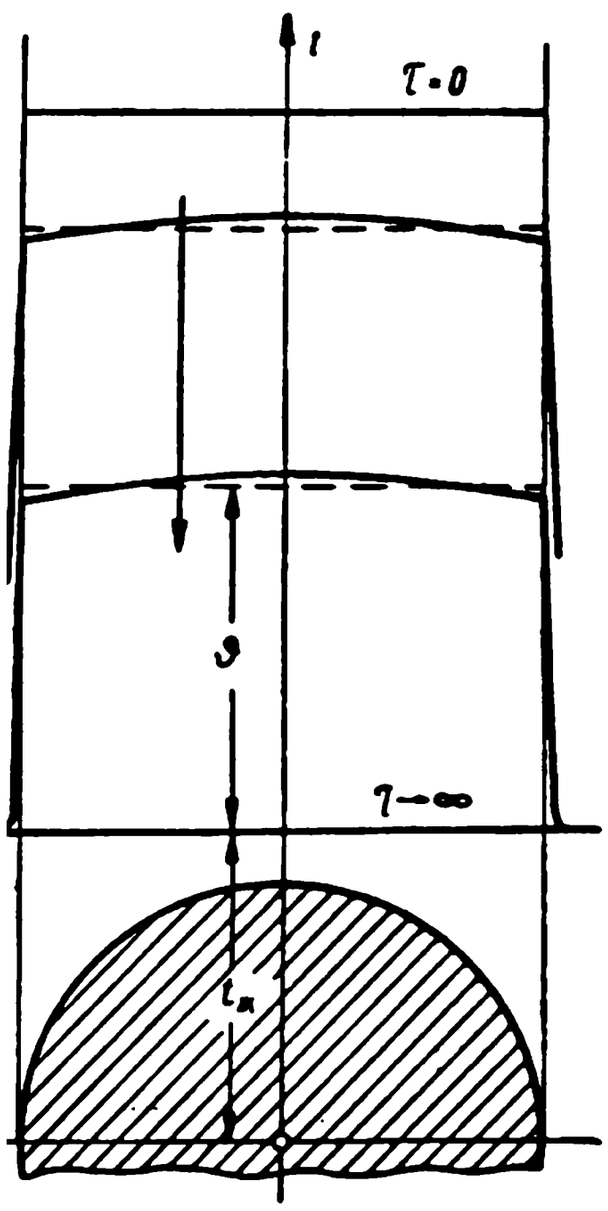

Fig. 31. Distribution of temperature in a body with a small Biot number.

The distribution of the temperature in a body with a small Biot number is practically uniform, and therefore the temperature $t(x, y, z, \tau)$ at any point (x, y, z) of the body at a given instant of time τ differs only slightly from the temperature of the body $\bar{t}^V(\tau)$ averaged over the volume V at the same instant of time (Fig. 31). The temperature $t|_{n=0}$ on the surface of the body differs little from the mean temperature $\bar{t}^V$. Consequently,

$$t(x, y, z, \tau) \approx \bar{t}^V(\tau) \equiv \frac{1}{V}\int_0^V t(x, y, z, \tau)\, dV \approx t|_{n=0}. \qquad (7.5)$$

These considerations enable us to establish the dependence of the mean temperature of the body ${}^{V}\bar{t}$ on the time τ. The temperature distribution in the body obeys the Fourier equation

$$\frac{\partial t}{\partial \tau} = a_v \cdot \operatorname{div} \operatorname{grad} t$$

and the boundary condition

$$\lambda_F \frac{\partial t}{\partial n}\Big|_{n=0} = \alpha (t|_{n=0} - t_\infty).$$

Integrating the Fourier equation over the volume V we obtain

$$\int_V \frac{\partial t}{\partial \tau} dV = a_v \int_V \operatorname{div} \operatorname{grad} t \cdot dV.$$

On the basis of the Ostrogradskiy theorem, and also on the basis of Eq. (7.5) and the boundary condition of the third kind, we obtain

$$V \cdot \frac{d^V \bar{t}}{d\tau} = \frac{\lambda}{C_v} \oint_F \frac{\partial t}{\partial n}\Big|_{n=0} dF = -\frac{1}{C_v} \oint_F \alpha (t|_{n=0} - t_\infty) dF.$$

In turn

$$\oint_F \alpha \cdot (t|_{n=0} - t_\infty) dF = {}^{F}\bar{\alpha} \cdot F \cdot ({}^{V}\bar{t} - t_\infty).$$

Substituting the last equation in the preceding one, we obtain

$$V \frac{d^V t}{d\tau} = -\frac{{}^{F}\bar{\alpha}}{C_v} \cdot F \cdot ({}^{V}\bar{t} - t_\infty)$$

or

$$\frac{d^V \bar{\theta}}{d\tau} = -m \cdot {}^{V}\bar{\theta}, \tag{7.6}$$

where

$${}^{V}\bar{\theta} \equiv {}^{V}\bar{t} - t_\infty;$$

$$m \equiv \frac{{}^{F}\bar{\alpha} \cdot F}{C_v \cdot V}. \tag{7.7}$$

The general integral of Eq. (7.5) has the form

$${}^{V}\bar{\theta} = c_1 \cdot e^{-m\tau}.$$

Consequently, in this case one observes an exponential regular mode: the quantity m is the rate of change of the temperature field

which is regular in time. Thus, to determine the average heat-transmission coefficient $\overline{F}_{\alpha}$ on the surface of a body with a small Biot number Bi $\ll$ 1 it is necessary to know the values of the specific heat of the body C_V, its volume V, and its surface F, as well as the rate $\underline{m}$ of the regular mode. Then

$$\overline{F}_{\alpha} = m \frac{C_v \cdot V}{F}. \qquad (7.8)$$

The rate $\underline{m}$ is determined with a galvanometer and two thermocouples connected to "buck" each other and used to measure the excess temperature $\vartheta \equiv t - t_{zh}$ at any part of the body as a function of the time τ. The results of the measurements are plotted in the form of a connection between ln ϑ and τ (see Fig. 30). The value of $\underline{m}$ is determined as the slope of the straight line on this plot.

This method of obtaining the average value of the heat-transfer coefficient on the surface of the body can be used only at small temperature differences, when the thermal-physical parameters of the liquid and of the body can be regarded with sufficient accuracy as constant. In addition, this method gives the correct results if the local values of the coefficient of heat transmission on the surface of the body differ little from its average value $\overline{F}_{\alpha}$.

The method of regular mode can be used also for the determination of the heat transmission in the case when liquid flows in tubes [50]. This possibility is of particular interest in those cases when the duration of the experiment must be made as short as possible for various reasons (for example, when measuring the heat transmission of corrosive media).

The use of the regular mode for the determination of the heat-transfer coefficient under ordinary laboratory conditions with flow outside the body is limited to relatively small values of this

coefficient ($^{F}\overline{\alpha} \leq 1300$). Indeed, if we take as convenient values $\delta \leq$ ≤ 0.025 m and $\lambda_F \leq 340$, then we obtain from the inequality $^{F}\overline{\alpha}\delta/\lambda_F \leq 0.1$

$$^{F}\overline{\alpha} < \frac{0,1 \cdot \lambda_F}{0,025} = \frac{0,1 \cdot 340}{0,025} = 1360 \text{ kcal/m}^2\text{-hr-deg.}$$

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[Footnotes]

377 This procedure can be used only for linear differential equations and for linear initial and boundary conditions.

389 This condition is satisfied with good accuracy for many substances, for example for chamotte [19a].

391 The self-similarity of the excess-temperature field can be observed also in a stationary heat-conduction process, for example along the axis of a rod. Consequently, the regular thermal mode can also arise under stationary conditions.

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[List of Transliterated Symbols]

372 ж = zh = zhidkost' = liquid

383 Б = B

390 макс = maks = maksimum = maximum

390 мин = min = minimum = minimum

394 гр = gr = granitsa = boundary

Chapter 4

DIMENSIONALITY AND SIMILARITY ANALYSIS METHODS

§8. METHOD OF ANALYSIS OF DIMENSIONALITIES OF PHYSICAL QUANTITIES

Reduction of Equations to Dimensionless Form by the Method of Scale Transformation

Natural phenomena do not depend on the choice of the measurement units and dimensions of the units. The equations of mathematical physics, which reflect the properties of the phenomena, should possess similar properties. This requirement is satisfied if an equation can be transformed to dimensionless form. Therefore the dimensionless form of equations is the most natural. In addition, dimensionless equations have many useful properties.

To reduce equations to dimensionless form one uses the analysis of dimensionalities of physical quantities. The method of scale transformation [41a] is one of the modifications of the dimensionality method. Let us demonstrate the use of this method with a particular example of heating of a thin plate. The nonstationary field of the excess temperature $\vartheta \equiv t_{zh} - t$ of a thin plate during the course of its heating is described by the following equations

$$\left.\begin{aligned} &\frac{\partial \vartheta}{\partial \tau} = a_v \frac{\partial^2 \vartheta}{\partial x^2};\ \ \vartheta|_{\tau=0;\ 0<x<\delta} = \Delta t; \\ &\vartheta|_{\tau\to\infty;\ 0<x<\delta} \to 0;\ \ \left.\frac{\partial \vartheta}{\partial x}\right|_{x=0;\ \tau>0} = 0; \\ &\left|\frac{\partial \vartheta}{\partial x} + \frac{\alpha}{\lambda}\vartheta\right|_{x=+\delta-0} = 0. \end{aligned}\right\} \qquad (8.1)$$

Let us assume that the problem (8.1) is such that its analytical solution cannot be obtained at the present time. Therefore it is necessary to calculate the heating of the plate by numerical methods or

to determine a solution by experimental means. In either case it is necessary to determine ϑ as a function of six dimensional quantities, since it follows from Eq. (8.1) that

$$\vartheta = \vartheta\left(x, \tau, a, \frac{a}{\lambda}, \delta, \Delta t\right).$$

However, the dimensionless form of the analytical solution of the problem, as shown in §6, contains only four dimensionless quantities and has the following structure:

$$\Theta = \Theta(X, Fo, Bi). \qquad (8.2)$$

In this case it is necessary to obtain the dependence of the dimensionless temperature Θ on only three dimensionless quantities X, Fo, and Bi, which greatly facilitates the experimenter's labor. Moreover, the dimensionless solution of the problem can be extended to include many other nonstationary heat-conduction processes in plates, similar to the investigated process.* These advantages of dimensionless equations induce us to seek methods for obtaining Formula (8.2), bypassing the analytical solution of Eqs. (8.1) which, by assumption, cannot be obtained.

The transformation of equations to dimensionless form by the scale-transformation method consists of the following.

For each variable entering into the mathematical description of the process, we set up its dimensionless expression. The problem of heating a plate contains three variables ϑ, $\underline{x}$, and τ. We therefore write expressions for the dimensionless temperature $\vartheta/\Delta t \equiv \Theta$, the dimensionless abscissa $x/\delta \equiv X$, and the dimensionless time $\tau/\tau_0 \equiv T$. In these relations the quantities Δt, δ, and τ_0 have the meanings of measurement units or scales of the temperature, distance, and the time. We note that the time scale τ_0 is not specified by the conditions of the problem. From the last relations we obtain

$$\vartheta = \Delta t \cdot \Theta, \; x = \delta \cdot X \quad \text{and} \quad \tau = \tau_0 \cdot T. \tag{8.3}$$

Substituting Eqs. (8.3) into Eq. (8.1) and transforming them, we obtain

$$\left.\begin{gathered} \frac{\partial \Theta}{\partial T} = \frac{a \cdot \tau_0}{\delta^2} \cdot \frac{\partial^2 \Theta}{\partial X^2}; \quad \Theta|_{T=0;\ 0<X<1} = 1; \\ \vartheta|_{T\to\infty;\ 0<X<1} \to 0; \\ \left.\frac{\partial \Theta}{\partial X}\right|_{X=0;\ T>0} = 0; \quad \left|\frac{\partial \Theta}{\partial X} + Bi \cdot \Theta\right|_{X=+1-0} = 0. \end{gathered}\right\} \tag{8.4}$$

Equations (8.4) contain the following dimensionless quantities:

$$\Theta = \Theta\left(X, T, Bi, \frac{a\tau_0}{\delta^2}\right). \tag{8.5}$$

However, this solution cannot depend on the time scale τ_0, which does not enter into the mathematical description of the process. Therefore the dimensionless quantities must be contained in Eq. (8.5) in such a way that the scale τ_0 cancels out. This will be attained if $T \equiv \tau/\tau_0$ and $a_v \cdot \tau_0/\delta^2$ enter into the solution in the form of the product

$$\frac{\tau}{\tau_0} \cdot \frac{a_0 \cdot \tau_0}{\delta^2} = \frac{a_0 \cdot \tau}{\delta^2} \equiv Fo.$$

Consequently, the dimensionless solution of the problem of heating of a plate contains the following dimensionless quantities:

$$\Theta = \Theta(X, Bi, Fo).$$

Among the advantages of the scale-transformation method are the following: the possibility of obtaining exactly as many dimensionless quantities as follow from the analytical solution, provided the latter is feasible; the clarity of obtaining dimensionless quantities. We see here clearly the origin of the particular dimensionless quantity. For example, from Eqs. (8.4) it follows that the Biot number results from the boundary condition of the third kind, while Fo originates from the differential equation. This property of the method of scale transformation facilitates the analysis of the limiting cases of the process.

Among the shortcomings of the scale-transformation method are the need of having a system of differential equations describing the in-

vestigated process, and that the application of this method to the solution of complicated problems necessitates cumbersome derivations and consumes a relatively long time.

Reduction of Equations to Dimensionless Form by Analysis of the Dimensionalities of the Physical Quantities

The use of dimensional analysis [4; 45] will be illustrated using the heating of a thin plate as a particular example. For this purpose it is necessary to have the system of quantities defining the investigated process. For heating of a plate we have

$$\theta\left(\vartheta, x, \tau, \frac{\lambda_F}{a}, a_v, \delta, \Delta t\right) = 0. \qquad (8.6)$$

Let us write down the measurement units of all the dimensional quantities in Eq. (8.6)

$$[\vartheta] = [\Delta t] = \text{deg}; \quad [x] = [\delta] = [\lambda_F/\alpha] = \text{m}; \quad [\tau] = \text{hr}; \quad [a_v] = \text{m}^2/\text{hr}.$$

We choose for them the smallest number of the units that are independent of one another, in terms of which we can express all other units used to set up Eq. (8.6). We can choose as the independent measurement units degrees, meters, and hours.

Let us change to new independent measurement units (m', hr', and deg'), which are A_l, A_τ, and A_ϑ times smaller than the preceding units, respectively:

$$\text{m}' = \text{m}/A_l, \quad \text{r}' = \text{hr}/A_\tau, \quad \text{deg}' = \text{deg}/A_\vartheta.$$

Then the quantities ϑ, $\underline{x}$, τ, ... in Relation (8.6) will change their numerical values and the equation will assume the form

$$\theta\left(\vartheta \cdot A_\vartheta; \; x \cdot A_l; \; \tau \cdot A_\tau; \; \frac{\lambda_F}{a} \cdot A_l; \; a_v \cdot \frac{A_l^2}{A_\tau}; \; \Delta t \cdot A_\vartheta; \; \delta \cdot A_l\right) = 0. \qquad (8.7)$$

The heat conduction in the plate does not depend on the dimension of the units used to measure the quantities ϑ, $\underline{x}$, τ, ... Therefore Eq. (8.7) can retain its structure for different values of the coefficients A_l, A_ϑ, and A_τ. We use this consideration to choose A_l, A_ϑ, and A_τ

such that Eq. (8.7) has the smallest number of terms. We choose A_l, A_v, and A_τ such as to make

$$\Delta t \cdot A_0 = 1;\ \delta \cdot A_l = 1;\ a_0 \frac{A_l^2}{A_\tau} = 1.$$

Then

$$A_0 = \frac{1}{\Delta t};\ A_l = \frac{1}{\delta};\ A_\tau = \frac{a}{\delta^2}. \qquad (8.8)$$

Substituting (8.8) in (8.7) we obtain the solution of the problem

$$\Phi\left(\frac{\Phi}{\Delta t},\ \frac{x}{\delta},\ \frac{\lambda_F}{a \cdot \delta},\ \frac{a_0 \cdot \tau}{\delta^2},\ 1,\ 1,\ 1\right) = 0$$

or

$$\Theta(\Theta,\ X,\ Bi,\ Fo) = 0.$$

Among the advantages of the method of dimensional analysis is the possibility of obtaining dimensionless quantities without the differential equations that describe the investigated process. When using this method it is sufficient to have only a system of defining parameters. This possibility may turn out to be decisive in the investigation of new phenomena for which the mathematical description has not yet been made up.

It is further necessary to point out that in some cases an investigation of the analysis of the dimensionalities makes it possible to obtain immediately a solution of differential equations, apart from a factor, and in other cases it is possible to transform partial differential equations into total differential equations. It must also be noted that the solution of complicated problems with the aid of dimensional analysis in the presence of differential equations does not call for cumbersome derivations and consumes relatively little time. Dimensional analysis does not make it possible to obtain precisely the same dimensionless quantities as arise in the analytical solution of the problem. The different variants of dimensionless quantities ob-

tained with the aid of dimensional analysis are formally equivalent to one another, and the choice of any particular variant is governed by considerations of physical or applied character. Finally, the use of the system of defining parameters may turn out to be less reliable than an analysis of the differential equations. However, in the investigation of little-known phenomena, for which differential equations have not yet been set up, the dimensional method is the natural one for obtaining dimensionless quantities. Such quantities are also obtained by the similarity-constant method developed in §9.

The Π Theorem

This theorem determines the number of dimensionless quantities which should be obtained after reducing a differential equation or a system of defining parameters to dimensionless form [45].

If the equation

$$f(a_1, a_2, \ldots, a_k, b_{k+1}, \ldots, b_n) \tag{8.9}$$

contains $n \geq 2$ dimensional quantities and $k \geq 1$ groups of homogeneous quantities with units that do not depend on one another then when reduced to dimensionless form this equation will contain $n - k$ dimensionless quantities π, each of which is a product of the quantities a and b to various powers:

$$F(\pi_1, \pi_2, \ldots, \pi_{n-k}) = 0. \tag{8.10}$$

The content of this theorem is clearly illustrated by the example of heating a thin plate. In this case the dimensional equation has the form

$$\vartheta\left(\vartheta, \Delta t', x, \delta, \frac{\lambda_F}{a}, \tau, a_v\right) = 0.$$

$$[\vartheta] = [\Delta t] = \deg \quad [x] = [\delta] = \left[\frac{\lambda_F}{a}\right] = m;$$

$$[a_v] = m^2/hr; \quad [\tau] = hr.$$

The total number of dimensional quantities is equal to seven here,

and the number of groups of homogeneous quantities with independent units is three. Consequently, the number of dimensionless quantities is four:

$$n - k = 7 - 3 = 4.$$

The π theorem should be used to check on the correctness of the reduction of equations to dimensionless form.

Particular Cases of the π Theorem

Let us indicate three particular cases of the π theorem which are of certain interest to the applications of dimensional analysis to the study of specific processes.

The first case corresponds to the condition

$$n - k = 0, \tag{8.11}$$

which indicates that the system of defining parameters cannot be reduced to dimensionless form. Such a system of parameters is in contradiction with dimensional analysis. In this system there are some quantities lacking to be able to reduce it to dimensionless form. Such systems are called incomplete. When compiling a system of defining parameters for processes that have not been thoroughly investigated, it is necessary to check the completeness of this system. It must be noted, however, that the system of parameters satisfying the main requirement of dimensional analysis, namely the completeness requirement ($n - k \geq 1$), may turn out to be erroneous from the physical point of view.

The second case corresponds to the condition

$$n - k = 1, \tag{8.12}$$

for which the dimensionless equation (8.10) contains only one dimensionless quantity π_1

$$F(\pi_1) = 0. \tag{8.13}$$

Consequently, π_1 is a root of Eq. (8.13)

$$\pi_1 = \text{const.} \tag{8.14}$$

As indicated above, the dimensionless quantity π_1 is a product of dimensional quantities to different degrees. Among these quantities there is one sought quantity, so that it follows from (8.14) that the sought quantity is connected with the specified quantities by means of a power-law equation, which is known accurate to a constant factor.

The condition $n - k = 1$ is of interest to the researcher, since it reduces the solution of the corresponding problem to a determination of only one constant factor. A. Einstein [4; 62] states that the value of this factor should be not too large and not too small compared with unity. This rule has apparently an empirical character. It enables us to set up in most rational fashion a system of defining parameters for a little-investigated phenomenon when $n - k = 1$.

The third case corresponds to the condition

$$n - k = 2, \tag{8.15}$$

for which the dimensionless equation (8.10) contains only two dimensionless quantities π_1 and π_2

$$F(\pi_1, \pi_2) = 0. \tag{8.16}$$

Consequently,

$$\pi_1 = \Phi(\pi_2). \tag{8.17}$$

Equation (8.17) is convenient for a graphical representation of the function Φ and its interpolation with the aid of approximate formulas. In addition, the condition $n - k = 2$ simplifies the solution of some problems in mathematical physics, since it enables us to transform partial differential equations into total differential equations [9a].

§9. METHOD OF SIMILARITY OF PHYSICAL PROCESSES

Concept of Physical Similarity

The method of physical similarity is used to investigate by means of models processes that are difficult to calculate or to study under

natural conditions; to extend the results of investigation of one process to a set of processes similar to it; to study and to use self-similar processes.

Two physical processes 1 and 2 are called similar if 1) both processes have qualitatively identical uniqueness conditions; 2) the comparable quantities of both processes are connected with one another by the similarity transformation; 3) the mathematical descriptions of both processes are also connected with one another by means of the similarity transformation.*

An aggregate of processes for any pair of which indicated requirements are satisfied is known as a group of similar processes.

Qualitatively identical uniqueness conditions coincide in all but the numerical values of the physical quantities of which they are made up. It is assumed here that the uniqueness conditions have been set up with the aid of a single coordinate system and a single system of measurement units.

A similarity transformation is called a linear relation connecting comparable quantities

$$\varphi_1 = C_\varphi \cdot \varphi_2. \qquad (9.1)$$

The quantities φ_1 and φ_2 are called comparable if they have an identical measurement unit, an identical physical meaning, and characterize the processes 1 and 2 from comparable points of space and at comparable instants of time.

In a single reference frame, comparable points of space have coordinates that are proportional to one another

$$\left.\begin{aligned} x_1 &= C_l \cdot x_2, \\ y_1 &= C_l \cdot y_2, \\ z_1 &= C_l \cdot z_2, \end{aligned}\right\} \; l_1 = C_l \cdot l_2.$$

Comparable instants of time have the same starting point and are proportional to each other

$$\tau_1 = C_\tau \cdot \tau_2.$$

The proportionality coefficient C_φ in the similarity transformation is called the similarity constant. Similarity constants are dimensionless, positive, finite, and generally speaking are not equal to unity; they depend on neither the coordinates nor the time, nor the direction; they are the same for all homogeneous quantities and generally speaking are different for inhomogeneous quantities. The values of the similarity constants indicate by how many times it is necessary to change the parameters of one process in order to obtain the parameters of another process that is similar to the first.

There are known generalizations of the concept of similarity of physical processes. Thus, for example, for an analysis of the similarity conditions of heat conductivity processes in anisotropic media, similarity constants that depend on the direction have been introduced [6a].

The use of physical analogies [8; 13] is based on a generalization of the similarity concept such that the compared processes have a different physical nature, but formally they have identical uniqueness conditions.

The Method of Similarity Constants

By way of an example of physical similarity let us consider two processes involving the heating of thin plates (Fig. 32). Assume that these processes are similar to each other, i.e., that the requirements of the definition of similarity are satisfied for them.

It follows from the first requirement that the heat-conduction processes occurring in both plates are qualitatively the same. Let the plates be thin, homogeneous, isotropic, without internal sources, and let them be heated in liquid streams. The mathematical descriptions of these processes coincide, apart from numerical values of the physical

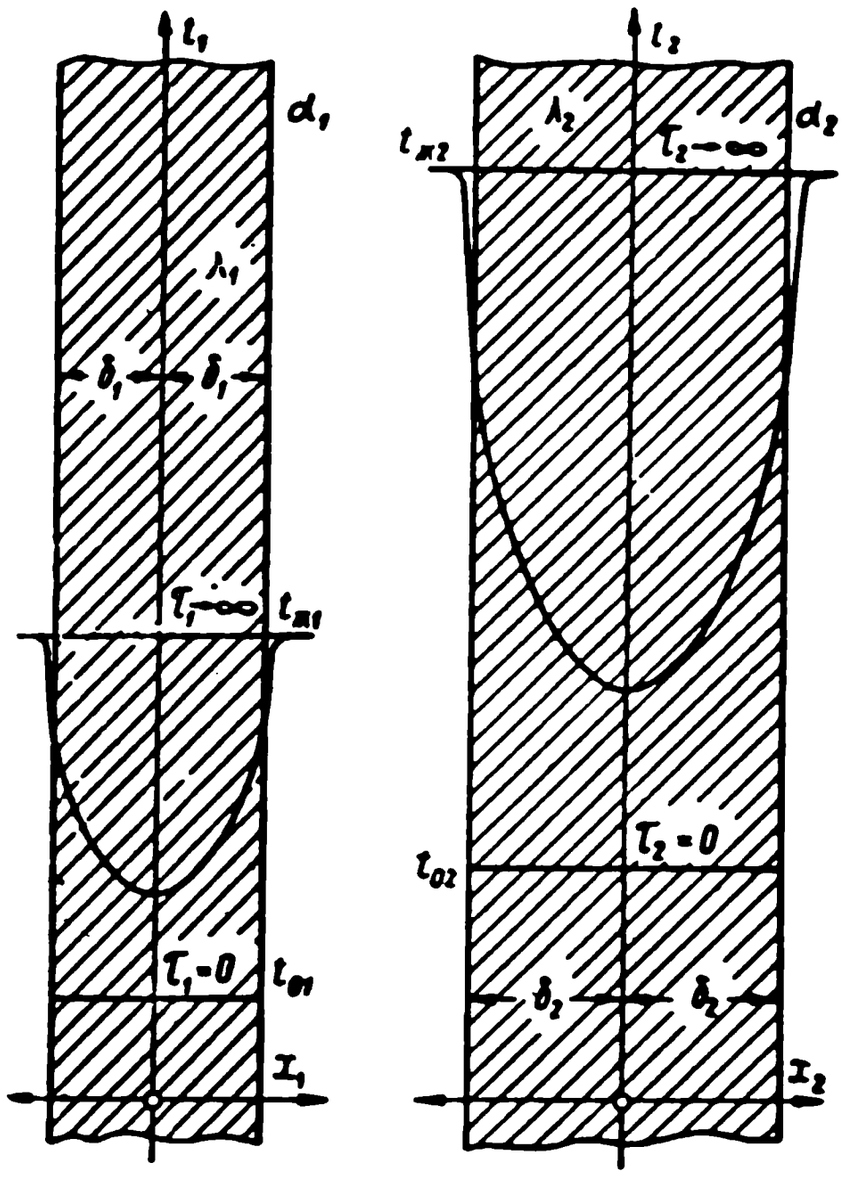

Fig. 32. Similarity of non-stationary temperature fields in the heating of two plates.

quantities

$$\left.\begin{aligned}
&\frac{\partial \vartheta_1}{\partial \tau_1} = a_{v1}\frac{\partial^2 \vartheta_1}{\partial x_1^2};\\
&\vartheta_1|_{\tau_1=0;\ 0<x_1<\delta_1} = \Delta t_1;\\
&\vartheta_1|_{\tau_1\to\infty;\ 0<x_1<\delta_1} \to 0;\\
&\frac{\partial \vartheta_1}{\partial \tau_1}\Big|_{x_1=0;\ \tau_1>0} = 0;\\
&\left[\frac{\partial \vartheta_1}{\partial x_1} + \frac{\alpha_1}{\lambda_{F_1}}\cdot\vartheta_1\right]_{x_1=+\delta-0} = 0;
\end{aligned}\right\} \qquad (9.2)$$

$$\left.\begin{aligned}
&\frac{\partial \vartheta_2}{\partial \tau_2} = a_{v2}\frac{\partial^2 \vartheta_2}{\partial x_2^2};\\
&\vartheta_2|_{\tau_2=0;\ 0<x_2<\delta_2} = \Delta t_2;\\
&\vartheta_2|_{\tau_2\to\infty;\ 0<x_2<\delta_2} \to 0;\\
&\frac{\partial \vartheta_2}{\partial \tau_2}\Big|_{x_2=0;\ \tau_2>0} = 0;\\
&\left[\frac{\partial \vartheta_2}{\partial x_2} + \frac{\alpha_2}{\lambda_{F_2}}\cdot\vartheta_2\right]_{x_2=+\delta-0} = 0.
\end{aligned}\right\} \qquad (9.3)$$

The values of the quantities in these equations are generally speaking not the same, i.e., $x_1 \neq x_2$, $\tau_1 \neq \tau_2$, $a_{v1} \neq a_{v2}$, $\vartheta_1 \neq \vartheta_2$, $\delta_1 \neq \delta_2$, $\Delta t_1 \neq \Delta t_2$, $\alpha_1 \neq \alpha_2$, $\lambda_1 \neq \lambda_2$.

From the second requirement it follows that the comparable quantities are related with one another by the similarity transformation

$$\left.\begin{aligned}
\vartheta_1 &= C_\vartheta\cdot\vartheta_2; & \Delta t_1 &= C_\vartheta\cdot\Delta t_2;\\
a_{v1} &= C_a\cdot a_{v2}; & \tau_1 &= C_\tau\cdot\tau_2;\\
\delta_1 &= C_l\cdot\delta_2; & \alpha_1 &= C_\alpha\cdot\alpha_2;\\
x_1 &= C_l\cdot x_2; & \lambda_{F_1} &= C_\lambda\cdot\lambda_{F_2}.
\end{aligned}\right\} \qquad (9.4)$$

The similarity constants C_φ are not equal to one another in this case, nor are they equal to unity, i.e.,

$$0 < C_\vartheta + C_\tau + C_l + C_a + C_\lambda + C_\alpha + 1.$$

If all the C_φ are equal to unity, then the two processes of heating the plates are completely identical.

It follows from the third requirement that if the mathematical description of the first process is subjected to a similarity transformation, i.e., if Eqs. (9.4) are substituted in (9.2), then we obtain the mathematical description of the second process, i.e., Eqs.

(9.3), which do not contain the similarity constant.

Substituting (9.4) in (9.2) and transforming them, we obtain

$$\left.\begin{gathered}\frac{\partial\vartheta_2}{\partial\tau_2}=a_{v2}\frac{\partial^2\vartheta_2}{\partial x_2}\cdot\frac{C_a\cdot C_\tau}{C_l^2};\ \left.\frac{\partial\vartheta_2}{\partial\tau_2}\right|_{x_2=0;\ \tau_2>0}=0;\\ \vartheta_2\Big|_{\tau_2=0;\ 0\le x_2\le\delta_2}=\Delta t_2;\ \left|\frac{\partial\vartheta_2}{\partial x_2}+\frac{\alpha_2}{\lambda_2}\cdot\vartheta_2\cdot\frac{C_aC_l}{C_\lambda}\right|_{x_2=+\delta-0}=0;\\ \vartheta_2|_{\tau_2\to\infty;\ 0<x_2<\delta_2}\to 0.\end{gathered}\right\}\qquad(9.5)$$

From a comparison of (9.5) with the mathematical description of the second process, i.e., with Eq. (9.3), we obtain

$$\frac{C_a\cdot C_\tau}{C_l^2}=1;\ \frac{C_a\cdot C_l}{C_\lambda}=1.\qquad(9.6)$$

Thus, the fulfillment of the requirements of physical similarity of the processes leads to the deduction that certain limitations are imposed on the choice of the similarity constants, in the form of Eqs. (9.6) for the particular case considered here.

The foregoing considerations are not only an example of physical similarity, but make it also possible to obtain dimensionless quantities that are characteristic of the heating of the plate. Indeed, if the equation $\vartheta_1 = C_\vartheta\cdot\vartheta_2$ is divided by $\Delta t_1 = C_\vartheta\cdot\Delta t_2$, we obtain

$$\frac{\vartheta_1}{\Delta t_1}=\frac{\vartheta_2}{\Delta t_2}.\qquad(9.7)$$

Analogously we obtain

$$x_1/\delta_1 = x_2/\delta_2.\qquad(9.8)$$

If Eqs. (9.4) are substituted in Eq. (9.6) then we obtain after transformation

$$\frac{a_{v1}\cdot\tau_1}{\delta_1^2}=\frac{a_{v2}\cdot\tau_2}{\delta_2^2}\ \text{ and }\ \frac{\alpha_1\cdot\delta_1}{\lambda_{F_1}}=\frac{\alpha_2\cdot\delta_2}{\lambda_{F_2}}.\qquad(9.9)$$

Consequently, we have found all the dimensionless quantities characterizing the heating of the plate, namely $\vartheta/\Delta t$, x/δ, $a_v\cdot\tau/\delta^2$, $\alpha\cdot\delta/\lambda_F$. Such a method of determining the dimensionless quantities is called the method of similarity constants.

The Direct and Inverse Theorems of the Similarity Method

The direct theorem of the similarity method indicates the consequences that follow from the definition of the concept of similarity of physical processes, i.e., indicates the necessary conditions for similarity.

If processes are similar to one another, then their like dimensionless quantities π_1 have identical values

$$\pi_1 = \text{idem};$$
$$\cdots\cdots$$
$$\pi_{n-k} = \text{idem},$$

and the processes themselves are described by one and the same similarity equation.

To explain the meaning of this theorem let us use the example with the heating of the plates and let us rewrite Relations (9.7)-(9.9) in the customary notation

$$\Theta_1 = \Theta_2, \quad X_1 = X_2, \quad Fo_1 = Fo_2, \quad Bi_1 = Bi_2.$$

These relations were obtained as a consequence of the assumption that the two heat-conduction processes in the plates are similar to each other.

If this assumption is satisfied, then the dimensionless quantities Θ, X, Fo, and Bi acquire the meaning of invariant quantities. In this case, the like invariants π have identical values for similar processes. For this reason the dimensionless numbers π are customarily called the similarity invariants or the similarity criteria. Since similar heat-conduction processes have qualitatively identical uniqueness conditions and numerically identical similarity criteria, then one and the same dimensionless equation $\Theta(\Theta, X, Fo, Bi) = 0$ describes a set of nonstationary similar processes of heating or cooling of thin plates. Therefore the dimensionless equations are called the similarity equations.

The similarity criteria are classified as defining and nondefining. Defining criteria consist of quantities that are specified beforehand by the uniqueness condition (for example the Biot criterion $Bi \equiv \equiv \alpha \cdot \delta/\lambda_F$), but can contain also independent variables (for example, the criterion $Fo \equiv a_v \cdot \tau/\delta^2$ contains the time, or the dimensionless abscissa $X \equiv x/\delta$ contains $\underline{x}$). Nondefining criteria contain the sought quantity (for example the dimensionless temperature $\Theta \equiv (t_{zh} - t)/(t_{zh} - t_0)$ contains $\underline{t}$).

The inverse theorem of the similarity method indicates the sufficient conditions for the similarity of physical processes [14, 22]:

<u>in order for physical processes to be similar to each other, it is sufficient that the uniqueness conditions of these processes be qualitatively the same and that like defining criteria of similarity have the same values</u>.

This theorem was first formulated by M.V. Kirpichev and A.A. Gukhman [22] and is named after them. The inverse theorem is the basis for simulation of physical processes in general and heat-engineering devices in particular.

As applied to the particular example of heating the plates, the inverse theorem states the following:

in order for nonstationary heat-conduction processes in two plates to be similar to each other, it is sufficient that the uniqueness conditions for them be qualitatively identical, and that the like defining similarity criteria X, Fo, and Bi have identical values $X_1 = X_2$, $Fo_1 = Fo_2$, $Bi_1 = Bi_2$.

When these conditions are satisfied, the nondefining similarity criterion (the dimensionless temperature Θ) has the same value for both plates. The last deduction is a consequence of the fact that $\Theta_1 = \Theta(X_1, Fo_1, Bi_1)$ and $\Theta_2 = \Theta(X_2, Fo_2, Bi_2)$, and the form of the func-

tional dependence is the same in both cases because the uniqueness conditions are qualitatively the same.

Concept of Self-Similarity of Physical Processes

The concept of self-similarity is used to investigate processes in hydrodynamics, gasdynamics, and heat transfer [23, 45].

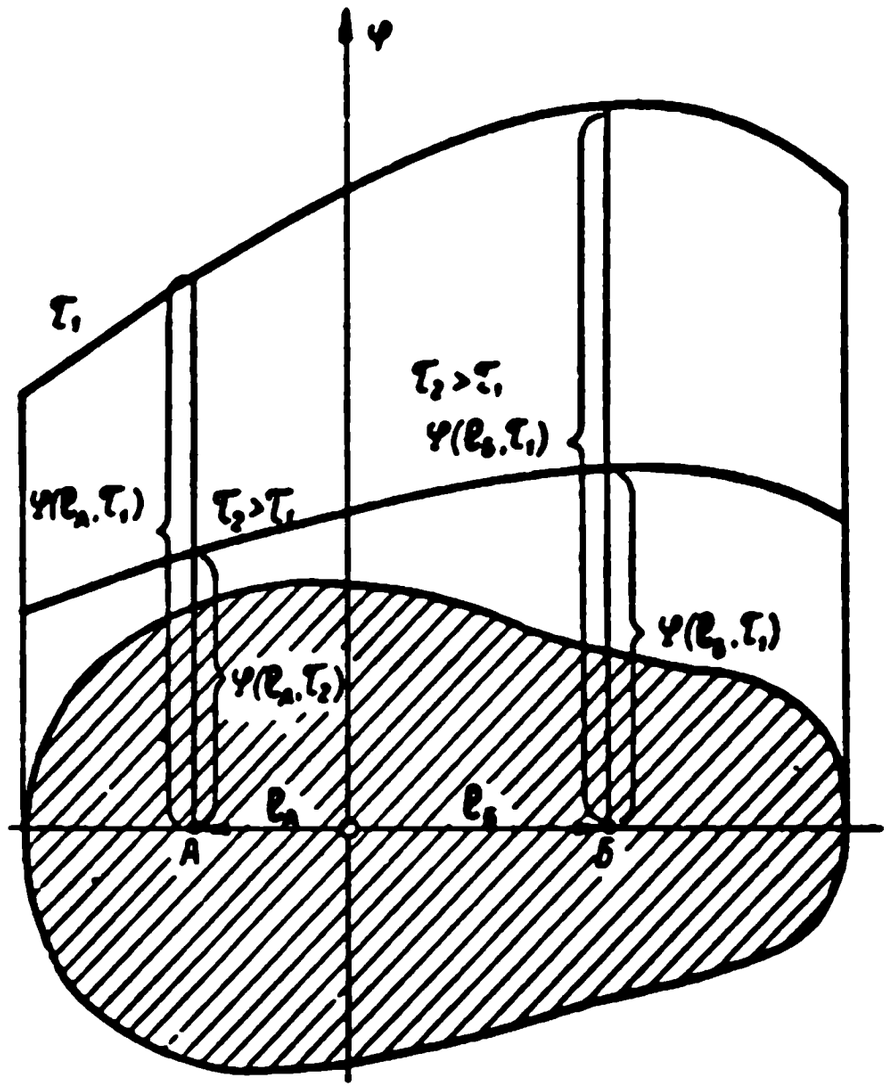

Fig. 33. Self-similar time variation of a quantity φ.

A physical process is called self-similar in time or in coordinate if the distribution of the quantity $\varphi(\tau, x, y, z)$ characterizing this process remains similar to itself as time goes on or as the coordinate is varied. From this definition follows the self-similarity condition for the process, for example with respect to time. Assume that in some body there is a nonstationary distribution of the quantity φ, which is self-similar in time (Fig. 33). This means that for two arbitrary instants of time τ_1 and $\tau_2 > \tau_1$ one observes two distributions of φ that are similar to each other. If one fixes in the body two points A and B with radius vectors $\vec{r}_A$ and $\vec{r}_B$, respectively, then for each of them the similarity transformation (9.1) should hold true for the quantity

$$\left.\begin{aligned}\varphi(l_A,\ \tau_1) &= C_\varphi \cdot \varphi(l_A,\ \tau_2);\\ \varphi(l_B,\ \tau_1) &= C_\varphi \cdot \varphi(l_B,\ \tau_2).\end{aligned}\right\} \qquad (9.10)$$

Dividing the first equation by the second and taking into account the independence of the similarity constant C_φ of the coordinates and of the time, we obtain

$$\frac{\varphi(l_A,\ \tau_1)}{\varphi(l_B,\ \tau_1)} = \frac{\varphi(l_A,\ \tau_2)}{\varphi(l_B,\ \tau_2)}.$$

The instants of time τ_1 and τ_2 were chosen arbitrarily, so that the last equation can be rewritten for any τ for which self-similarity exists. Consequently,

$$\frac{\varphi(l_A,\ \tau)}{\varphi(l_B,\ \tau)} \neq f(\tau). \qquad (9.11)$$

Equation (9.10) is the self-similarity condition in time for the distribution of the quantity $f(\vec{l},\ \tau)$. This condition will be fulfilled if

$$\varphi(l,\ \tau) = L(l) \cdot T(\tau), \qquad (9.12)$$

where $L(\vec{l})$ is the function of the coordinates only and $T(\tau)$ is a function of the time only.

The character of the time dependence, i.e., the form of the function $T(\tau)$ in Eq. (9.12) is determined by the differential equations and by the uniqueness conditions of the investigated process. Therefore for different differential equations and uniqueness conditions the self-similarity equation of the process (9.11) will contain different functions. Thus, for example, for the heating of a plate with boundary conditions of the third kind we have

$$\vartheta = L(x) \cdot e^{-m\tau}.$$

If the plate is heated by a constant heat flux, then the equation for the temperature field that is self-similar in time assumes the form

$$t = L_1(x) + m_1\tau.$$

In conclusion we note that if the condition $n - k = 1$ is satis-

fied for the distribution of the quantity φ, then we certainly have self-similarity with respect to each of the independent variables, for in this case the equation for φ should have the character of a product of parameters and arguments to certain powers. However, not every self-similar process has n − k = 1.

The foregoing considerations pertain to the simplest case of self-similarity.

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[Footnotes]

401 The conditions for the similarity of physical processes are considered in §9.

408 The third requirement is the consequence of the first two.

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[List of Transliterated Symbols]

400 ж = zh = zhidkost' = liquid

Chapter 5

FORMULATION OF THE MAIN PROBLEMS OF CONVECTIVE HEAT EXCHANGE

§10. DIFFERENTIAL EQUATIONS OF CONVECTIVE HEAT EXCHANGE

Phenomenological Method of Investigating Processes of Convective Heat Exchange

The phenomenological method of investigating processes of convective heat exchange is characterized by the following features:

1. The substance that participates in the convective heat exchange is regarded as a continuous medium. Its molecular structure, and also the microscopic mechanism of convective heat transport is disregarded.

2. To set up the mathematical description of convective heat-exchange processes one uses the first law of thermodynamics, the law of conservation of matter, and the law of conservation of momentum.

3. To set up a closed system of differential equations one uses the Fourier-Ostrogradskiy hypothesis concerning the vector $\vec{q}$ of convective heat exchange, the Biot-Fourier hypothesis concerning heat conduction in a liquid, and the Newton hypothesis concerning viscous friction in liquids.

4. The physical parameters of the liquid (the viscosity μ, the mass density ρ_{zh}, the specific heat C_{pzh}, and the heat conductivity λ_{zh}) are determined by experiment.

Fundamental Concepts and Limitations

Following the first feature of the phenomenological method, we introduce the fundamental concepts of the theory of convective heat exchange without the use of microscopic representations.

The transport of heat by the heat carrier itself is called convection. A viscous moving body whose temperature differs from that of the bodies surrounding it is known as a heat carrier.

Heat carriers are subdivided into those that are readily mobile and solid ones. Readily mobile heat carriers are those which acquire finite deformations under infinitesimally small loads (gases, drops of liquids, and vapors). Solid heat carriers are not considered here; readily mobile heat carriers are unified under a single term "fluid."

In real conditions the convection of heat is always accompanied by heat conductivity, and sometimes also by radiant heat exchange. The joint process of convection and heat conductivity is called convective heat exchange. Convective heat exchange between a liquid stream and a solid heated surface through which heat is transmitted is called heat transfer.

The principal practical purpose of studying heat transfer lies in determining the connection between the heat rate of the heated surface q_F (kcal/m^2-hr), the temperature of this surface t_F, and the temperature of the fluid t_{zh}, i.e., in the determination of the type of the function

$$f(t_{zh}, t_F, q_F) = 0. \qquad (10.1)$$

To accomplish this purpose it is necessary to know the temperature field in the fluid stream, i.e., to know the function $t_{zh} = t_{zh}(x, y, z, \tau)$.

The heat-transmission process is called stationary if the temperature field $t(\vec{l})$ does not depend on the time, and nonstationary if the distribution of the temperature in the fluid stream depends on the time. Experience has shown that in laminar flow the temperature field in the fluid stream is continuous. The distribution of the time-averaged temperature in a turbulent stream also apparently has the

properties of a continuous field. We shall therefore consider henceforth the temperature distribution in the fluid stream as being a continuous field, for which the concept of the temperature gradient grad $\underline{t}$ and the heat-flux density vector $\vec{q}$ remain meaningful. For registration of the fluid and heat fluxes we use the Euler reference frame.

Study of heat transmission will be henceforth based on the following limitations:

1) all the characteristics of the process of convective heat exchange in a specified volume are continuous functions of the coordinates and of the time;

2) the heat carrier adheres to the heating surface;

3) there is no radiant heat exchange between the heating surface and the fluid stream;

4) we consider only isobaric heat-transmission processes;

5) the thermal-physical parameters of the liquid do not depend on the temperature (in some cases this limitation will be lifted completely or partially);

6) the heat due to viscous friction in the fluid stream is negligibly small compared with the change in its enthalpy;

7) the kinetic energy of the stream is negligibly small compared with the changes in the enthalpy of the fluid;

8) the "thermal contact resistance" between the fluid and the heating surface is zero.

Heat Rate of the Heating Surface and the Heat-Transfer Coefficient

The main purpose of the study of processes of heat transmission lies in the determination of the connection (10.1) between q_F, t_F, and t_{zh}. For this purpose it is necessary to know the distribution of the temperature in the fluid stream. Indeed, if the fluid adheres to the heating surface and there is no radiant heat exchange, then the

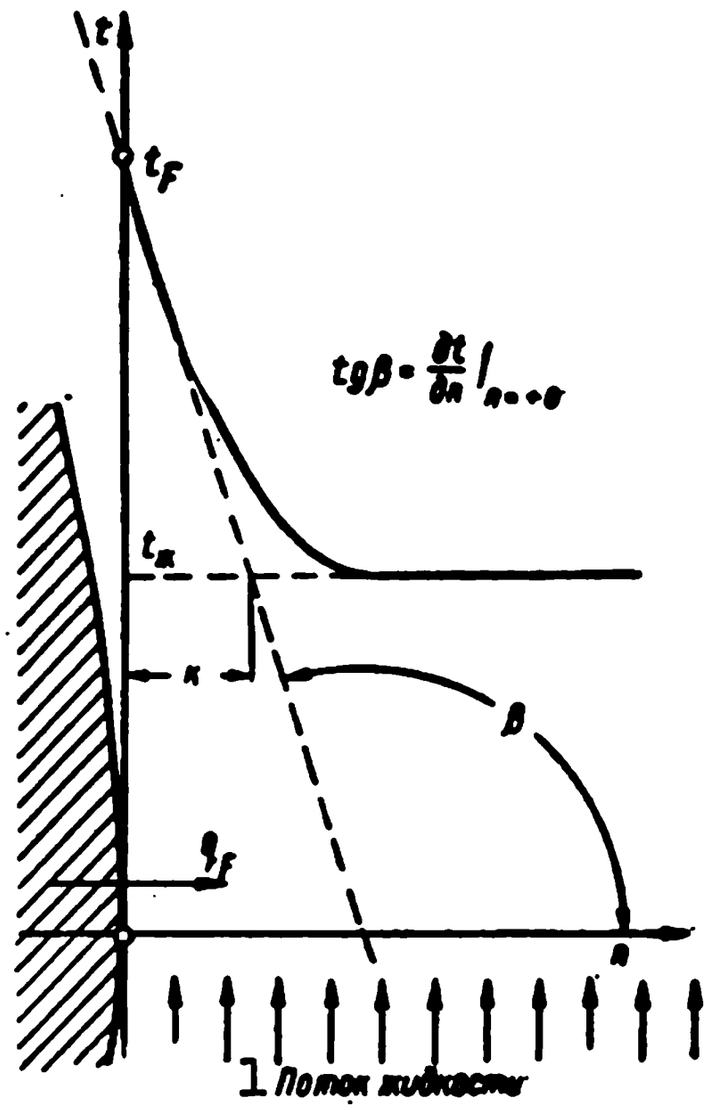

Fig. 34. Determination of the heat rate on the surface of a body cooled by a fluid with the aid of the Biot-Fourier law. 1) Fluid stream.

heat can pass through the separation boundary between the fluid and the wall via a layer of stationary fluid that has adhered to the wall only by heat conduction. This layer is quite thin. However, it certainly exists since if the fluid adheres to the stationary heating surface, then we have directly on this surface, i.e., at n = +0, a stationary liquid and there is no convective heat transport here. Therefore, to determine the value of the heat rate on the wall we can use the Biot-Fourier law for heat conduction through a fluid that is stationary on the wall

$$q_F = -\lambda_{zh} \cdot \frac{\partial t}{\partial n}\Big|_{n=+0}. \qquad (10.2)$$

It must be emphasized that this equation is meaningful only when n = +0 (Fig. 34), i.e., directly on the heating surface, where the liquid is stationary, and loses its meaning even at small distances from the wall, for when n > 0 the liquid already is moving and we have here convective heat transport along with heat conduction.

It follows from Eq. (10.2) that to determine the connection (10.1) between q_F, t_F, and t_{zh} it is necessary to know the temperature field $t_{zh}(\vec{l}, \tau)$ in the fluid stream.

Before we proceed to determine this field phenomenologically, let us introduce an auxiliary quantity, the coefficient of heat transmission α.* If the right half of Eq. (10.2) is divided and multiplied by the temperature difference $\Delta t = t_F - t_{zh}$, we obtain

$$q_F = \frac{-\lambda_{ж}\left.\frac{\partial t}{\partial n}\right|_{n=+0}}{t_F - t_{ж}} \cdot (t_F - t_{ж}). \qquad (10.3)$$

The coefficient of proportionality in Eq. (10.3) does not depend on the temperature of the wall t_F or of the fluid t_{zh}, if the thermal-physical parameters of the liquid do not depend on the temperature,* and the fluid itself does not change its aggregate state.

For this reason, a special name and symbol is used for the coefficient of proportionality in Eq. (10.3)

$$\frac{-\lambda_{ж}\left.\frac{\partial t}{\partial n}\right|_{n=+0}}{t_F - t_{ж}} \equiv \alpha. \qquad (10.4)$$

Consequently, the heat rate on the heating surface, cooled (or heated) by a stream of a single-phase fluid with constant thermal-physical properties is directly proportional to the temperature difference

$$q_F = \alpha \cdot (t_F - t_{ж}). \qquad (10.5)$$

Equation (10.5) is called the Newton-Rikhman heat-transmission law. In many modern heat-power installations, the thermal-physical parameters of the fluid change appreciably in value within a specified temperature interval $t_{zh} \leq t \leq t_F$, and the fluid changes its aggregate state. The Newton-Rikhman law becomes meaningless here, for in these cases the heat-transmission coefficient depends on the temperatures t_{zh} and t_F so that the direct proportionality between the heat rate q_F and the temperature difference Δt is violated.

In spite of this, the value of the heat-transmission coefficient is extensively used in modern heat engineering, since knowledge of this quantity greatly simplifies the calculation of heat transfer in heat-exchange installations. In addition, α under real conditions has a more stable value than the heat rate q_F. Thus, for example, for gases with natural convection we have $\alpha \approx 5$-30 kcal/m^2-hr-deg; for

forced flow the value is 10-500; for water under the same conditions it is 500-$2 \cdot 10^4$; for boiling water it is $2 \cdot 10^3$ to $4 \cdot 10^4$; and for liquid metals it is 10^2-$3 \cdot 10^4$.

The use of the Newton-Rikhman law is also quite convenient in the theory of heat conduction, since it leads to a linear boundary condition of the third kind.

The coefficient of heat transfer α is numerically equal to the heat rate q_F on the heating surface, referred to a unit temperature difference.

The value of the heat-transmission coefficient depends on many factors. The most important of these are: the cause of fluid motion (natural or forced convection); the fluid flow conditions (laminar, transition, or turbulent); the velocity of the fluid; the thermodynamic parameters of the fluid (λ_{zh}, μ, C_{pzh}, ρ_{zh}); the geometric shape and dimensions of the heating surface, and also its location in the stream and in the gravitational field.

Let us estimate the character of the influence of some factors on the value of α. For this purpose we make use of the fact that usually the total change in the temperature of the fluid from t_F to t_{zh} is concentrated in a certain region $\underline{k}$, adjacent to the heating surface. This region is called <u>the thermal boundary layer</u>.

Then

$$\left.\frac{\partial t}{\partial n}\right|_{n=+0} \approx -\frac{t_F - t_\infty}{k}$$

and

$$\alpha = -\frac{\lambda_\infty \left.\frac{\partial t}{\partial n}\right|_{n=+0}}{t_F - t_\infty} \approx \frac{\lambda_\infty}{k}. \qquad (10.6)$$

Relation (10.6) is quite approximate; it is used here only for a qualitative estimate of the influence of certain factors on the value of

the heat-transfer coefficient. It follows from (10.6) that to increase the heat-transmission coefficient it is necessary to use fluids with high heat conductivity, and also to decrease the thickness of the thermal boundary layer. Therefore to increase α it is necessary to increase the velocity of the fluid, its density, the hydrodynamic disturbances of the fluid, the roughness of the heating surface, while the viscosity of the liquid and the characteristic dimension of the surface (for example the tube diameter) should be decreased.

First Law of Thermodynamics for Isobaric Heat-Transport Processes

Following the second feature of the phenomenological methods, let us express the first law of thermodynamics for the convective heat-exchange process.

We consider an unevenly heated stream of fluid. To be definite, we assume that the temperature increases with time at all points of the stream and that heat is released in the stream as a result of the action of internal sources $q_V > 0$. In this stream we isolate an arbitrary volume V with the aid of an imaginary stationary and permeable control surface F, on which is located an Euler observer, who registers the fluxes of heat and fluid passing through this surface.

Let us set up an equation for the first law of thermodynamics for the volume V over a short time interval $d\tau$. Neglecting the kinetic energy of the stream and thermal friction, we obtain

$$d\tilde{Q}_F + d\tilde{Q}_V = dU + dL.$$

The quantities entering into this equation have the same physical meaning as in the analogous equation for the heat-conduction process.

For an isobaric convective heat-exchange process we have

$$dL = ApdV = Ad(pV),$$

where dV is the change in the volume of the fluid passing through the control surface F during the time $d\tau$. Therefore

$$dU + dL = d(U + ApV) = dI.$$

Here dI is the increase in the enthalpy of the fluid in the volume V, over the time dτ. Therefore the equation of the first law of thermodynamics for the isobaric process assumes the following form [11]:

$$d\tilde{Q}_F + d\tilde{Q}_V = dI. \tag{10.7}$$

As for heat-conduction processes, we have

$$\left.\begin{aligned} d\tilde{Q}_F &= -d\tau \int_V \operatorname{div} \mathbf{q}\, dV; \\ d\tilde{Q}_V &= d\tau \int_V q_V dV. \end{aligned}\right\} \tag{10.8}$$

Let us calculate the increment in the enthalpy of the fluid dI in the volume V during the time dτ:

$$dI = \int_V di \cdot dV.$$

where $\underline{i}$ is the enthalpy per unit volume of the fluid; dV is the volume element bounded by the stationary and permeable surface dF. The enthalpy of the substance $\underline{i}$ is a function of the parameters of its thermodynamic state $i = i(T, p)$. Consequently,

$$di = \left(\frac{\partial i}{\partial T}\right)_p dT + \left(\frac{\partial i}{\partial p}\right)_T dp.$$

For an isobaric convective heat-exchange process

$$\left(\frac{\partial i}{\partial p}\right)_T dp = 0.$$

In addition, the quantity $(\partial i/\partial T)_p$ is (by definition) the isobaric specific heat C_p per unit volume:

$$(\partial i/\partial T)_p \equiv C_p \text{ kcal/m}^3\text{-deg}.$$

Therefore

$$di = C_p \cdot dT = C_p \cdot \frac{\partial T}{\partial \tau} d\tau$$

and

$$dI = d\tau \int_V C_p \cdot \frac{\partial T}{\partial \tau} \cdot dV. \tag{10.9}$$

Substituting (10.8) and (10.9) in (10.7), we obtain

$$\int_V \left\{C_p \cdot \frac{\partial T}{\partial \tau} + \operatorname{div} \mathbf{q} - q_V\right\} dV = 0.$$

It was assumed above that the volume V, bounded by the control surface F, is arbitrary, and that all the characteristics of the process are continuous functions of the coordinates and of the time. Under these conditions the integrand in the last equation is equal to zero:

$$C_p \frac{\partial t}{\partial \tau} + \operatorname{div} \mathbf{q} - q_V = 0. \qquad (10.10)$$

Equation (10.10) contains five unknown quantities: the temperature t, the three components of the vector $\vec{q}$ ($\vec{q}_x$, $\vec{q}_y$, $\vec{q}_z$), and the specific heat C_p.

The Fourier-Ostrogradskiy Hypothesis and Differential Equation

Following the third feature of the phenomenological method, we introduce the Fourier-Ostrogradskiy hypothesis concerning the additional connection between the unknown quantities in Eq. (10.10):

the heat flux density vector $\vec{q}$ due to convective heat exchange at a given point of a nonisothermal fluid stream and at a given instant of time is equal to the sum of the heat flux density vectors due to the "pure" heat conduction $\vec{q}_t$ and the "pure" convection $\vec{q}_k$ at the same point of the stream and at the same instant of time. Here the vector $\vec{q}_t$ obeys the Biot-Fourier law $\vec{q}_t = -\lambda_{zh} \cdot \text{grad } t$, while the vector $\vec{q}_k$ is equal to $\vec{q}_k = i \cdot \vec{u}$.

Here i and $\vec{u}$ are the enthalpy of the fluid and the velocity vector at the same point of the stream and at the same instant of time. Thus, this hypothesis can be written in the form of the following equation:

$$\mathbf{q} = -\lambda_{zh} \cdot \operatorname{grad} t + i \cdot \mathbf{u}. \qquad (10.11)$$

As far as is known, experiments on a check of the validity of Formula

(10.11) were not carried out, apparently because this equation can be regarded as obvious. The good agreement between experiment and the results of the theory of convective heat exchange (for example the theory of the thermal boundary layer) is a confirmation of the presented hypothesis.

Following the fourth feature of the phenomenological method, it is necessary to determine experimentally the heat conduction λ_{zh}, the enthalpy $\underline{i}$, and the specific heat C_p for the given fluid as a function of the parameters of its thermodynamic state $\underline{p}$ and $\underline{t}$.

To obtain the differential equation for the convective heat exchange we substitute Eqs. (10.11) into Eqs. (10.10). As a result we obtain

$$C_p \cdot \frac{\partial t}{\partial \tau} + \operatorname{div}(i \cdot \mathbf{u} - \lambda_{zh} \cdot \operatorname{grad} t) - q_V = 0$$

or

$$C_p \cdot \frac{\partial t}{\partial \tau} + \operatorname{div}(i \cdot \mathbf{u}) = \operatorname{div}(\lambda_{zh} \cdot \operatorname{grad} t) + q_V. \qquad (10.12)$$

In accordance with the rules of vector analysis

$$\operatorname{div}(i \cdot \mathbf{u}) = i \cdot \operatorname{div} \mathbf{u} + (\mathbf{u}, \operatorname{grad} t). \qquad (10.13)$$

If the fluid has a constant density, then

$$\operatorname{div} \mathbf{u} = 0 *. \qquad (10.14)$$

In addition

$$\operatorname{grad} i = \left(\frac{\partial i}{\partial T}\right)_p \operatorname{grad} t = C_p \cdot \operatorname{grad} t. \qquad (10.15)$$

Substituting (10.14) and (10.15) into (10.13), we obtain

$$\operatorname{div}(i \cdot \mathbf{u}) = C_p \cdot (\mathbf{u}, \operatorname{grad} t). \qquad (10.16)$$

If Eq. (10.16) is substituted in (10.12), we obtain the Fourier-Ostrogradskiy differential equation for the isobaric processes of convective heat exchange

$$C_p \left[\frac{\partial t}{\partial \tau} + (\mathbf{u}, \operatorname{grad} t)\right] = \operatorname{div}(\lambda_{zh} \cdot \operatorname{grad} t) + q_V. \qquad (10.17)$$

This equation can be rewritten in different form, which in some cases is more convenient to use:

$$C_p\left[\frac{\partial t}{\partial \tau} + \operatorname{div}(t \cdot \mathbf{u})\right] = \operatorname{div}(\lambda_{zh} \cdot \operatorname{grad} t) + q_v. \qquad (10.18)$$

Equations (10.17) and (10.18) are valid for fluids which have a constant density ρ and variable specific heats $C_p = C_p(t)$ and thermal conductivities $\lambda_{zh} = \lambda_{zh}(t)$. If the fluid is homogeneous, i.e., if its thermodynamic parameters λ_{zh} and C_p are constant, then the Fourier-Ostrogradskiy differential equation assumes the simpler form

$$\left.\begin{aligned} &\frac{\partial t}{\partial \tau} + (\mathbf{u}, \operatorname{grad} t) = a_p \cdot \nabla^2 t + \frac{q_v}{C_p} \\ \text{or} \quad &\frac{\partial t}{\partial \tau} + \operatorname{div}(t \cdot \mathbf{u}) = a_p \cdot \nabla^2 t + \frac{q_v}{C_p}. \end{aligned}\right\} \qquad (10.19)$$

The differential equation of convective heat exchange contains the velocity vector $\vec{u}$. Consequently, the temperature distribution $\underline{t}$ in the flux depends on the velocity field of the fluid, while the convective heat exchange and the heat-transmission process depend on the hydrodynamic conditions observed in the stream. Experimental investigations of these processes offer evidence of a considerable influence of the hydrodynamic characteristics of the stream on the heat-transmission processes.

It must be emphasized that the vector $\vec{u}$ is the velocity of the heat carrier relative to the bodies surrounding it, as follows from the definition of the heat carrier. Therefore the quantity $\vec{u}$ is a relative velocity.

If there is no relative velocity, then the term $\operatorname{div}(\vec{u} \cdot t)$, characterizing the convective heat transport drops out from Eqs. (10.17), (10.18), or (10.19), and the Fourier-Ostrogradskiy equations are converted into Fourier equations for the heat conduction in a medium all parts of which are stationary relative to one another. Thus, for prac-

tical utilization of the Fourier-Ostrogradskiy differential equation it is necessary to have information regarding the velocity distribution in the fluid stream.

Law of Conservation of Matter for the Fluid Stream

Following the second feature of the phenomenological method, we make use of the law of conservation of matter to obtain the velocity field in the stream of the viscous fluid. For this purpose we separate in the fluid stream an arbitrary volume V with the aid of a control surface F. We set up the equation for the conservation of mass of fluid for the volume V over a small time interval $d\tau$

$$dM_F = dM. \tag{10.20}$$

During the time $d\tau$ there accumulates in the volume V a mass of fluid dM_F, resulting from the influx and outflow of fluid through the control surface F. The mass obtained leads to an increase in the fluid mass dM in the volume V resulting from the increase in the mass density ρ.

If we introduce the mass flow density vector $\vec{m}$, we obtain the following expression for dM_F:

$$dM_F = -d\tau \int_V \operatorname{div} \mathbf{m} \cdot dV^*. \tag{10.21}$$

The accumulation of the mass dM in the volume V as a result of the increase in mass density ρ is

$$dM = \int_V d\varrho \cdot dV = d\tau \int_V \frac{\partial \varrho}{\partial \tau} \cdot dV. \tag{10.22}$$

Substituting (10.21) and (10.22) in (10.20), we get

$$\int_V \left(\frac{\partial \varrho}{\partial \tau} + \operatorname{div} \mathbf{m} \right) dV = 0. \tag{10.23}$$

The volume V is selected arbitrarily, and all properties of the liquid flow are continuous functions of the coordinates and time.

Consequently, it follows from Equation (10.23) that

$$\frac{\partial \rho}{\partial \tau} + \operatorname{div} \mathbf{m} = 0. \qquad (10.24)$$

The equation of the law of conservation of matter contains four unknown quantities: the mass density ρ and three components of the mass flux density vector m_x, m_y, and m_z.

Let us express the vector $\vec{m}$ in terms of the sought quantity, namely the velocity vector $\vec{u}$:

$$\mathbf{m} = \rho \cdot \mathbf{u}. \qquad (10.25)$$

It is then necessary to determine by experimental means the mass density of the given fluid, ρ, as a function of the parameters of its thermodynamic state.

Let us substitute Eq. (10.25) in (10.24):

$$\frac{\partial \rho}{\partial \tau} + \operatorname{div} \rho \cdot \mathbf{u} = 0. \qquad (10.26)$$

The resultant equation is called the differential continuity equation. For an incompressible, i.e., homogeneous fluid, the density of which is constant, ρ = const, the continuity equation goes over into the incompressibility equation

$$\operatorname{div} \mathbf{u} = 0. \qquad (10.27)$$

The single scalar incompressibility equation contains three unknowns – the three components of the velocity vector u_x, u_y, and u_z. Consequently, this equation is not sufficient for the determination of the field of the velocity $\vec{u}$ in the stream.

Law of Momentum Conservation. The Newton Hypothesis Concerning Viscous Friction of Liquids and the Viscosity Coefficient

Following the second feature of the phenomenological method, one uses the law of momentum conservation to study the velocity field in the fluid stream [33]. For this purpose an arbitrary volume V is isolated in the fluid stream with the aid of a control surface F.

The law of momentum conservation for a fluid flowing through a volume V states that the rate of change of the principal momentum vector of the fluid in the volume V is equal to the principal vector of the weight forces and the surface forces acting on the surface F.

As a result of application of this law one obtains the dynamics equation of the fluid, expressed in terms of the stresses $\vec{P}$:

$$\varrho \frac{\partial u}{\partial \tau} + \varrho (u, \operatorname{grad}) u = \varrho g + \frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z}.$$

This is a vector equation and can be written in the form of three equations in terms of its projections:

$$\left.\begin{aligned}
&\varrho \left(\frac{\partial}{\partial \tau} + u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z}\right) u_x = \\
&\quad = \varrho g_x + \frac{\partial P_{xx}}{\partial x} + \frac{\partial P_{yx}}{\partial y} + \frac{\partial P_{zx}}{\partial z}; \\
&\varrho \left(\frac{\partial}{\partial \tau} + u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z}\right) u_y = \\
&\quad = \varrho g_y + \frac{\partial P_{xy}}{\partial x} + \frac{\partial P_{yy}}{\partial y} + \frac{\partial P_{zy}}{\partial z}; \\
&\varrho \left(\frac{\partial}{\partial \tau} + u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z}\right) u_z = \\
&\quad = \varrho g_z + \frac{\partial P_{xz}}{\partial x} + \frac{\partial P_{yz}}{\partial y} + \frac{\partial P_{zz}}{\partial z},
\end{aligned}\right\} \qquad (10.28)$$

these equations contain 12 unknowns: the three velocity vector components u_x, u_y, u_z and the nine stress-tensor components P_{xx}, P_{xy}, P_{xz}, P_{yx}, P_{yy}, P_{yz}, P_{zx}, P_{zy}, P_{zz}.

Following the third feature of the phenomenological method, we introduce a hypothesis concerning the connection between the stresses $\vec{P}$ and the velocity field $\vec{u}$. This hypothesis was proposed by Newton, and for the simplest case of planar flow of a viscous liquid parallel to the Ox axis it is formulated as follows:

<u>the tangential stress of friction in a planar stream of a viscous liquid is directly proportional to the derivative of the velocity</u> u_x <u>along the normal z to the direction of the stream</u>:

$$P_{xz} \equiv \tau_{xz} \sim \frac{du_x}{dz}.$$

Numerous experiments have confirmed the correctness of this hypothesis for most liquids used in technology.* It was established that the proportionality coefficient in the Newton equation is a physical parameter, since for a given liquid at specified temperature and pressure this coefficient has a unique value. This physical parameter characterizes the viscous friction in the fluid, and is therefore called the dynamic viscosity coefficient and usually designated μ (kg-sec/m^2). The ratio μ/ρ is called the kinematic viscosity coefficient and is designated ν (m^2/sec).

Following the fourth feature of the phenomenological method, one determines experimentally the viscosity coefficient for each substance as a function of t and p. The results of measurements of the viscosity are found in many books and handbooks, for example in [7].

The viscosity coefficients of liquids and gases depend on the temperature. For dropping liquids, these coefficients decrease with increasing temperature, while for gases they increase (Figs. 35 and 36). The viscosity gas as a function of the temperature is described by the Sutherland formula

$$\mu \sim \frac{T^{3/2}}{T+114}.$$

In particular, for air this formula assumes the form

$$\mu = 14{,}65\,\frac{T^{3/2}}{T+114}.$$

For approximate calculation a power-law equation is used

$$\frac{\mu}{\mu_0} = \left(\frac{T}{T_0}\right)^n,$$

where n depends principally on the nature of the gas and also on its temperature.

The Newton hypothesis has by now been generalized for three-dimensional flows of both incompressible drop liquids, and for com-

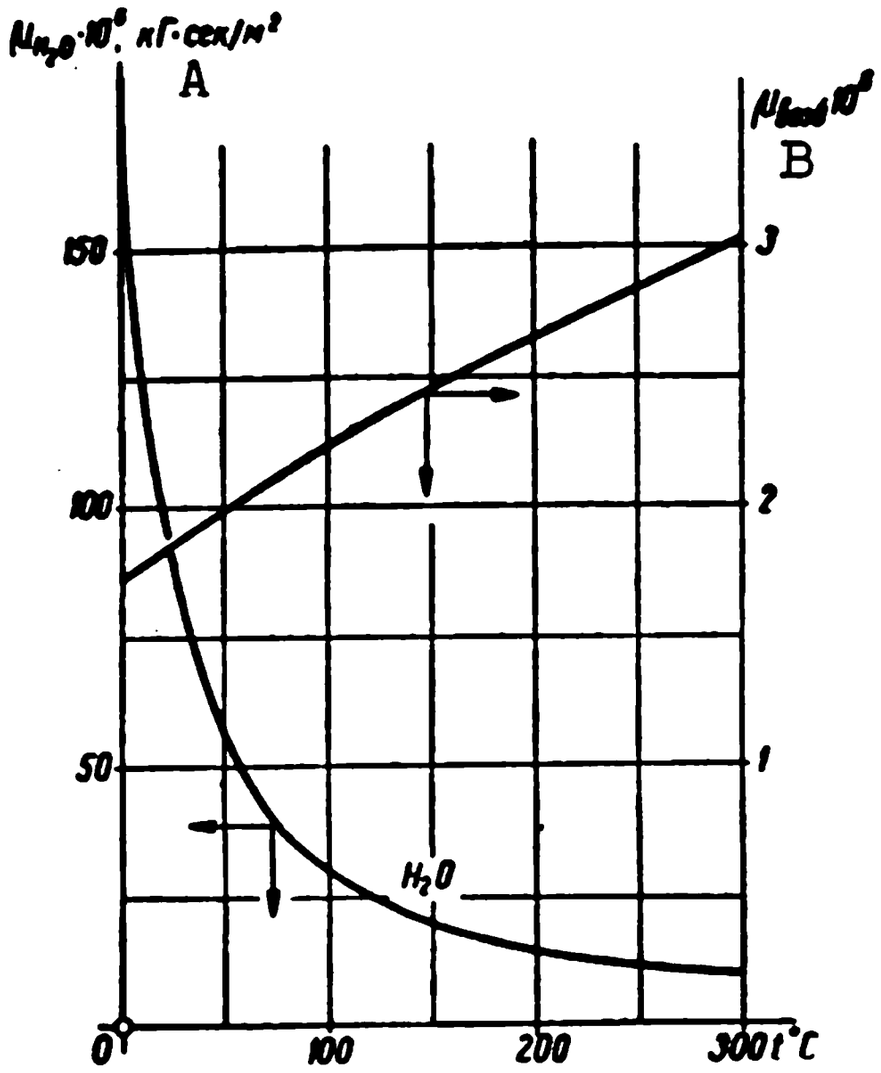

Fig. 35. Temperature dependence of the dynamic viscosity of water and air.

A) kg-sec/m^2; B) air.

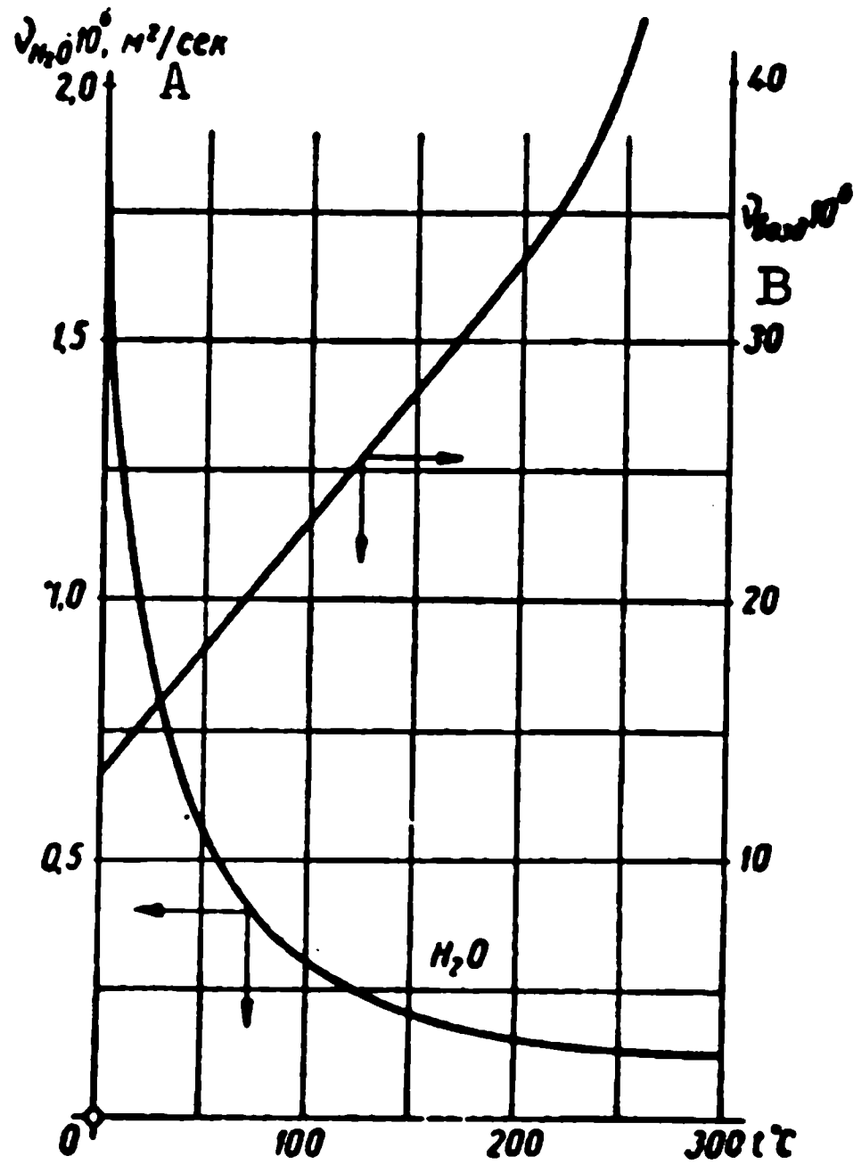

Fig. 36. Temperature dependence of the kinetic viscosity of water and air.

A) m^2/sec; B) air.

pressible gases. It is a law that serves as the basis of modern mechanics of viscous fluids.

The generalized Newton's law represents a linear dependence of the stresses P_{xx}, P_{xy}, ... on the rate of deformation of the liquid; it can be written in the following form [33]:

$$P_{ij} = \begin{cases} \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) & \text{for } j \neq i, \\ -p + 2\mu \frac{\partial u_i}{\partial x_i} & \text{for } j = i. \end{cases} \qquad (10.29)$$

Here p is the pressure of the fluid at any point of the stream; the coordinates x, y, and z are designated by x_i ($i = 1, 2, 3$), respectively.

The Navier-Stokes and Helmholz Differential Equations

The differential equations of the dynamics of a viscous fluid in terms of stresses (10.28), in conjunction with the generalized Newton law (10.29) in terms of the incompressibility equation (10.27) con-

stitute a closed system of equations. By eliminating the stresses P_{ij} from this system we obtain the Navier-Stokes equations

$$\frac{\partial \mathbf{u}}{\partial \tau} + (\mathbf{u},\ \mathrm{grad})\,\mathbf{u} = \mathbf{g} - \frac{1}{\varrho}\cdot \mathrm{grad}\, p + \nu \cdot \nabla^2 \mathbf{u}, \qquad (10.30)$$

where the symbol $\vec{\nabla^2 u}$ denotes a vector with components $\nabla^2 u_x$, $\nabla^2 u_y$, and $\nabla^2 u_z$. The nonlinear differential equation (10.30) in conjunction with the incompressibility equation (10.27) characterizes the distribution of the velocity $\vec{u}$ and of the pressure p in the stream of a viscous, homogeneous, and isotropic fluid.

Equation (10.30) is a vector equation, so that in terms of components it can be written as three equations:

$$\left.\begin{aligned}
\left(\frac{\partial}{\partial \tau} + u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z}\right) u_x &= \\
= g_x - \frac{1}{\varrho}\frac{\partial p}{\partial x} + \nu \cdot \nabla^2 u_x;& \\
\left(\frac{\partial}{\partial \tau} + u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z}\right) u_y &= \\
= g_y - \frac{1}{\varrho}\frac{\partial p}{\partial y} + \nu \cdot \nabla^2 u_y;& \\
\left(\frac{\partial}{\partial \tau} + u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z}\right) u_z &= \\
= g_z - \frac{1}{\varrho}\frac{\partial p}{\partial z} + \nu \cdot \nabla^2 u_z.&
\end{aligned}\right\} \qquad (10.31)$$

The system of equations (10.27)-(10.31) consists of four equations and contains the four unknowns p, u_x, u_y, u_z. The left half of the Navier-Stokes equation characterizes the inertial forces acting in the fluid stream. The linear term $\partial\vec{u}/\partial\tau$ is called the local part, while the nonlinear one $(\vec{u},\ \mathrm{grad})\ \vec{u}$ the convective part of the inertial forces. The first term of the right half of the equation, $\vec{g}$, characterizes the action of the weight forces, while the second term $[-(1/\rho)\ \mathrm{grad}\ p]$ characterizes the action of the difference in pressure forces and the third term $\nu \cdot \vec{\nabla^2 u}$ the action of the difference in the viscous-friction forces.

The Navier-Stokes equation can be written in a different form if its curl is taken. Taking the curl of the right and left halves of

(10.30) we obtain

$$\mathrm{rot}\left[\frac{\partial \mathbf{u}}{\partial \tau}+(\mathbf{u},\ \mathrm{grad})\,\mathbf{u}\right]=\mathrm{rot}\left(\mathbf{g}-\frac{1}{\varrho}\,\mathrm{grad}\,p\right)+\nu\cdot\mathrm{rot}\,\nabla^2\mathbf{u}.$$

In this equation rot $[\vec{g} - (1/\rho)\ \mathrm{grad}\ p] = 0$, since $\vec{g}$ is a constant vector, and the curl of the gradient of a scalar quantity is equal to zero. As a result we obtain the Helmholz differential equation

$$\mathrm{rot}\left[\frac{\partial \mathbf{u}}{\partial \tau}+(\mathbf{u},\mathrm{grad})\,\mathbf{u}-\nu\cdot\nabla^2\mathbf{u}\right]=0. \qquad (10.32)$$

Equation (10.32) contains neither the weight forces nor the pressure-difference forces and therefore is kinematic in character. It describes the velocity distribution in the stream of a viscous fluid in those cases when the weight forces and pressure forces are of no importance to the flow. This is observed, for example, in forced streams of incompressible liquids, which have no free surfaces.

§11. UNIQUENESS CONDITIONS FOR HEAT-TRANSMISSION PROCESSES

Differential Equations for Heat-Transmission Processes

In nuclear power one encounters various conditions under which convective heat exchange takes place. It is possible to separate from this variety two types of heat-transfer processes which are encountered most frequently and are at the same time relatively simple. The first of them is connected with determining the heat rate q_F on a heating surface, if the temperature t_F of this surface is specified along with the conditions under which heat is removed by heat transmission. Such a heat-transmission process is most frequently encountered in the design of heat-exchange apparatus in which the heat from a hotter liquid is transferred to a cooler one through a separating wall. The second type of heat-transmission process is connected with the determination of the temperature t_F of the surface of a solid, if the heat rate q_F on this surface and the conditions under which heat is removed from it are specified. Such processes are encountered principally in the

calculation of the cooling of fuel elements for nuclear power reactors.

In the first case the heat rate q_F can be expressed by means of the Biot-Fourier law, written for the heating surface on the side of the fluid stream:

$$q_F = -\lambda_{ж} \cdot \frac{\partial t}{\partial n}\Big|_{n=+0}. \qquad (11.1)$$

In the second case the wall temperature is determined from the condition that the temperatures of the wall and of the fluid are equal on their contact surface

$$t_F = t|_{n=+0}. \qquad (11.2)$$

The temperature distribution t(x, y, z, τ) in the fluid stream is described by the Fourier-Ostrogradskiy equation

$$\frac{\partial t}{\partial \tau} + (\mathbf{u}, \operatorname{grad} t) = a_{ж} \cdot \nabla^2 t, \qquad (11.3)$$

while the velocity distribution $\vec{u}$ is described by the incompressibility and Navier-Stokes equations

$$\operatorname{div} \mathbf{u}; \qquad (11.4)$$

$$\frac{\partial \mathbf{u}}{\partial \tau} + (\mathbf{u}, \operatorname{grad})\,\mathbf{u} = \mathbf{g} - \frac{1}{\varrho} \cdot \operatorname{grad} p + \nu \cdot \nabla^2 \mathbf{u}. \qquad (11.5)$$

If the weight and pressure forces do not influence the velocity distribution, one uses in lieu of (11.4) and (11.5) the Helmholz equation

$$\operatorname{rot}\left[\frac{\partial \mathbf{u}}{\partial \tau} + (\mathbf{u}, \operatorname{grad})\,\mathbf{u} - \nu \cdot \nabla^2 \mathbf{u}\right] = 0. \qquad (11.6)$$

This system is closed, since it contains six unknowns (q_F or t_F, $\underline{t}$, $\underline{p}$, u_x, u_y, u_z) and consists of six equations.

Uniqueness Conditions for Heat-Transmission Processes

The system of differential equations (11.1)-(11.6) describes a whole set of convective heat-exchange processes. In order to separate a specific process it is necessary to formulate the uniqueness conditions. As for heat-conduction, the uniqueness conditions of the heat-

transmission processes contain geometrical, physical, time, and boundary conditions.

The geometrical conditions determine the form and the dimensions of the solid body, on the surface of which it is necessary to determine q_F or t_F, the location of the heating surface in the fluid stream, and also the shape and dimensions of the fluid boundaries of the stream.

The physical conditions determine the numerical values of the thermodynamic parameters μ, ρ_{zh}, λ_{zh}, and C_{pzh}, and also contain information on the internal heat sources acting in the stream (information on the presence, intensity, and distribution of the sources).

The time conditions indicate the features governing the course of the heat transmission with time (whether it is stationary or not, the initial distribution of the temperature $\underline{t}$ and of the velocity $\vec{u}$ in the stream, etc.).

The boundary conditions determine the distribution of the velocity $\vec{u}$, the temperature $\underline{t}$, or the heat rate q_F on the solid and liquid boundaries of the stream. In this case the velocity of the fluid on the solid boundaries is equal to zero in the phenomenological analysis [12]

$$u|_{n=0} = 0,$$

i.e., the fluid adheres to the wall.

The boundary conditions on the heating surface specify its temperature distribution

$$t_F = t_F(x_F, y_F, z_F, \tau), \qquad (11.7)$$

if it is necessary to determine the heat rate q_F on it (design of heat-exchange apparatus).

If it is required to determine the temperature of the heating surface t_F, then the boundary conditions specify the distribution of the heat rate on the surface

$$-\lambda_{*}\cdot\frac{\partial t}{\partial n}\Big|_{l=+0}=q_F(x_F,\ y_F,\ z_F,\ \tau) \qquad (11.8)$$

(design of cooling for fuel elements).

It must be noted that a correct formulation of convective heat-exchange problems is possible at the present time only for a few particular fluid-flow conditions, for which there are known rigorous proofs of the uniqueness and existence conditions for the solutions of the corresponding equations [15, 67]. Even these proofs, however, are confined to the hydrodynamic part of the process and pertain only to laminar flow. Therefore the correctness of the formulation of problems in convective heat exchange is usually checked by experiment.

Scheme of the Phenomenological Method of Investigating Heat-Transmission Processes

The scheme of the phenomenological method for the investigation of heat-transmission processes is shown in Fig. 37. As a result of using the four stages of the phenomenological method one obtains the differential equations for the convective heat exchange (the Fourier-Ostrogradskiy, incompressibility, and the Navier-Stokes or Helmholz equations). It is necessary to add to these equations the geometrical, physical, time, and boundary conditions.

The problem formulated is solved at present principally by experiment using dimensionality and similarity methods to generalize the measurement results. The correctness of the formulation of the problem is likewise checked by experiment. The analytical and numerical methods of investigation of heat-transmission processes are used to some extent and in occasional cases they lead to satisfactory results. By way of an example of an effective utilization of the analytical method we can present the theory of the thermal boundary layer of finite thickness [26a]. However, the analytical and numerical methods of investigating heat transmission have found relatively narrow application ow-

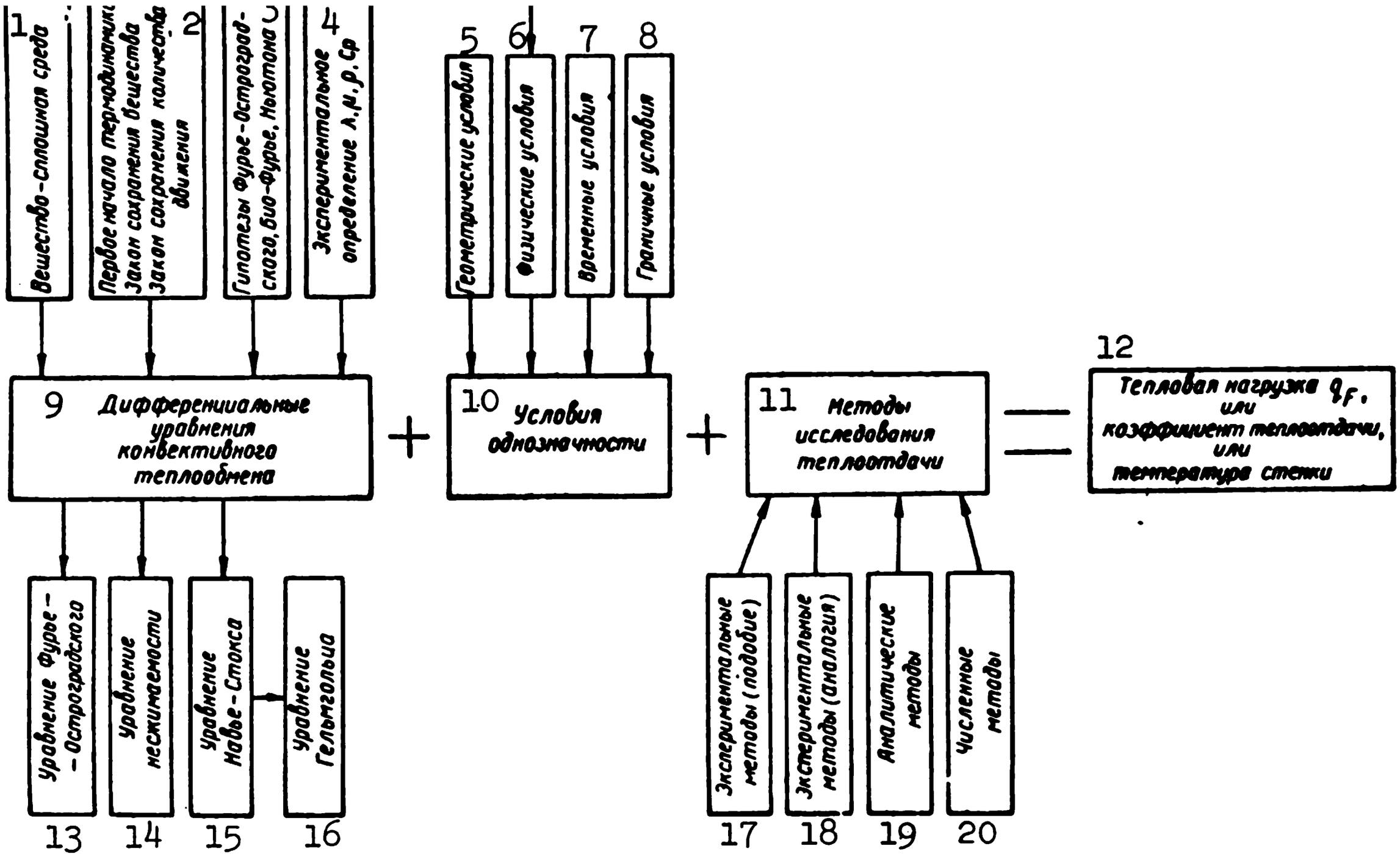

Fig. 37. Scheme of phenomenological method for the investigation of convective heat transmission. 1) Substance – continuous medium; 2) first law of thermodynamics. Law of conservation of matter. Law of conservation of momentum; 3) Fourier-Ostrogradskiy, Biot-Fourier, and Newton hypotheses; 4) experimental determinations of λ, μ, ρ, and C_p; 5) geometrical conditions; 6) physical conditions; 7) time conditions; 8) boundary conditions; 9) differential equations of convective heat exchange; 10) uniqueness conditions; 11) methods of investigating heat transmission; 12) heat rate q_F or heat-transmission coefficient or wall temperature; 13) Fourier-Ostrogradskiy equation; 14) incompressibility equation; 15) Navier-Stokes equation; 16) Helmholz equation; 17) experimental methods (similarity); 18) experimental methods (analogy); 19) analytical methods; 20) numerical methods.

ing to the complexity and nonlinearity of the system of initial differential equations and owing to the need for appreciably simplifying the problem, leading in many cases to insufficiently reliable results. The situation becomes appreciably aggravated by the lack of a sufficiently general theory of turbulent heat transfer. For these reasons, the experimental investigation of heat-transmission processes using similarity and dimensionality methods for the generalization of the measure-

ment results has found greatest application. However, the mathematical formulation of the problem is an important means, which makes it possible to obtain a system of quantities that are essential for the investigated process, and also many conclusions that explain the meaning of the experimental investigations and indicate the most advantageous methods of presenting the heat-transmission measurement data.

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[Footnotes]

420 The properties of the heat-transmission coefficients are repeated here, see §2.

421 This property of the proportionality coefficient in Formula (10.3) will be proved in Chapter 6.

426 For proof see below.

428 The derivation of Eq. (10.21) for $d\underset{\sim}{M}_F$ is analogous to the derivation of the expression for dQ_F in the first law of thermodynamics for heat-conduction or convective heat-exchange processes.

431 Such liquids are customarily called Newtonian to distinguish them from liquids for which the Newton hypothesis is not satisfied (colloidal solutions, oil paint, etc.).

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[List of Transliterated Symbols]

417 ж = zh = zhidkost' = liquid

425 т = t = teploprovodnost' = heat conduction

425 к = k = konvektsiya = convection

434 rot = curl

Chapter 6

HEAT TRANSMISSION FROM A CYLINDER SITUATED IN A FLOWING LIQUID

§12. SIMILARITY CONDITIONS FOR THE HEAT TRANSMISSION FROM A BODY OF ARBITRARY SHAPE SITUATED IN A FORCED STREAM OF A HOMOGENEOUS LIQUID

Setup and Mathematical Formulation of the Problem

Let us consider the similarity conditions of heat transmission on a surface F of some body, the shape and dimensions of which are known.

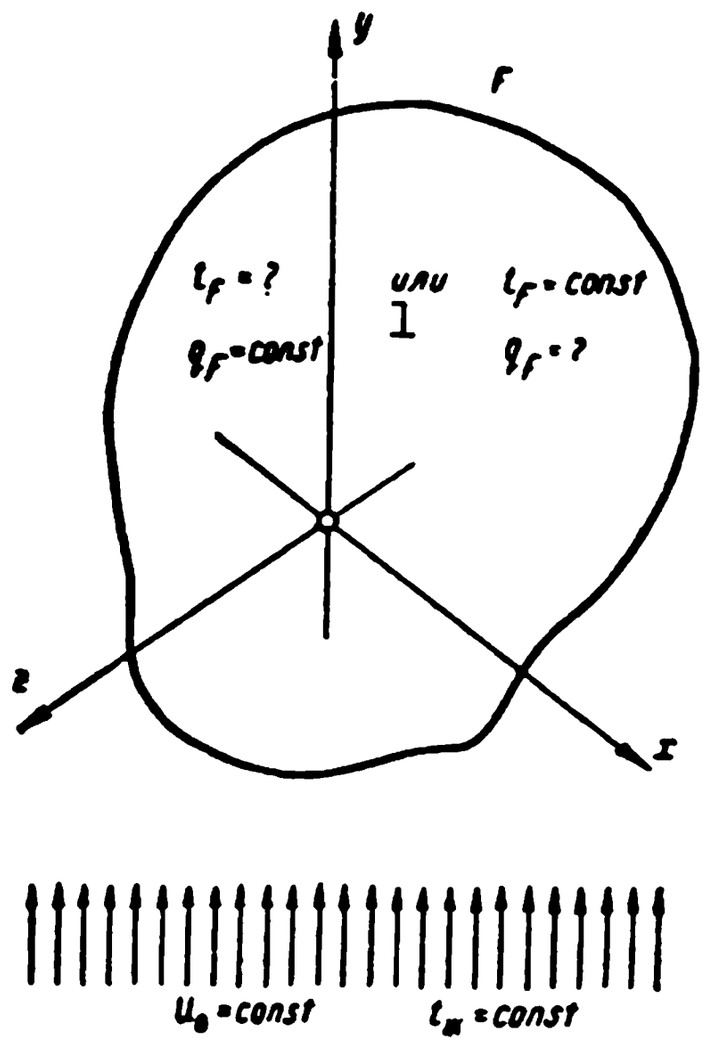

Fig. 38. Diagram of flow around a body by a forced stream of liquid. 1) Or.

The body is placed in a forced unlimited stream of a homogeneous liquid (Fig. 38). Remote from the body the temperature and velocity of the liquid are, respectively, t_{zh} and $\vec{u}_0$. It is required to consider the similarity conditions for two characteristic heat-transmission processes. The first of them is connected with the determination of the heat rate q_F on the surface F of the body, if the constant temperature on this surface $t_F > t_{zh}$ is known. The second heat-transmission process occurs when a constant heat rate q_F = const is maintained on the surface F of the body. It is then necessary to determine the distribution of the temperature t_F on the surface F. To determine the similarity conditions it is necessary to set up a mathematical description of the heat-transmission process.

Let us formulate the uniqueness conditions for the first process,

when the sought quantity is the heat rate.

The geometrical conditions specify the equation for the surface of the body

$$F(l_F, l_1, l_2, \ldots, l_m) = 0,$$

and also the direction of the liquid stream away from the body, i.e., the direction angles ψ_x and ψ_y of the velocity vector. These angles are called the angles of attack. Here $\vec{l}_F$ is the radius vector of any point on the surface, and l_1, l_2, ..., l_m are the characteristic dimensions of the body.

The physical conditions specify the constant values of the heat conductivity λ_{zh}, the temperature conductivity a_{pzh}, and the kinematic viscosity ν, and also indicate that there are no internal heat sources in the stream $q_V = 0$.

The time conditions indicate that the heat-transmission process is stationary, i.e., $\partial t/\partial\tau = 0$ and $\partial\vec{u}/\partial\vec{\tau} = 0$.

The boundary conditions specify the distribution of the temperature and velocity on the solid boundaries of the stream, i.e., on the surface F,

$$u|_{n=0} = 0 \text{ and } t|_{n=0} = \text{const} = t_F$$

and on the liquid boundaries of the stream

$$u|_{n\to\infty} = u_0; \quad t|_{n\to\infty} \to t_\infty < t_F.$$

If we introduce the excess temperature $\vartheta \equiv t - t_{zh}$, then the differential equations (10.2), (10.19), (10.32) and the boundary conditions for the first heat-transmission process, with allowance for the uniqueness conditions, can be written in the form of the equations

$$\begin{gathered}
q_F = -\lambda_\infty \cdot \frac{\partial\vartheta}{\partial n}\Big|_{n=+0}; \\
(u, \operatorname{grad}\vartheta) - a_{p\infty}\cdot\nabla^2\vartheta = 0; \\
\operatorname{rot}[(u, \operatorname{grad})u - \nu\cdot\nabla^2 u] = 0; \\
u|_{n=0} = 0; \quad \vartheta|_{n=0} = t_F - t_\infty \equiv \Delta t; \\
u|_{n\to\infty} \to u_0; \quad \vartheta|_{n\to\infty} \to 0;
\end{gathered} \tag{12.1}$$

$$F(l_F, l_1, l_2, \ldots, l_m) = 0. \qquad (12.1)$$

The system (12.1) is closed, since it contains five unknowns q_F, ϑ, u_x, u_y, u_z and consists of five equations. In this system the quantities λ_{zh}, a_{pzh}, Δt, $\vec{u}_0$, l_1, ..., l_m are specified and it is required to determine $q_F = q_F(\vec{l}_F)$.

Let us consider another characteristic heat-transmission process, when one maintains on the surface of the body a constant heat rate q_F = const. It is required to determine the temperature distribution $t_F = t_F(\vec{l}_F)$ on this surface.

The mathematical formulation of the problem assumes in this case the following form:

$$\left.\begin{gathered}\vartheta_F = \vartheta|_{n=0};\quad (\mathbf{u}, \operatorname{grad}\vartheta) - a_{pж}\cdot\nabla^2\vartheta = 0;\\ \operatorname{rot}[(\mathbf{u}, \operatorname{grad})\mathbf{u} - \nu\nabla^2\mathbf{u}] = 0;\\ -\lambda_ж\cdot\left.\frac{\partial\vartheta}{\partial n}\right|_{n=+0} = q_F;\\ \mathbf{u}|_{n\to\infty}\to\mathbf{u}_0;\quad \vartheta|_{n\to\infty}\to 0;\\ F(l_F, l_1, l_2, \ldots, l_m) = 0.\end{gathered}\right\} \qquad (12.2)$$

The system of equations (12.2) is also closed. The quantities λ_{zh}, q_F, a_{pzh}, ν, $\vec{u}_0$, l_1, l_2, ..., l_m are specified and it is required to find the distribution $\vartheta_F = \vartheta_F(\vec{l}_F)$.

Derivation of Similarity Criteria by the Method of Transforming the Scales

To reduce Eqs. (12.1) and (12.2) to dimensionless form, we shall use the method of scale transformation. For this purpose we introduce the dimensionless temperature $\Theta \equiv \vartheta/\Delta t$, the dimensionless velocity $\vec{U} \equiv \vec{u}/u_0$ ($u_0 \equiv |\vec{u}_0|$), the dimensionless radius vector of any point both in the stream $\vec{L} \equiv \vec{l}/l_1$ and on the surface of the body $\vec{L}_F \equiv \vec{l}_F/l_1$, and the dimensionless normal to the surface $N \equiv n/l_1$. From these relations we obtain

$$\left.\begin{gathered}\vartheta = \Delta t\cdot\Theta;\quad \mathbf{u} = u_0\cdot\mathbf{U};\quad \mathbf{l} = l_1\cdot\mathbf{L};\\ \mathbf{l}_F = l_1\cdot\mathbf{L}_F;\quad n = l_1\cdot N.\end{gathered}\right\} \qquad (12.3)$$

obtain the dimensionless mathematical description of the first heat-transmission process

$$\left.\begin{gathered}\frac{q_F \cdot l_1}{\Delta t \cdot \lambda_\infty} = -\frac{\partial \Theta}{\partial N}\Big|_{N=+0}; \quad \frac{u_0 \cdot l_1}{a_{p\infty}}(\mathbf{U}, \operatorname{Grad}\Theta) - \Delta^2\Theta = 0; \\ \operatorname{Rot}\left[(\mathbf{U}, \operatorname{Grad})\,\mathbf{U} \cdot \frac{u_0 \cdot l_1}{\nu} - \nabla^2\mathbf{U}\right] = 0; \\ \mathbf{U}|_{N=0} = 0; \quad \Theta|_{N=0} = 1; \\ \mathbf{U}|_{N\to\infty} \to \frac{u_\infty}{u_0}; \quad \Theta|_{n\to\infty} \to 0; \\ F(L_F, L_2, L_3, \ldots, L_m) = 0.\end{gathered}\right\} \qquad (12.4)$$

Here the symbols Grad, Rot, and ∇^2 denote the dimensionless vector operations of taking the gradient, the curl, and the Laplace operator. The solution of the problem (12.4) consists of the following dimensionless quantities:

$$\frac{q_F \cdot l_1}{\Delta t \cdot \lambda_\infty} = f\left(\frac{l_F}{l_1}, \frac{l_2}{l_1}, \ldots, \frac{l_m}{l_1}, \psi_x, \psi_y, \frac{u_0 \cdot l_1}{a_{p\infty}}, \frac{u_0 \cdot l_1}{\nu}\right). \qquad (12.5)$$

To reduce Eqs. (12.2) to dimensionless form it is necessary to have a temperature scale. We introduce such a scale a certain quantity ϑ_0, which has the dimension of temperature. Then

$$\Theta = \frac{\vartheta}{\vartheta_0}.$$

The scales for the remaining variables remain unchanged.

Reducing the system (12.2) to dimensionless form, we obtain

$$\left.\begin{gathered}\frac{\vartheta_F}{\vartheta_0} = \Theta\Big|_{N=0}; \quad \frac{u_0 \cdot l_1}{a_{p\infty}}(\mathbf{U}, \operatorname{Grad}\Theta) - \nabla^2\Theta = 0; \\ \operatorname{Rot}\left[(\mathbf{U}, \operatorname{Grad})\,\mathbf{U} \cdot \frac{u_0 \cdot l_1}{\nu} - \nabla^2\mathbf{U}\right] = 0; \\ \mathbf{U}|_{N=0} = 0; \quad \frac{q_F \cdot l_1}{\lambda_\infty \cdot \vartheta_0} = -\frac{\partial\Theta}{\partial N}\Big|_{N=+0}; \\ \mathbf{U}|_{N\to\infty} \to \frac{u_\infty}{u_0}; \quad \Theta|_{N\to\infty} \to 0.\end{gathered}\right\} \qquad (12.6)$$

The solution of the problem (12.6) consists of the following dimensionless quantities:

$$\frac{\vartheta_F}{\vartheta_0} = f_1\left(\frac{l_F}{l_1}, \frac{l_2}{l_1}, \ldots, \frac{l_m}{l_1}, \psi_x, \psi_y, \frac{u_0 \cdot l_1}{a_{p\infty}}, \frac{u_0 \cdot l_1}{\nu}, \frac{q_F \cdot l_1}{\lambda_\infty \cdot \vartheta_0}\right).$$

However, the distribution of the temperature ϑ_F on the surface of the body cannot depend on the arbitrary scale ϑ_0. Therefore the quantities ϑ_F/ϑ_0 and $(q_F \cdot l_1)/(\lambda_{zh} \cdot \vartheta_0)$, which contain ϑ_0, should be contained in the solution of the problem (12.6) in the form of a ratio

$$\frac{\frac{\vartheta_F}{\vartheta_0}}{\frac{q_F \cdot l_1}{\lambda_{zh} \cdot \vartheta_0}} = \frac{\vartheta_F \cdot \lambda_{zh}}{q_F \cdot l_1},$$

which no longer contains ϑ_0. Thus, the solution of the problem (12.6) has the following dimensionless structure:

$$\frac{\vartheta_F \cdot \lambda_{zh}}{q_F \cdot l_1} = f_2\left(\frac{l_F}{l_1}, \frac{l_2}{l_1}, \ldots, \frac{l_m}{l_1}, \frac{u_0 \cdot l_1}{a_{p zh}}, \frac{u_0 \cdot l_1}{\nu}\right). \qquad (12.7)$$

This equation is analogous to the previously obtained relation (12.5) for the first heat-transmission process, when the thermal load q_F was considered as the sought quantity.

Derivation of Similarity Criteria with the Aid of Dimensional Analysis of the Physical Quantities

Dimensionless quantities for the heat-transmission process of a body situated in a stream of liquid can also be obtained by dimensional analysis of the physical quantities. For this purpose it is necessary to use a system of defining parameters. Such a system can be readily set up with the aid of the differential equations and the boundary conditions (12.1).

Indeed, from the Fourier-Ostrogradskiy equation and from the boundary conditions for the temperature field we obtain

$$\vartheta = \varphi_1(l, a_{p zh}, u, \Delta t, l_1, l_2, \ldots, l_m). \qquad (12.8)$$

The distribution of the velocity $\vec{u}$, as follows from the Helmholz equation and from the boundary conditions, depends on the following parameters:

$$u = \varphi_2(l, \nu, u_0 l_1, l_2, \ldots, l_m, \psi_x, \psi_y). \qquad (12.9)$$

Substituting (12.9) in (12.8) we obtain

$$\vartheta = \varphi_3(l, l_1, l_2, \ldots, l_m, \psi_x, \psi_y, \nu, a_{pm}, u_0, \Delta t). \qquad (12.10)$$

Substituting Expression (12.10) into Eq. (12.8) for the sought quantity, we obtain

$$q_F = -\lambda_m \cdot \frac{\partial \vartheta}{\partial n}\Big|_{n=+0} =$$

$$= -\lambda_m \cdot \frac{\partial}{\partial n} \varphi_3(l, l_1, l_2, \ldots, l_m, \psi_x, \psi_y, \nu, a_{pm}, u_0, \Delta t)|_{n=+0}.$$

After transformation we obtain the system of defining parameters

$$\frac{q_F}{\lambda_m} = \varphi_4(l_F, l_1, l_2, \ldots, l_m, \psi_x, \psi_y, \nu, a_{pm}, u_0, \Delta t). \qquad (12.11)$$

Equation (12.11) contains n = 6 + m dimensional quantities and two angles ψ_x and ψ_y, which can be regarded as dimensionless quantities. The dimensional quantities have the following units:

$$[q_F/\lambda_{zh}] = \text{deg/m}; \quad [\nu] = [a_{pzh}] = \text{m}^3/\text{hr}; \quad [\Delta t] = \text{deg};$$

$$[\vec{l}_F] = [l_1] = \ldots = [l_m] = \text{m}; \quad [u_0] = \text{m/hr}.$$

The measurement units (deg, m, m^2/hr) can be chosen to be independent of one another for the process under consideration. Consequently, the number of groups of homogeneous quantities with units that are independent of one another is three (k = 3). In accordance with the π theorem, the total number of similarity criteria is n − k = 6 + m − 3 = = 3 + m. This is precisely the number of dimensionless quantities (disregarding ψ_x and ψ_y) contained in Eq. (12.5), so that this calculation is a check on the correctness of the results obtained above by scale transformation.

To reduce Eq. (12.11) to dimensionless form it is necessary to change over to new measurement units that are independent of one another [deg', m', $(\text{m}^2/\text{hr})'$], which are smaller than the preceding units by factors A_ϑ, A_l, and A_ν, respectively:

$$\text{deg}' = \text{deg}/A_\vartheta, \quad \text{m}' = \text{m}/A_l, \quad (\text{m}^2/\text{hr})' = (\text{m}^2/\text{hr})/A_\nu.$$

Then the quantities q_F/λ_{zh}, $\vec{l}$, l_1, ..., l_m change their numerical values and Eq. (12.11) assumes the form

$$\frac{q_F}{\lambda_m}\frac{A_\Phi}{A_l} = \varphi_4\left(l_F\cdot A_l,\ l_1\cdot A_l,\ l_2\cdot A_l,\ \ldots\right.$$

$$\left.\ldots,\ l_m\cdot A_l,\ \psi_x\cdot\psi_y,\ \nu\cdot A_\nu,\ a_{pm}\cdot A_\nu,\ u_0\cdot\frac{A_\nu}{A_l},\ \Delta t\cdot A_\Phi\right). \qquad (12.12)$$

The process of heat transmission from the body does not depend on the dimensions of the units used to measure the parameters and the variable quantities. Therefore Eq. (12.12) should have a special mathematical structure, which is conserved when the coefficients A_l, A_ϑ, A_ν are changed. We use this consideration to choose the quantities A_l, A_ϑ, and A_ν such that Eq. (12.12) retains the smallest number of independent terms, separated from one another by a comma.

We choose A_l, A_ϑ, and A_ν such that $A_l\cdot l_1 = 1$; $A_\vartheta\cdot\Delta t = 1$; $\nu\cdot A_\nu = 1$. Then

$$A_l = \frac{1}{l_1};\ A_\Phi = \frac{1}{\Delta t}\ ;\ A_\nu = \frac{1}{\nu}.$$

Substituting these values of A_l, A_ϑ, and A_ν in Eq. (12.12), we obtain

$$\frac{q_F\cdot l_1}{\lambda_m\Delta t} = \varphi_5\left(\frac{l_F}{l_1},\ 1,\ \frac{l_2}{l_1},\ \ldots,\ \frac{l_m}{l_1},\ \psi_x,\ \psi_y,\ 1,\ \frac{a_m}{\nu},\ \frac{u_0\cdot l_1}{\nu},\ 1\right).$$

In accordance with the π theorem we have obtained $n - k = 3 + m$ similarity criteria (not counting ψ_x and ψ_y, which enter into the initial dimensional equations). The last equation can be rewritten in different form

$$\frac{q_F\cdot l_1}{\lambda_m\cdot\Delta t} = \varphi_5\left(\frac{1}{l_1},\ \frac{l_2}{l_1},\ \ldots,\ \frac{l_m}{l_1},\ \psi_x,\ \psi_y,\ \frac{\nu}{a_{pm}},\ \frac{u_0\cdot l_1}{\nu}\right).$$

This example shows that transformation of equations to dimensionless form with the aid of dimensional analysis of the physical quantities makes it possible to obtain the same similarity criteria (or their equivalents) as the method of scale transformation.

Similarity Criteria for Heat-Transmission Processes of a Body Situated in a Forced Stream of Liquid

Let us consider the dimensionless quantities that are contained in Eqs. (12.5) and (12.7). The quantity $(q_F\cdot l_1)/(\lambda_{zh}\cdot\Delta t) = (\alpha\cdot l_1)/\lambda_{zh}$

is called the Nusselt number and is designated

$$\frac{\alpha \cdot l_1}{\lambda_{zh}} \equiv Nu.$$

This criterion of thermal similarity is nondefining, since it contains the sought quantity q_F or α. Its physical meaning is determined by the first equation of the system (12.4). The Nusselt number is the gradient of the dimensionless temperature in a stream of liquid on a heating surface. It recalls in its structure the Biot number. The difference between them lies in the physical meaning; in addition, the defining Biot number contains the coefficient of heat conductivity λ_F of the solid body, while the nondefining Nusselt number contains the heat conductivity λ_{zh} of the fluid.

The quantity $\vec{L} = \vec{l}_F/l_1$ is the dimensionless radius vector of any point on F, the quantities $L_2, \ldots, L_m$ are the dimensionless dimensions of the body itself, while ψ_x and ψ_y are the angles of attack of the body by the stream of liquid. Thus, $\vec{L}_F, L_2, \ldots, L_m, \psi_x, \psi_y$ are defining criteria of the geometrical similarity of the body-stream system.

The dimensionless quantity $u_0 l_1/a_{pzh}$ is called the Peclet number and is designated

$$Pe \equiv \frac{u_0 l_1}{a_{pzh}}.$$

This criterion of thermal similarity is defining, since it consists of the quantities u_0, l_1, and a_{pzh}, which are specified by the uniqueness conditions. The Peclet number can be regarded as a measure of the ratio of the convective heat transport to the heat transported by thermal conductivity in the liquid.

The quantity $u_0 l_1/\nu$ is called the Reynolds number and is designated

$$Re \equiv \frac{u_0 l_1}{\nu}.$$

This criterion of hydrodynamic similarity is also defining, since it consists of specified quantities u_0, l_1, and ν. The Reynolds number determines the kinematic similarity of streams, i.e., the similarity in the distributions of the velocity $\vec{u}$. It can be considered as a measure of the ratio of the inertia forces to the viscosity forces. The Reynolds number characterizes also the degree of stability of the stream relative to its perturbations. Experimental investigations of this question lead to the conclusion that the flow conditions of the liquid depend appreciably on the Reynolds number. It is therefore called a criterion for the flow conditions.

Let us rewrite Eq. (12.5) in the assumed notation:

$$Nu = f(L_F, L_2 \ldots, L_m, \psi_x, \psi_y, Pe, Re). \qquad (12.13)$$

Relation (12.13) contains all the similarity criteria of the process of heat transmission from the body under consideration. It is therefore a similarity condition, which describes a group of processes that are similar to one another.

The processes of heat transmission from a body around which a forced unbounded stream of homogeneous liquid flows are similar if the conditions of geometrical similarity of the body-stream system are satisfied, and if the conditions of kinematic and thermal similarity of the liquid streams are satisfied.

From the presented dimensionless differential equations we can see clearly the origin of each of the criteria. Thus, the Nusselt number has been obtained from the first initial equation of the system (12.4), which defines the sought quantity q_F (or α), the Peclet number comes from the Fourier-Ostrogradskiy equation, while the Reynolds number comes from the Helmholz equation.

Equations (12.13) are frequently represented in a different equivalent form:

$$Nu = f_3\left(L_F, L_2, \ldots, L_m, \psi_x, \psi_y, Re, \frac{Pe}{Re}\right).$$

The ratio of the Peclet and Reynolds numbers is

$$\frac{Pe}{Re} = \frac{u_\infty l_1}{a_{p\infty}} \cdot \frac{\nu}{u_\infty l_1} = \frac{\nu}{a_{p\infty}}.$$

The dimensionless quantity ν/a_{pzh} is called the Prandtl number and is designated

$$\frac{\nu}{a_{p\infty}} \equiv Pr.$$

Then the similarity equation for heat transmission from the body assumes the following form:

$$Nu = f_3(L_F, L_2, \ldots, L_m, \psi_x, \psi_y Re, Pr). \qquad (12.14)$$

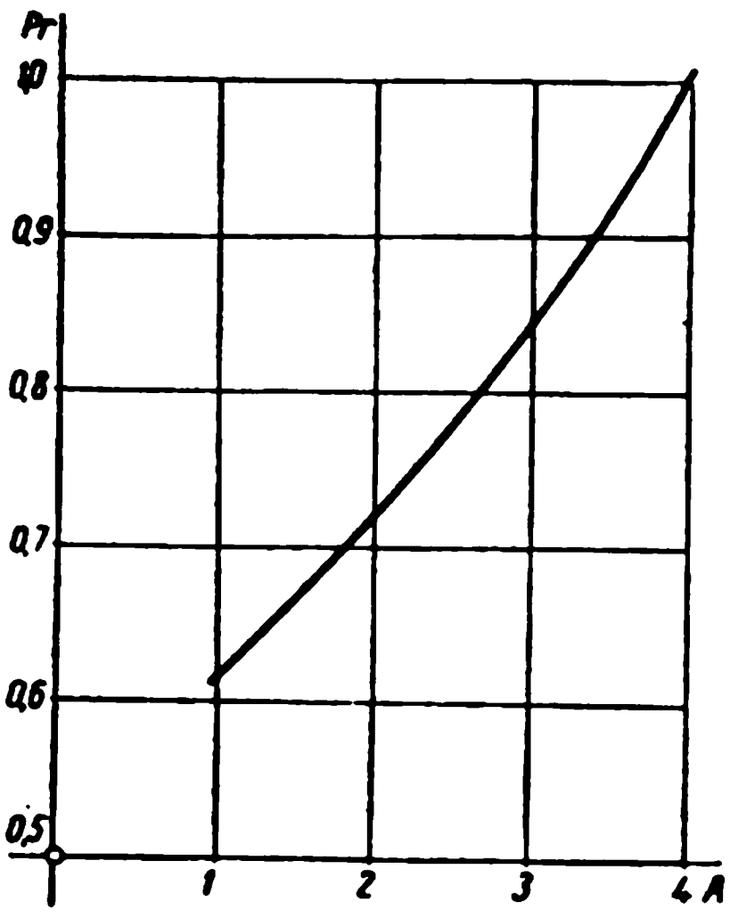

Fig. 39. Character of the dependence of the Prandtl number for gases on their atomic weight.

The Prandtl numbers cannot be obtained directly with the aid of an analysis of dimensionless differential equations and boundary conditions by transforming scales, and in this sense it is an "artificial" formation. The Prandtl number is a defining similarity criterion which consists of the quantities ν and a_{pzh}, specified by the uniqueness conditions, and can therefore be regarded as a dimensionless thermodynamic parameter, characterizing the ratio of the intensity of the molecular transports of momentum and heat.

The value of the Prandtl number, as well as of any physical parameter, is determined in the general case by the physical nature of the fluid, its temperature, and its pressure. For gases, however, the Prandtl number is determined principally by the atomic weight, and is practically independent of the temperature or the pressure over a wide

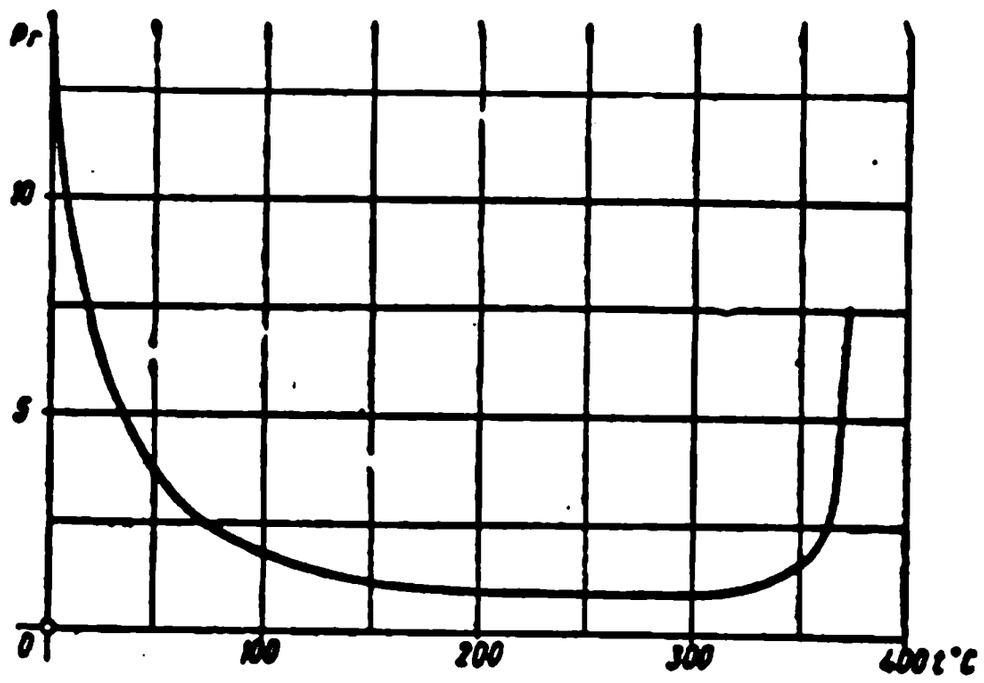

Fig. 40. Temperature dependence of the Prandtl number for water on the saturation line.

range of their variation. Thus, for example, the Prandtl number for air ranges from 0.72 to 0.74 as the temperature is increased from 0 to 1800°C. The dependence of this number for gases on their atomic weight is shown in Fig. 39. For gases Pr ~ 1. The Prandtl number for non-metallic liquids depends appreciably on the temperature (Fig. 40). The largest Prandtl number is possessed by oils, for which Pr ~ 10^2-10^3. The smallest value of this number (Pr ~ $10^{-2} \ll 1$) is possessed by molten metals. The character of the connection between the similarity criteria in Eqs. (12.13) or (12.14) is at present determined most reliably by experiment.

The Newton-Rikhman Law

An analysis of the similarity conditions for the heat-transmission processes of a body situated in a forced fluid stream can be considered also as a general proof of the heat-transmission law of Newton and Rikhman. Indeed, if we use to determine the heat rate q_F on the surface of a body any of the obtained dimensionless equations, for example Eq. (12.5), then we get

$$q_F = \frac{\lambda_\infty}{l_1} \cdot f\left(\frac{l_F}{l_1}, \frac{l_2}{l_1}, \ldots, \frac{l_m}{l_1}, \psi_x, \psi_y, \frac{u_0 l_1}{a_{p\infty}}, \frac{u_0 l_1}{\nu}\right) \times \times (t_F - t_\infty). \tag{12.15}$$

It follows from (12.15) that the heat rate q_F at any point $\vec{l}_F$ on

the surface F is directly proportional to the temperature drop $t_F - t_{zh}$. The proportionality coefficient

$$\frac{\lambda_{zh}}{l_1} \cdot f\left(\frac{l_F}{l_1}, \frac{l_2}{l_1}, \ldots, \frac{l_m}{l_1}, \psi_x, \psi_y, \frac{u_0 l_1}{a_{pzh}}, \frac{u_0 l_1}{\nu}\right) \equiv \alpha \qquad (12.16)$$

depends under these conditions neither on the temperatures t_F and t_{zh} nor on the rate q_F, for in accordance with the uniqueness conditions the fluid has constant thermodynamic parameters λ_{zh}, a_{pzh}, and ν, and therefore a special name (coefficient of heat transfer) and a special symbol (α) are used for the proportionality coefficient. Now we can write Eq. (12.15) in the form of the Newton-Rikhman law:

$$q_F = \alpha (t_F - t_{zh}).$$

It is thus obvious that this law is satisfied only if the thermodynamic parameters of the single-phase fluid are constant. If in a given temperature interval $t_{zh} < t < t_F$ the physical parameters of the fluid change noticeably in magnitude, this law loses its validity, since the direct proportionality between q_F and Δt is violated. Analogous conclusions can be drawn also as the result of an analysis of the dimensionless equation (12.7), obtained for the heat-transmission process when the heat rate q_F is specified on the surface F, and the temperature t_F is the unknown quantity.

It follows from Eq. (12.16) that the local heat-transmission coefficient α in the case of a forced stream of a homogeneous fluid flowing around a body depends on the heat conductivity λ_{zh}, the viscosity ν, the thermal diffusivity a_{pzh} of the fluid, its velocity u_0 remote from the body, the angles of attack ψ_x and ψ_y of the body by the stream, the shape and dimensions of the body l_1, l_2, ..., l_m, and on the position of the point $\vec{l}_F$ on its surface.

§13. HEAT TRANSMISSION FROM A CYLINDER SITUATED IN A FORCED STREAM OF FLUID

Similarity Conditions for Heat Transmission by the Cylinder

Let us consider the similarity conditions of the stationary processes of heat transmission from a long cylinder, which is in a trans-

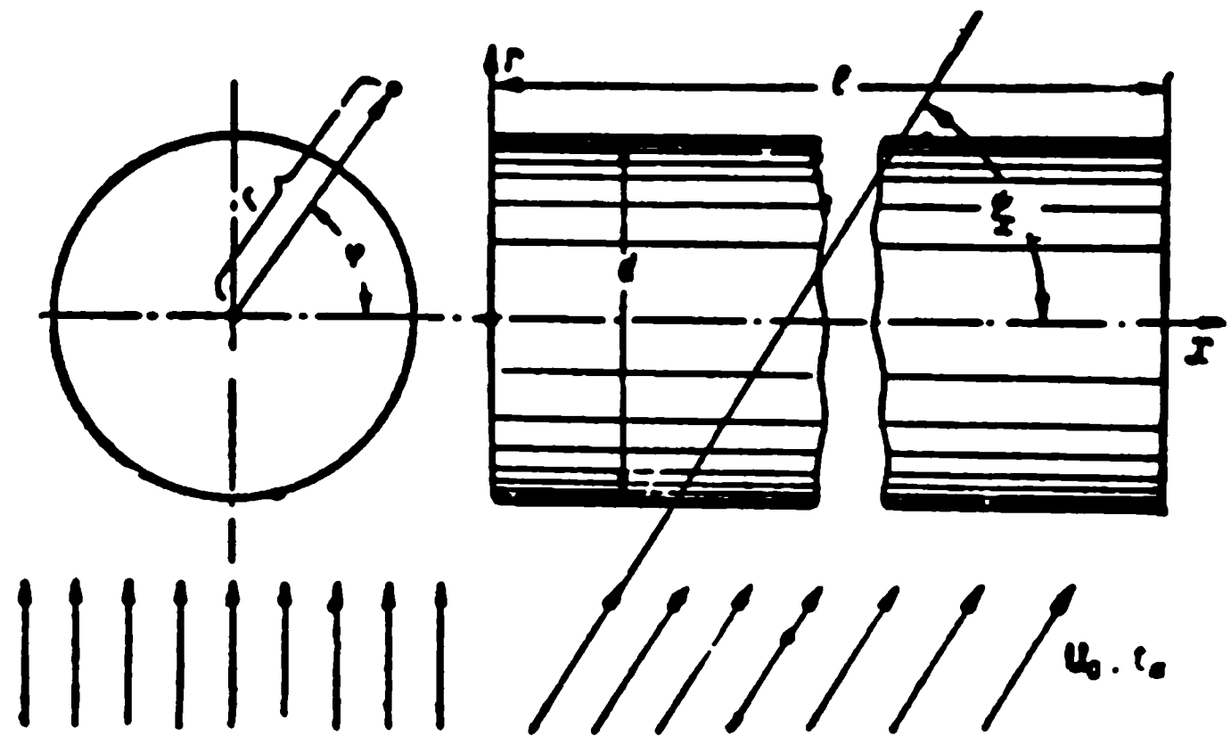

Fig. 41. Diagram showing flow of transverse fluid stream around a long cylinder.

verse force stream of a homogeneous fluid. Far from the cylinder the temperature and velocity of the fluid are, respectively, t_{zh} and $\vec{u}_0$ (Fig. 41). The similarity conditions for this process of heat transmission are a particular case of the more general conditions (see §12).

In this particular case the two angles of attack ψ_y and ψ_z are fixed, and only the angle ψ_x can vary; in addition, it is advantageous to choose as the characteristic dimension of the body l_1 the cylinder diameter d, since the cylinder is long ($l \gg d$) and the value of l does not exert any noticeable influence on the heat transmission. For the coordinates of the point $\vec{l}_F$ on the surface of the cylinder it is necessary to choose its radius d/2 and the angle φ. Under these conditions the similarity equation (12.14) assumes the form

$$\frac{q_F d}{\lambda_\infty \Delta t} = \frac{\alpha_\varphi d}{\lambda_\infty} = f_4\left(\varphi, \psi_x, \frac{u_0 d}{\nu}, \frac{\nu}{a_{p\infty}}\right) \qquad (13.1)$$

or

$$Nu_\varphi = f_4(\varphi, \psi_x, Re, Pr).$$

Equation (13.1) characterizes the similarity in the distribution of the local values of the heat-transmission coefficient at any point φ on the surface of the cylinder. To carry out the technical calculations it is necessary to have information on the heat-transmission coefficient $^{\varphi}\overline{\alpha}$ averaged over the perimeter. This quantity is best defined such that the Newton-Rikhman law is satisfied for the average values of $^{\varphi}\overline{q}_F$, $^{\varphi}\overline{\Delta t}$, and $^{\varphi}\overline{\alpha}$. Therefore

$$^{\varphi}\overline{\alpha} = \frac{^{\varphi}\overline{q}_F}{^{\varphi}\overline{\Delta t}},$$

where

$$^{\varphi}\overline{q}_F \equiv \frac{1}{2\pi}\int_0^{2\pi} q_F d\varphi \; ; \quad ^{\varphi}\overline{\Delta t} \equiv \frac{1}{2\pi}\int_0^{2\pi} \Delta t d\varphi.$$

If a constant temperature t_F = const is maintained on the surface of the cylinders, then $^{\varphi}\overline{\Delta t} = \Delta t$ and $^{\varphi}\overline{\alpha} = \overline{q}_F/\Delta t$. For another characteristic process

$$^{\varphi}\overline{q}_F = q_F \quad \text{and} \quad ^{\varphi}\overline{\alpha} = \frac{q_F}{^{\varphi}\overline{\Delta t}}.$$

For these cases of heat transmission, the similarity conditions simplify:

$$^{\varphi}\overline{Nu} = f_5(\psi_x, Re, Pr). \tag{13.2}$$

Thus, the Nusselt number averaged over the perimeter of the cylinder is determined by the angle of attack ψ_x and by the values of the Reynolds and Prandtl numbers.

Simplest Procedure for Experimental Investigation of Heat Transmission from a Cylinder

To determine the connection between the similarity criteria $^{\varphi}\overline{Nu}$, ψ_x, Re, and Pr in Eqs. (13.1) and (13.2) by experiment, it is necessary to measure all the quantities that are contained in these criteria. At the present time various methods for measuring these quanti-

ties are known [17, 41]. Figure 42 shows the schematic diagram of the simplest experimental setup for the measurement of the heat transfer from a cylinder to a transverse stream of air [41a].

A horizontal cylinder 8, on the surface of which the heat transmission is measured, is placed in a wind tunnel 4. The right end of this tunnel is connected to the intake opening of a fan 7, and the air enters the tunnel from its left end 2. The entrance to the tunnel has faired outlines in order for the distribution of the velocity in the air stream in the tunnel to be as uniform as possible. Used for the same purpose is the rectifying grid 5, which does not allow the air stream to twist in the fan and propagate against the stream. The temperature t_{zh} of the air incident on the cylinder 8 is measured with a thermometer 1, while the air velocity is measured with a Pitot tube 9, connected to a differential manometer 3. In order for the vibration of the ventilator 7 not to be transmitted to the cylinder 8, the aerodynamic tube 4 is connected to the ventilator by means of a rubber sleeve 6.

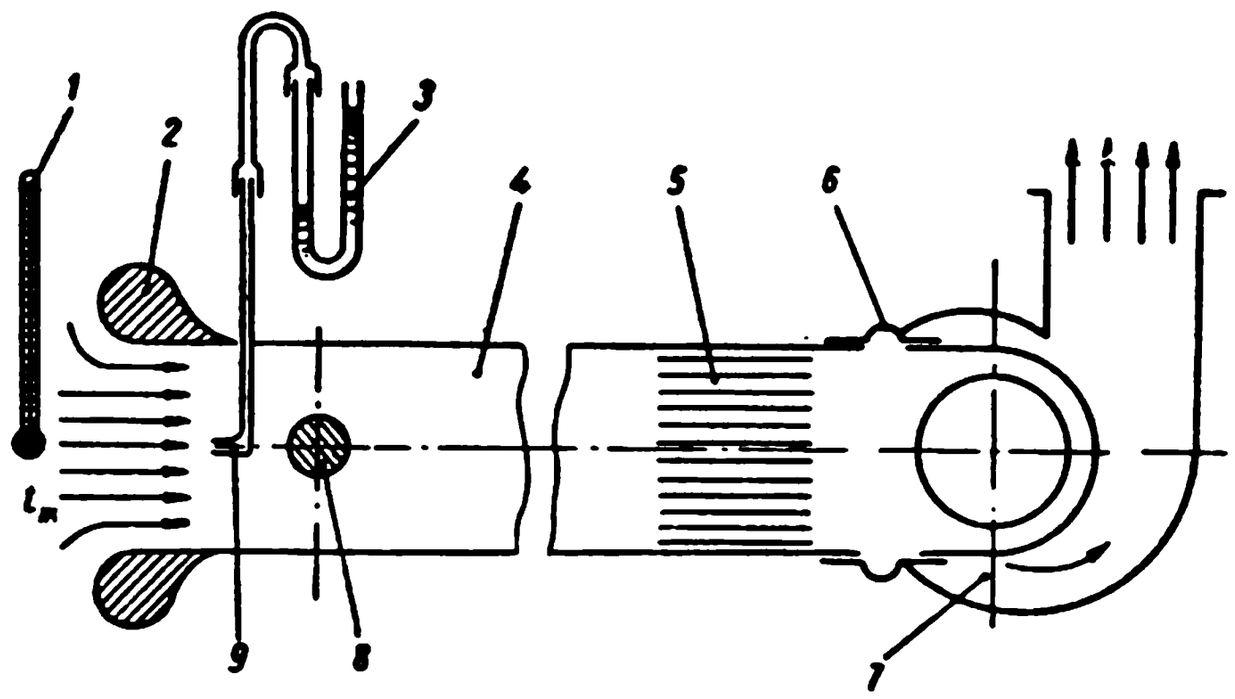

Fig. 42. Diagram of experimental setup for the measurement of heat transmission from a cylinder.

Figure 43 shows a diagram of the working portion of a cylinder, on the surface of which the heat transmission is measured. Cylinder 3 is made in the form of a copper tube, inside of which an electric

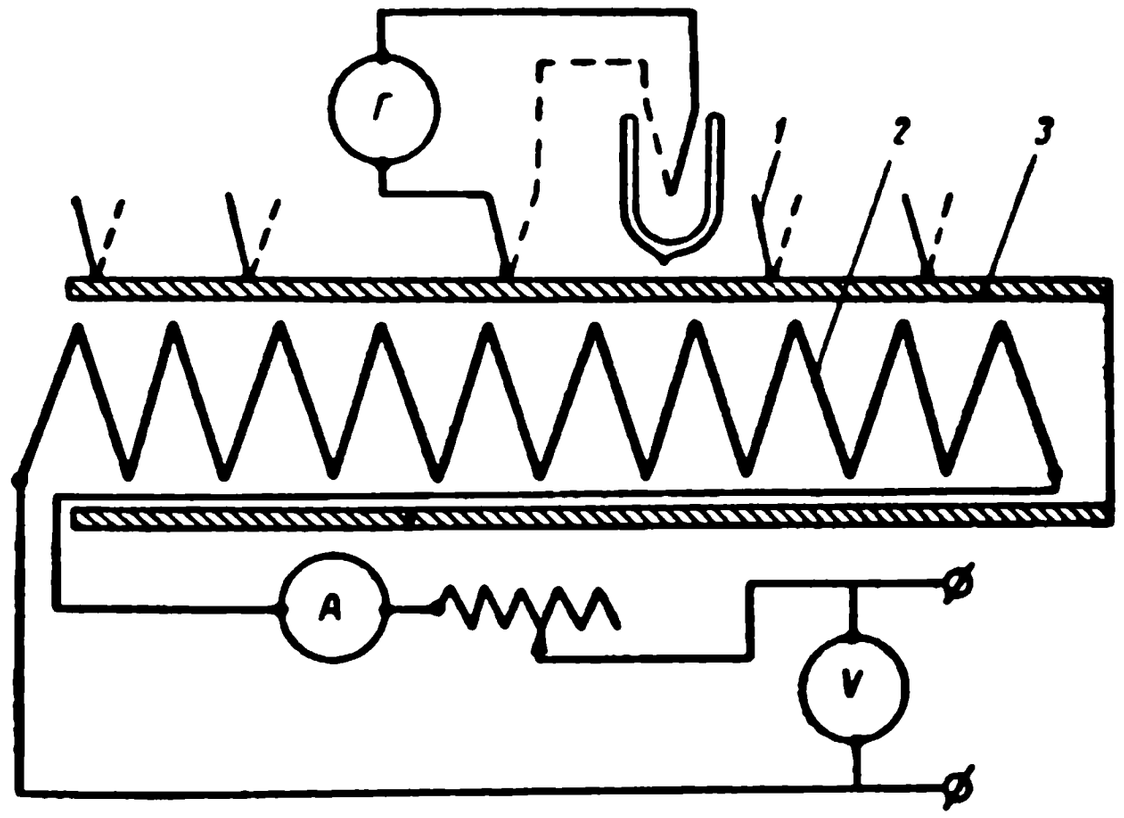

Fig. 43. Diagram of working portion of the experimental setup for the measurement of heat transmission from a cylinder.

heater 2 is placed. The heater is connected to an electricity source in the usual circuit, which makes it possible to measure the amount of heat released by the heater. The temperature on the heat-transmitting surface of the working portion of the cylinder is measured at different points by thermocouples 1. Such a scheme makes it possible to determine the parameters that enter into the similarity equation. This scheme should be regarded as a very simple example giving some idea of the experimental procedure of determining the heat transmission of the cylinder. More highly perfected schemes and also a procedure for the processing of the measurement results are detailed in [17, 28, 41a].

Principal Results of Measurements of Heat Transmission from a Cylinder and Computation Formulas [17]

In the experiments, the fluids employed were water, transformer oil, and air. The results of the measurements were represented in the form of the following equations:

for $Re_f = 10\text{-}10^3$

$$\overline{Nu}_f = 0{,}59 Re_f^{0.47} \cdot Pr_f^{0.36} \cdot \left(\frac{Pr_f}{Pr_w}\right)^{0.25} \cdot \varepsilon(\psi_z); \qquad (13.3)$$

for $Re_f = 10^3\text{-}2 \cdot 10^5$

$$\bullet\overline{Nu}_f = 0{,}21 Re_f^{0.62} \cdot Pr_f^{0.38} \cdot \left(\frac{Pr_f}{Pr_w}\right)^{0.25} \cdot \varepsilon(\psi_x). \qquad (13.3a)$$

For air these relationships simplify to the following form:

$$\left.\begin{aligned} &\text{for } Re_f = 10 \div 10^3 \\ &\qquad \bullet\overline{Nu}_f = 0{,}52 Re_f^{0.47} \cdot \varepsilon(\psi_x); \\ &\text{for } Re_f = 10^3 \div 2 \cdot 10^5 \\ &\qquad \bullet\overline{Nu}_f = 0{,}18 Re_f^{0.62} \cdot \varepsilon(\psi_x). \end{aligned}\right\} \qquad (13.4)$$

The influence of the angle of attack ψ_x on the average heat transmission from the cylinder is shown in Fig. 44 [46]. The dimensionless formulas (13.3) and (13.4) are similarity equations, since they include all the processes of heat transmission from cylinders, similar to those investigated. These formulas are valid for convective heat transmission without change in the aggregate state of the various fluids, which have Prandtl numbers from 0.6 to 800. The value of the Prandtl number for the investigated fluids decreases with increasing temperature, i.e., $(\partial Pr/\partial T)_p \leq 0$. Consequently, in the region of the thermodynamic crisis of the fluids, where $(\partial Pr/\partial T)_p > 0$, these equations must be checked.

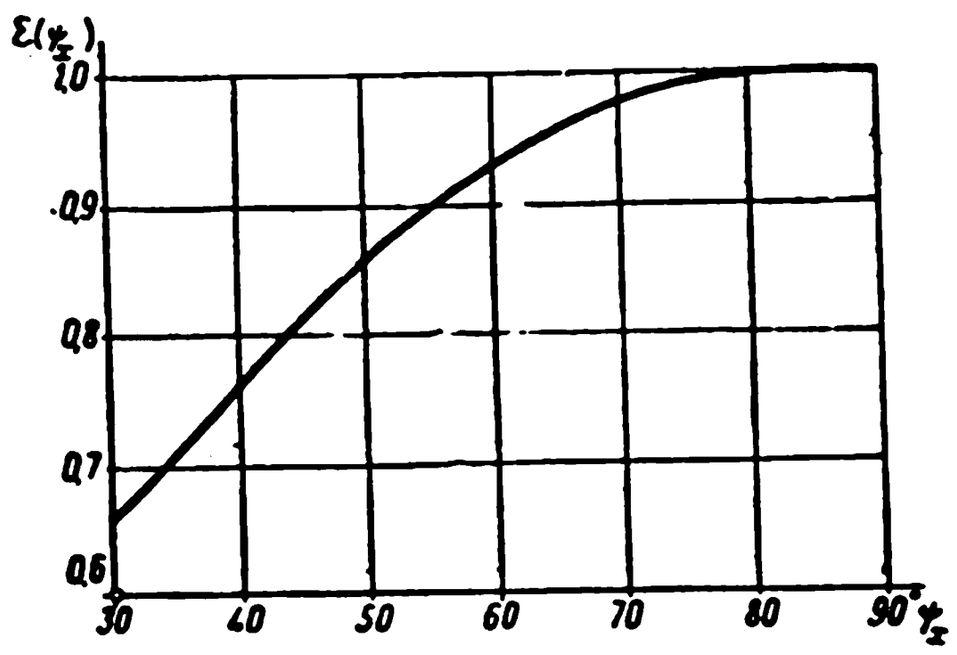

Fig. 44. Correction coefficient that takes into account the influence of the angle of attack on the average heat transmission from the cylinder.

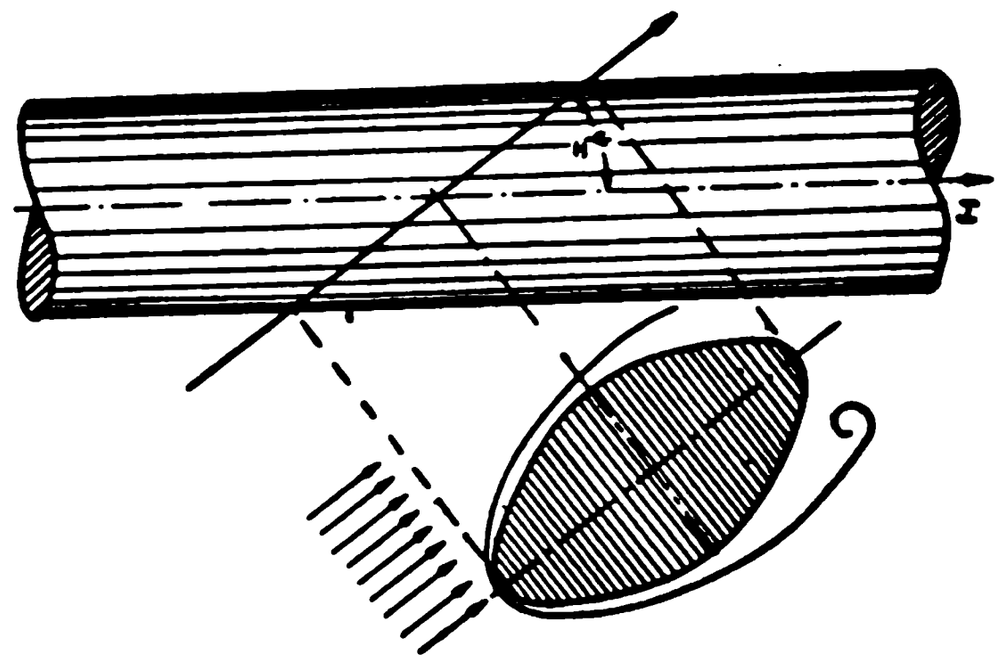

Fig. 45. Diagram of flow past a round cylinder at an acute angle of attack.

The experiments were carried out with air speeds which did not exceed 1/3 the speed of sound c. Consequently, these formulas are valid

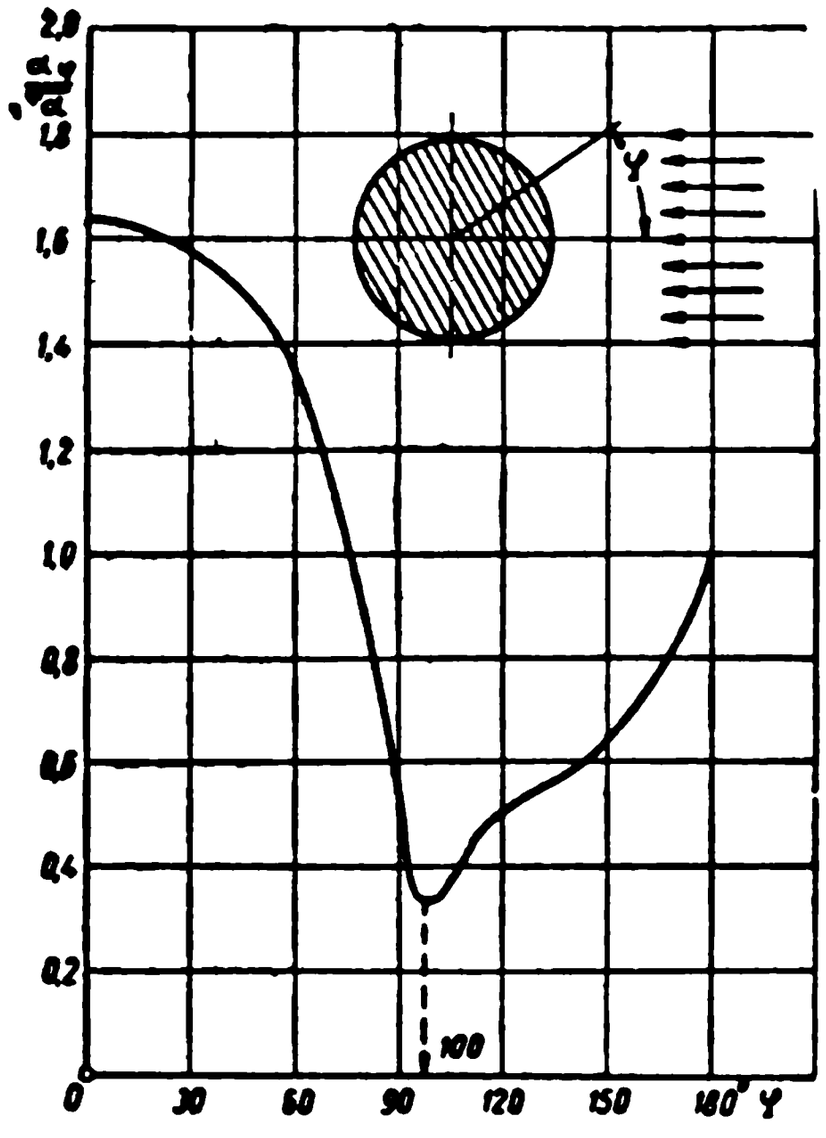

Fig. 46. Character of the distribution of the local coefficient of heat transmission over the circumference of a cylinder.

for $u_0/c \leq 1/3$. The quantity Pr_f/Pr_w has been introduced [35b] in order to take into account the dependence of the thermodynamic parameters of the fluid on the temperature. The indices w and f refer the thermodynamic parameters in the similarity criteria, respectively, either to the temperature t_F of the cylinder wall, or to the temperature t_{zh} of the fluid far from the cylinder. Therefore the quantity λ_{zhf} in the Nusselt number denotes the coefficient of heat conductivity of the fluid chosen at a temperature t_{zh}, while λ_{zhw} denotes the same coefficient but for t_F.

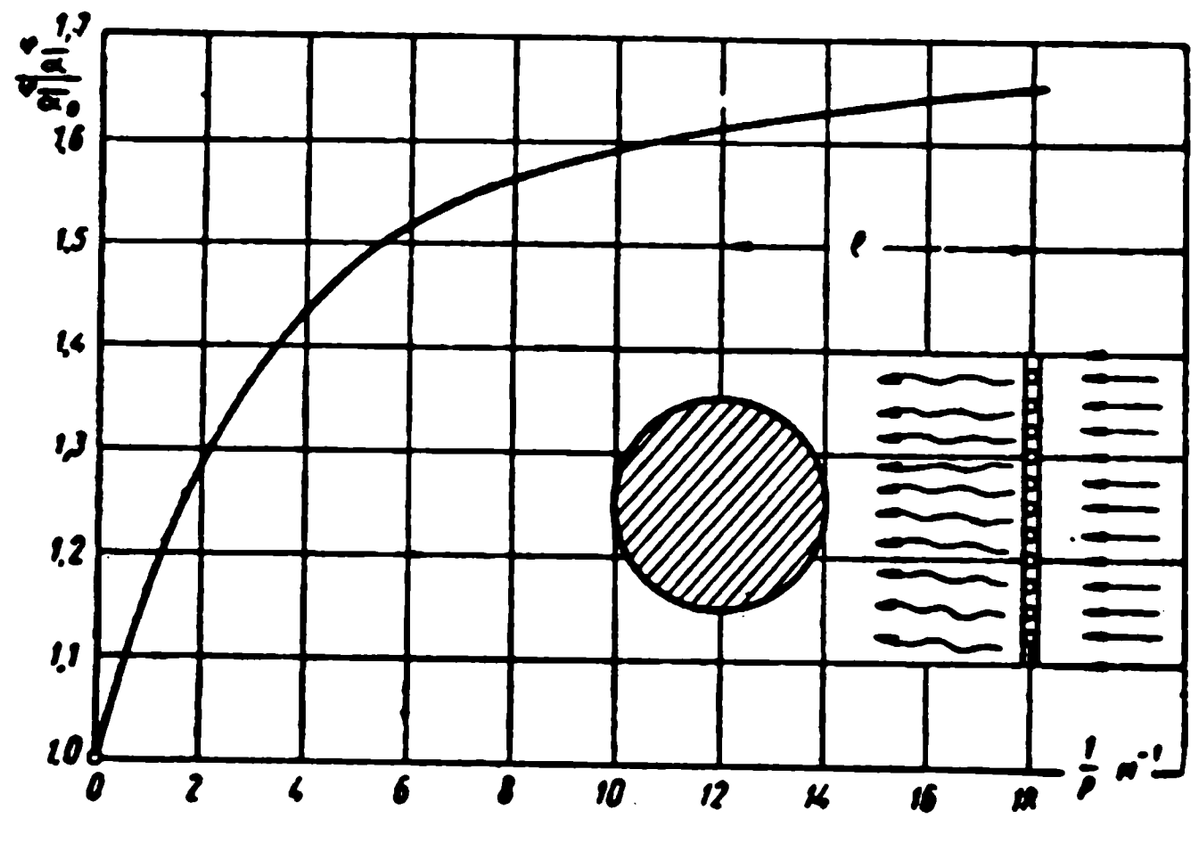

Fig. 47. Character of influence of the degree of turbulence of the incoming stream on the average heat transmission of a cylinder.

In carrying out the experiments, the turbulence of the incoming stream of fluid was reduced to a minimum with the aid of a quieting grid installed in the stream ahead of the cylinder. Formulas (13.3)

and (13.4) interpolate the measurement results with accuracy ±5%.

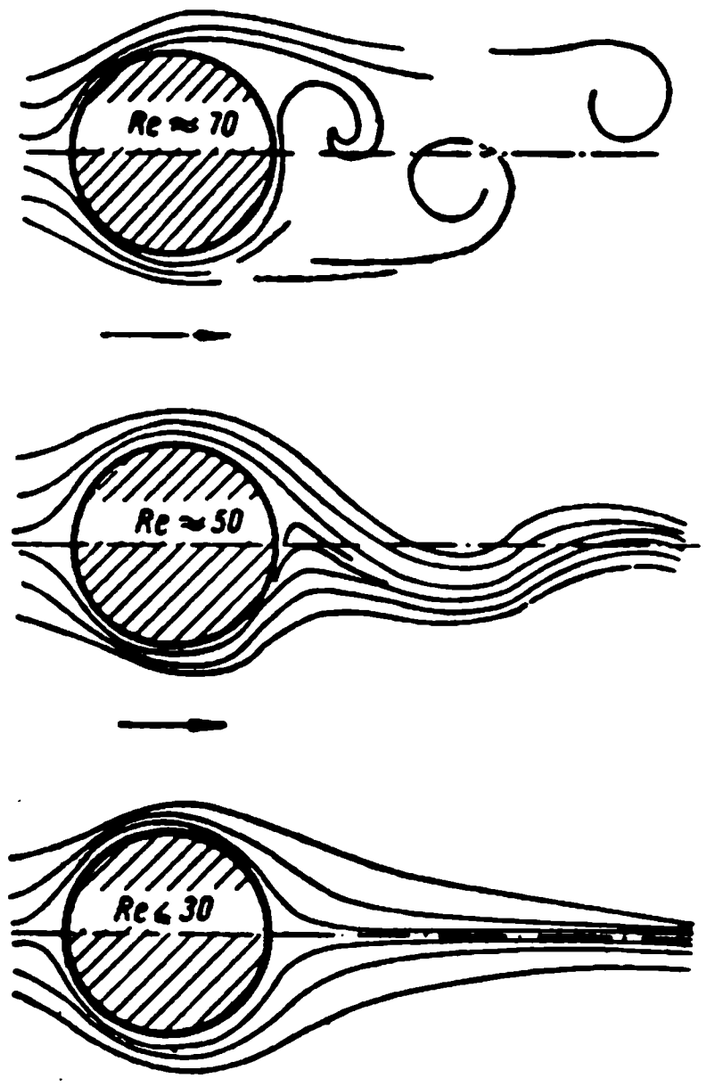

Fig. 48. Character of flow around a cylinder by a stream of fluid at different values of the Reynolds number.

Let us consider the influence of different factors on heat transmission from the cylinder. For this purpose we obtain from Eqs. (13.3) and (13.4) the value of $\varphi_{\bar{\alpha}}$ and represent it in the form of a dimensional dependence on the defining quantities. After transformations we obtain

$$\varphi_{\bar{\alpha}} \sim \frac{\mu_0^{(0.47+0.62)} \lambda_{ж\,w}^{0.25} \lambda_{ж\,f}^{0.37} \gamma_{ж\,f}^{(0.22+0.37)} \gamma_{ж\,w}^{0.25} \mu_{ж\,f}^{0.16+0.03} C_{p\,ж}^{0.63}}{d^{0.53+0.38} \mu_{ж\,w}^{0.25} C_{p\,ж\,w}^{0.25}}.$$

This formula shows clearly how the various parameters influence the average value of the heat-transfer coefficient. The influence of the angle of attack ψ_x on $\varphi_{\bar{\alpha}}$ is shown in Fig. 44. It is seen from this figure that an increase in the angle of attack from 30 to 90° increases the heat transfer from 60 to 100% of its value in the case of transverse flow. This is explained by the fact that when $\psi_x < \pi/2$ the stream flows around not a round cylinder, but an oval one. This leads to a tightening of the detachment of the boundary layer (Fig. 45). The development and detachment of the boundary layer on the surface of cylinder during transverse flow leads to a rather unique distribution of the local value of the coefficient of heat transmission over the circumference of the cylinder. This distribution is shown in Fig. 46 [28], where it is seen that on the frontal point of the cylinder at $\varphi = 90^\circ$ the largest value of α_φ corresponds to the smallest thickness of the boundary layer. With increasing φ the thickness of the layer increases, and the value of α_φ decreases sharply. The heat-transmission coefficient reaches a minimum value for $\varphi \approx 100^\circ$, where the thickness of the

boundary layer is largest. For $\varphi > 100^{\circ}$ the boundary layer is detached from the cylinder and its stern region becomes filled with a vortical stream, so that the heat transmission increases here. Figure 47 shows the character of the influence of the degree of turbulence of the incoming stream on the average heat transmission of the cylinder [24]. As a measure of the degree of turbulence use is made here of the reciprocal of the distance between the cylinder and the turbulizing grid. It is seen from the figure that the average value of the heat-transmission coefficient of the cylinder increases noticeably with increasing intensity of disturbance of the incoming stream, i.e., as the turbulizing grid is brought closer to the cylinder.

Figure 48 shows the character of flow around a cylinder by a fluid stream with different values of Reynolds number. With increasing value of this number the intensity of formation of vortices in the stern region of the cylinder increases, and the thickness of the boundary layer on the frontal part decreases. Consequently, the Nusselt number increases with increasing Reynolds number.

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[List of Transliterated Symbols]

440 ж = zh = zhidkost' = liquid

455 Г = G = generator = generator

Chapter 7

HEAT TRANSMISSION OF A CYLINDER IN THE CASE OF EXTERNAL NATURAL CONVECTION OF A FLUID

§14. SIMILARITY CONDITIONS FOR HEAT-TRANSMISSION PROCESSES OF A BODY WITH ARBITRARY SHAPE IN THE CASE OF EXTERNAL NATURAL CONVECTION

Setup and Mathematical Formulation of the Problem

Natural convection is defined as the motion of a fluid caused by the fact that its density ρ is different at different points of the

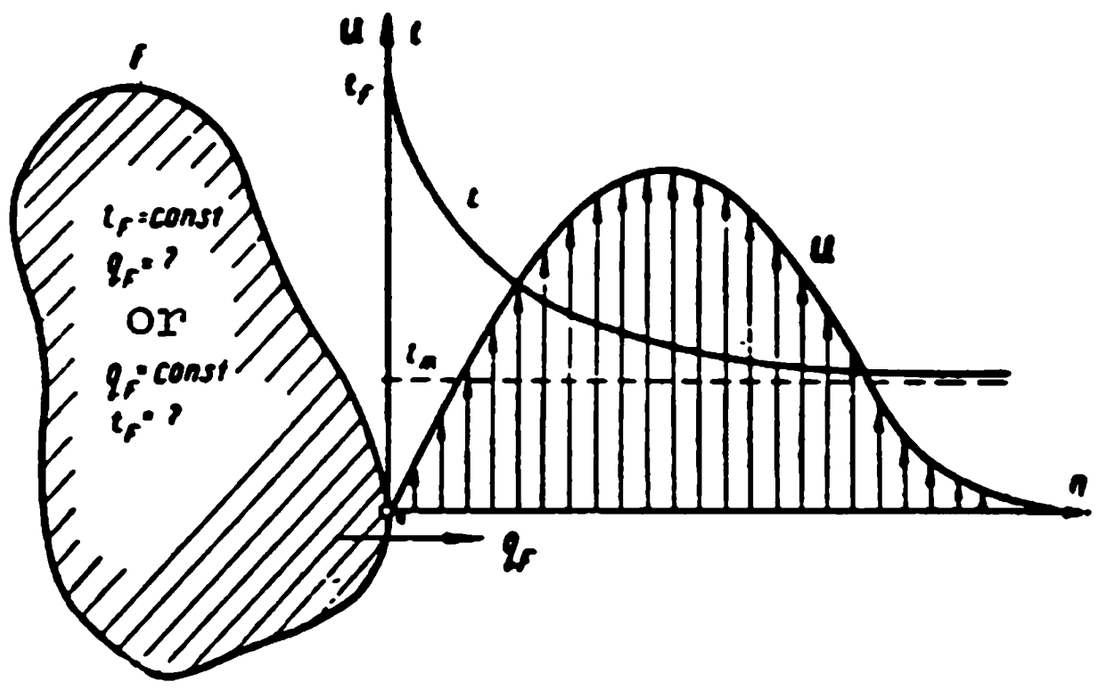

Fig. 49. Flow of a natural fluid stream around a body.

volume. This difference $\Delta\rho$ is due to the dependence of the density on the temperature and to the presence of a nonuniform field $t(\vec{r}, \tau)$ in the fluid. Let us consider the similarity conditions for two characteristic heat-transmission processes $t_F = \text{const}$ and $q_F = \text{const}$ for the case of external natural convection of a fluid in an unbounded volume around a certain body, the dimensions and shape of which are known (Fig. 49). Away from the body the fluid has a temperature $t_{zh} < t_F$ and a velocity $\vec{u}|_{n\to\infty} \to 0$.

Let us formulate the uniqueness conditions for the first heat-

transmission process, when the sought quantity is the heat rate q_F.

The geometrical conditions specify the shape and dimensions of the body in the form of an equation for its surface

$$F(l_F, l_1, l_2, \ldots, l_m) = 0,$$

and also the orientation of this body in the gravitational field. Let the ordinate axis y be directed vertically downward.

The physical conditions specify the constant values of the heat conductivity λ_{zh}, of the thermal diffusivity a_{pzh}, and of the viscosity μ of the fluid, and also indicate that there are no heat sources in the stream $q_V = 0$. The fluid density is specified in the form of a weak linear function of the temperature

$$\varrho(t) = \varrho_\infty[1 - \beta_\varrho(t - t_\infty)]. \qquad (14.1)$$

Here $\rho_{zh} = \rho_{zh}(t_{zh})$ is the density of the fluid away from the body at a temperature t_{zh}, and β_ρ is the coefficient of volume expansion (deg^{-1}).

The condition of weak dependence of ρ on t is specified in the form of the inequality

$$\beta_0(t_F - t_\infty) \ll 1.$$

Because of this condition, the quantity ρ should be regarded as constant if it is a factor or a divisor. If there is a difference $\Delta\rho$ between the values at two points in the stream, the dependence of ρ on the temperature must be taken into account.

The time conditions indicate that the heat-transmission process is stationary.

The boundary conditions determine the distribution of the temperature and of the velocity on the solid boundaries of the stream, i.e., on the surface F

$$u|_{n=0} = 0; \quad t|_{n=0} = t_F = \text{const},$$

and on the liquid boundaries of the stream

$$u|_{n\to\infty}\to 0;\quad t|_{n\to\infty}\to t_\infty<t_F.$$

The sought quantity is the heat rate q_F or the local value of the heat-transmission coefficient at any point on the surface F.

Using the assumption that the fluid adheres to the surface of the body and that there is no radiant heat exchange, we can write for the heat rate q_F the following expression:

$$q_F=-\lambda_\infty\frac{\partial t}{\partial n}\Big|_{n=+0}.$$

Consequently, to determine q_F it is necessary to know the distribution of the temperature $\underline{t}$ in the natural fluid stream around the body. The distribution of the temperature $\underline{t}$ obeys the Fourier-Ostrogradskiy equation

$$(\mathbf{u},\operatorname{grad}t)=a_{p,\infty}\nabla^2 t.$$

The velocity field $\vec{u}$ in the fluid stream is described by the Navier-Stokes equations

$$\varrho(\mathbf{u},\operatorname{grad})\mathbf{u}=\varrho\mathbf{g}-\operatorname{grad}p+\mu\nabla^2\mathbf{u},$$

where

$$\varrho=\varrho_\infty[1-\beta_0(t-t_\infty)].$$

Far from the body the fluid is stationary, and therefore the Navier-Stokes equation assumes here the form

$$\varrho_\infty\mathbf{g}-\operatorname{grad}p_\infty=0.$$

Subtracting the last equation from the preceding one and dividing the results by the density ρ, we obtain

$$(\mathbf{u},\operatorname{grad})\mathbf{u}=\left(\frac{\varrho-\varrho_\infty}{\varrho}\right)\mathbf{g}-\frac{1}{\varrho}\operatorname{grad}(p-p_\infty)+\nu\nabla^2\mathbf{u}.$$

Further

$$\frac{\varrho-\varrho_\infty}{\varrho}\approx-\beta_0(t-t_\infty),$$

therefore

$$(\mathbf{u},\operatorname{grad})\mathbf{u}=-\beta_0(t-t_\infty)\mathbf{g}-\frac{1}{\varrho}\operatorname{grad}(p-p_\infty)+\nu\nabla^2\mathbf{u}.$$

The last equation contains a term $-\beta_\rho(t - t_{zh})g$, which expresses the action of the lifting force that arises in the hotter layer of the liquid.

Taking the curl of the obtained expression, we obtain the Helmholz equation for weak natural convection

$$\mathrm{rot}\,[(\mathbf{u}, \mathrm{grad})\,\mathbf{u} + \beta_0 \vartheta\, \mathbf{g} - \nu\nabla^2 \mathbf{u}] = 0.$$

We further introduce the excess temperature $\vartheta \equiv t - t_{zh}$. Then the mathematical formulation of the problem concerning the first characteristic heat-transmission process for an external weak natural convection of the fluid in an unbounded volume assumes the form

$$\left.\begin{gathered} q_F = -\lambda_\infty \frac{\partial \vartheta}{\partial n}\Big|_{n=+0};\ (\mathbf{u}, \mathrm{grad}\,\vartheta) + a_{p\infty}\nabla^2\vartheta = 0; \\ \mathrm{rot}\,[(\mathbf{u}, \mathrm{grad})\,\mathbf{u} + \beta_0\vartheta\,\mathbf{g} - \nu\nabla^2\mathbf{u}] = 0; \\ \mathbf{u}|_{n=0} = 0;\ \vartheta|_{n=0} = t_F - t_\infty \equiv \Delta t; \\ \mathbf{u}|_{n\to\infty} \to 0;\ \vartheta|_{n\to\infty} \to 0; \\ F(l_F, l_1, l_2, \ldots, l_m) = 0. \end{gathered}\right\} \qquad (14.2)$$

The system of equations is closed, since it contains five unknowns q_F, ϑ, u_x, u_y, u_z and consists of five equations. The quantities λ_{zh}, a_{pzh}, β_ρ, Δt, l_1, l_2, ..., l_m are specified.

The mathematical formulation of the problem of the second characteristic heat-transmission process in natural convection, when a constant heat rate q_F = const is maintained on the surface F of the body, consists of the following system of equations

$$\left.\begin{gathered} \vartheta_F = \vartheta|_{n=0};\ (\mathbf{u}, \mathrm{grad}\,\vartheta) - a_{p\infty}\nabla^2\vartheta = 0; \\ \mathrm{rot}\,[(\mathbf{u}, \mathrm{grad})\,\mathbf{u} + \beta_0\vartheta\,\mathbf{g} - \nu\nabla^2\mathbf{u}] = 0; \\ \mathbf{u}|_{n=0} = 0;\ \ q_F = -\lambda_\infty \frac{\partial\vartheta}{\partial n}\Big|_{n=+0}; \\ \mathbf{u}|_{n\to\infty} \to 0;\ \vartheta|_{n\to\infty} \to 0; \\ F(l_F, l_1, l_2, \ldots, l_m) = 0. \end{gathered}\right\} \qquad (14.3)$$

The system (14.3) is also closed, since it contains five unknowns ϑ_F, ϑ, u_x, u_y, u_z and consists of five equations. The quantities λ_{zh}, a_{pzh}, β_ρ, q_F, l_1, l_2, ..., l_m are specified here. It is necessary to

Reduction of the Equations to Dimensionless Form

To reduce Eqs. (14.2) and (14.3) to dimensionless form we use the method of scale transformation and introduce with the aid of the quantities Δt and l_1 the dimensionless temperature $\Theta \equiv \vartheta/\Delta t$ and the dimensionless geometrical characteristics of the system

$$\mathbf{L} \equiv \frac{\mathbf{l}}{l_1}, \quad \mathbf{L}_F \equiv \frac{\mathbf{l}_F}{l_1}, \quad N \equiv \frac{n}{l_1}.$$

Consequently,

$$\vartheta = \Delta t \Theta; \quad \mathbf{l} = l_1 \mathbf{L}; \quad \mathbf{l}_F = l_1 \mathbf{L}_F; \quad n = l_1 N.$$

Substituting these relations into (14.2) and transforming the latter, we obtain a dimensionless mathematical description of the first heat-transfer process:

$$\left.\begin{gathered} \frac{q_F l_1}{\Delta t \lambda_{\text{ж}}} = -\frac{\partial \Theta}{\partial N}\Big|_{N=+0}; \quad (\mathbf{Pe}, \operatorname{Grad}\Theta) - \nabla^2\Theta = 0; \\ \operatorname{Rot}[(\mathbf{Pe}, \operatorname{Grad})]\mathbf{Pe} + \Theta\frac{g\beta_0\Delta t l_1^3}{\nu^2} \times \\ \times Pr^2\mathbf{j} - Pr\nabla^2\mathbf{Pe} = 0; \\ \mathbf{Pe}|_{N=0} = 0; \quad \Theta|_{N=0} = 1; \quad \mathbf{Pe}|_{N\to\infty} \to 0; \\ \Theta|_{N\to\infty} \to 0; \\ F(\mathbf{L}_F, L_2, \ldots, L_m) = 0, \end{gathered}\right\} \qquad (14.4)$$

where $\overrightarrow{\mathbf{Pe}} \equiv \vec{u} l_1/a_{\text{pzh}}$ is the local Peclet number.

The solution of the problem (14.4) consists of the following criteria:

$$\left.\begin{aligned} \frac{u l_1}{a_{p\text{ж}}} &= \varphi_1\left(\mathbf{L}, L_2, \ldots, L_m, Pr, \frac{g\beta_0\Delta t l_1^3}{\nu^2}\right); \\ \frac{q_F l_1}{\lambda_{\text{ж}}\Delta t} &= \varphi_2\left(\mathbf{L}_F, L_2, \ldots, L_m, Pr, \frac{g\beta_0\Delta t l_1^3}{\nu^2}\right). \end{aligned}\right\} \qquad (14.5)$$

In order to transform Eqs. (14.3) to dimensionless form, we introduce by way of a temperature scale an arbitrary quantity ϑ_0. Then $\vartheta = \vartheta_0\Theta$. The scales of the remaining variables are left unchanged.

Reducing the system (14.3) to dimensionless form, we obtain

$$\frac{\vartheta_F}{\vartheta_0} = \Theta|_{N=0}; \quad (\mathbf{Pe}, \operatorname{Grad}\Theta) - \nabla^2\Theta = 0;$$

$$\operatorname{Rot}\left[(\mathbf{Pe}, \operatorname{Grad})\mathbf{Pe} + \frac{g\beta_0\vartheta_0 l_1^3}{\nu^2} \times\right.$$

$$\times \Theta Pr^2 \mathbf{j} - Pr\nabla^2 \mathbf{Pe}\Big] = 0;$$

$$\mathbf{Pe}|_{N=0} = 0;\ \frac{q_F l_1}{\lambda_{zh}\vartheta_0} = -\frac{\partial\Theta}{\partial N}\Big|_{N=+0}; \qquad (14.6)$$

$$\mathbf{Pe}|_{N=0} = 0;$$

$$\mathbf{Pe}|_{N\to\infty} \to 0;\ \Theta|_{N\to\infty} \to 0;\ F(L_F, L_2 \ldots, L_m) = 0.$$

The solution of the problem (14.6) consists of the following criteria:

$$\frac{\vartheta_F}{\vartheta_0} = \varphi_3\left(L_F, L_2, \ldots, L_m, Pr, \frac{g\beta_0\vartheta_0 l_1^3}{\nu^2}, \frac{q_F l_1}{\lambda_{zh}\vartheta_0}\right).$$

However, the temperature ϑ_F on the surface of the body cannot depend on the scale ϑ_0, which was chosen arbitrarily. Therefore the three dimensionless quantities ϑ_F/ϑ_0; $q_F l_1/\lambda_{zh}\vartheta_0$, and $g\beta_\rho\vartheta_0 l_1^3/\nu^2$ should be contained in the solution of the problem (14.6) in the form of two combinations, none of which contain ϑ_0. This requirement will be satisfied in each of the following variants of these combinations:

$$\frac{\frac{\vartheta_F}{\vartheta_0}}{\frac{q_F l_1}{\lambda_{zh}\vartheta_0}} = \frac{\vartheta_F \lambda_{zh}}{q_F l_1};\ \frac{\frac{g\beta_0\vartheta_0 l_1^3}{\nu^2}}{\frac{q_F l_1}{\lambda_{zh}\vartheta_0}} = \frac{g\beta_0 l_1^4 q_F}{\nu^2\lambda_{zh}};$$

$$\frac{\frac{\vartheta_F}{\vartheta_0}}{\frac{g\beta_0\vartheta_0 l_1^3}{\nu^2}} = \frac{g\beta_0\vartheta_F l_1^3}{\nu^2};\ \frac{\frac{g\beta_0\vartheta_0 l_1^3}{\nu^2}}{\frac{q_F l_1}{\lambda_{zh}\vartheta_0}} = \frac{g\beta_0 l_1^4 q_F}{\nu^2\lambda_{zh}}.$$

Therefore the dimensionless solution of the problem (14.6) can be written, accurate to an unknown function, in the form of the following two equations

$$\frac{\vartheta_F\lambda_{zh}}{q_F l_1} = \varphi_4\left(L_F, L_2, \ldots, L_m, Pr, \frac{g\beta_0 l_1^4 q_F}{\nu^2\lambda_{zh}}\right) \qquad (14.7)$$

or

$$\frac{g\beta_0\vartheta_F l_1^3}{\nu^2} = \varphi_5\left(L_F, L_2, \ldots, L_m, Pr, \frac{g\beta_0 l_1^4 q_F}{\nu^2\lambda_{zh}}\right). \qquad (14.8)$$

Equations (14.5), (14.7), and (14.8) can be readily obtained also with the aid of dimensional analysis.

Violation of the Newton-Rikhman Law in Natural Convection

Natural convection arises under the influence of the lifting forces caused by the fact that the density of a liquid that is unevenly

heated at different points is also uneven. The dependence of the density on the temperature $\rho(t)$ even if all other thermodynamic parameters are constant leads to a violation of the Newton-Rikhman law. Indeed, from the second equation of (14.5) we obtain an expression for the heat rate on the surface of the body

$$q_F = \frac{\lambda_\infty}{l_1} \varphi\left(L_F, L_2, \ldots, L_m, Pr, \frac{g\beta_0 \Delta t l_1^3}{\nu^2}\right) \Delta t. \qquad (14.9)$$

It follows from Eq. (14.9) that the local heat rate is not directly proportional to the temperature difference $\Delta t \equiv t_F - t_{zh}$, since the heat-transmission coefficient

$$\alpha \equiv \frac{\lambda_\infty}{l_1} \varphi\left(L_F, L_2, \ldots, L_m, Pr, \frac{g\beta_0 \Delta t l_1^3}{\nu^2}\right)$$

depends on Δt. An analogous conclusion is reached also as a result of an analysis of the second characteristic process of heat transmission (q_F = const; t_F = ?). In this case the dependence of q_F on the local temperature drop $\vartheta_F \equiv t_F - t_{zh}$ can be obtained from Eq. (14.7):

$$q_F = \frac{\lambda_\infty}{l_1} \varphi_6\left(L_F, L_2, \ldots, L_m, Pr, \frac{g\beta_0 l_1^4 q_F}{\nu^2 \lambda_\infty}\right) \vartheta_F. \qquad (14.10)$$

It follows from this equation that the heat-transmission coefficient

$$\alpha \equiv \frac{\lambda_\infty}{l_1} \varphi_6\left(L_F, L_2, \ldots, L_m, Pr, \frac{g\beta_0 l_1^4 q_F}{\nu^2 \lambda_\infty}\right)$$

depends on the heat rate q_F. Consequently, there is likewise no direct connection between q_F and ϑ_F here.

Similarity Criteria of the Heat-Transmission Process in the Case of Natural Convection

Let us consider the dimensionless quantities that are contained in Eqs. (14.5), (14.7), and (14.8). The criteria $q_F l_1/\lambda_{zh}\Delta t \equiv Nu$; $\vec{L}_F$, L_2, ..., L_m, and Pr were considered in §12. The quantity $g\beta_\rho \Delta t l_1^3/\nu^2$ in Eq. (14.5) is called the Grashof number and is designated

$$\frac{g\beta_0 \Delta t l_1^3}{\nu^2} \equiv Gr.$$

This criterion of thermal similarity for the first characteristic

heat-transmission process (q_F = ?, t_F = const) is <u>defining</u>, since it consists of quantities that are specified by the uniqueness conditions. The Grashof number can be regarded as a measure of the ratio of the lifting forces due to the density difference $\Delta\rho$ at different points of a nonisothermal stream, to the viscosity forces.

The criterion $\vartheta_F\lambda_{zh}/q_F l_1$ in Eq. (14.7) is the reciprocal of the Nusselt number, which contains as an unknown quantity the value of the local temperature difference

$$\frac{\vartheta_F\lambda_{zh}}{q_F l_1} = \frac{1}{Nu}.$$

The dimensionless quantity $g\beta_\rho l_1^4 q_F/\nu^2\lambda_{zh}$ is called the Rayleigh number and is designated

$$\frac{g\beta_\rho l_1^4 q_F}{\nu^2\lambda_{zh}} = Ry.$$

This criterion of thermal similarity for the second characteristic heat-transmission process is defining, since it is made up of specified quantities.

It is interesting to note that the Grashof number in Eq. (14.8) for the second characteristic process is nondefining, since it contains the unknown quantity ϑ_F (the local temperature difference). It is also interesting to note that the Peclet number and the Reynolds number in Eqs. (14.4) and (14.6) for the first and second characteristic heat-transfer processes in the case of natural convection cease to be defining, since they contain an unknown quantity, namely the velocity vector $\vec{u}$, for which it is impossible to specify a single finite value.

In the assumed notation, Eqs. (14.5), (14.7), and (14.8) assume the following form:

for the first heat-transmission process (t_F = const; q_F = ?)

$$Nu = \varphi_2(L_F, L_2, \ldots, L_m, Pr, Gr); \qquad (14.11)$$

for the second process (q_F = const; t_F = ?)

$$Nu_\theta = \varphi_4(L_F, L_2, \ldots, L_m, Pr, Ry) \qquad (14.12)$$

or

$$Gr_\theta = \varphi_5(L_F, L_2, \ldots, L_m, Pr, Ry). \qquad (14.13)$$

From Eqs. (14.11)-(14.13) it follows that under conditions of natural convection, unlike the forced flow, the composition of the criteria of thermal similarity depends essentially on the character of the boundary conditions.

While the Grashof number defines thermal similarity for the first process, it becomes nondefining for a specified heat rate q_F, and the Rayleigh number is then included among the defining criteria.

Limiting Heat-Transmission Conditions in Natural Convection

Let us consider the similarity of heat-transmission processes for natural convection in the following three limiting cases.

Assume that the inertia forces in a natural stream of liquid are negligibly small compared with the viscosity and lifting forces.

Then

$$|(\mathbf{Pe}, \mathrm{Grad})\,\mathbf{Pe}| \ll |Pr\nabla^2 \mathbf{Pe}| + |\theta GrPr^2 \mathbf{j}|$$

and the dimensionless Helmholz equation (14.4) assumes the form

$$\mathrm{Rot}\,[\theta GrPr\,\mathbf{j} - \nabla^2 \mathbf{Pe}] = 0.$$

This dimensionless equation contains the Grashof and Prandtl numbers only in the form of their product. Therefore the solution of the problem (14.4) for the first heat-transmission process and for the limiting conditions under consideration assumes the following form:

$$Nu = \varphi_6(L_F, L_2, \ldots, L_m, GrPr). \qquad (14.14)$$

Most presently known experimental researches on the first heat-transmission process in natural convection of nonmetallic liquids having $Pr \geq 0.6$ have confirmed the correctness of the result obtained. The Grashof and Prandtl numbers are actually contained in the similar-

ity equation in the form of their product Gr·Pr. Consequently, these researches were carried out under conditions in which the inertia forces in the natural stream of liquid could be neglected.

For the second heat-transmission process (q_F = const; t_F = ?) the solution of the problem (14.6) can be written in the form of the following dimensionless equations:

or
$$\left.\begin{aligned} Nu_\Phi &= \varphi_7(L_F, L_2, \ldots, L_m, RyPr) \\ Gr_\Phi Pr &= \varphi_8(L_F, L_2, \ldots, L_m, RyPr). \end{aligned}\right\} \qquad (14.15)$$

As the second limiting conditions we consider a case when the viscosity forces can be neglected compared with the inertia forces and the lifting forces in the natural stream. Under these conditions the dimensionless Helmholz equation in the system of equations (14.4) simplifies to

$$\mathrm{Rot}\,[(\mathbf{Pe}, \mathrm{Grad})\,\mathbf{Pe} + \Theta GrPr^2 \mathbf{j}] = 0,$$

and the dimensionless solutions of the problems (14.4) and (14.6) can be written, accurate to unknown functions, in the form of the following relations:

for the first heat-transmission process

$$Nu = \varphi_9(L_F, L_2, \ldots, L_m, RyPr^2); \qquad (14.16)$$

for the second process

or
$$\left.\begin{aligned} Nu_\Phi &= \varphi_{10}(L_F, L_2, \ldots, L_m, RyPr^2) \\ Gr_\Phi Pr^2 &= \varphi_{11}(L_F, L_2, \ldots, L_m, RyPr^2). \end{aligned}\right\} \qquad (14.17)$$

As the third limiting condition we consider the natural flow of liquid in which the convective heat transport is negligibly small compared with the heat conductivity. Then the temperature distribution in the stream obeys the Laplace equation $\nabla^2\Theta = 0$, and the similarity equations degenerate to the following:

$$Nu = \varphi_{12}(L_F, L_2, \ldots, L_m). \qquad (14.18)$$

In some cases the form of the function φ_{12} in Eq. (14.18) can be de-

termined by calculation. Thus, for example, if the body is in the form of a sphere, then Eq. (14.18) becomes simplest:

$$Nu = 2.$$

In conclusion we note that the first limiting condition of heat transmission in natural convection is observed in gases and nonmetallic liquids, for which the Prandtl number has the order of unity and more: $Pr \geq 1$ [31]. The main change in temperature is concentrated in a relatively thin layer of fluid near the wall. It is precisely in this layer that the lifting forces and the viscosity forces predominate over the inertia forces, i.e., the first limiting condition for heat transmission in natural convection is observed.

For metallic liquids, as indicated above, the Prandtl number is small compared with unity, $Pr \ll 1$. Owing to the high heat conductivity of metals, the change of temperature in the liquid embraces regions of the stream that are far away from the surface of the solid body. In these regions the influence of the viscosity forces can be neglected compared with the influence of the inertia forces. Consequently, when $Pr \ll 1$ the second limiting condition of heat transmission in natural convection is valid.

§15. HEAT TRANSMISSION FROM A HORIZONTAL CYLINDER IN THE CASE OF EXTERNAL NATURAL CONVECTION OF FLUID

Similarity Conditions for the Heat-Transmission Process in a Horizontal Cylinder for External Natural Convection

Let us consider the similarity conditions of a stationary heat-transmission process for a long horizontal cylinder in the case of external natural convection of the fluid. Far from the cylinder the fluid is stationary, and its temperature is constant at $t_{zh} < t_F$ (Fig. 50).

The similarity conditions for this heat-transmission process are a particular case of the more general conditions (see §14). The charac-

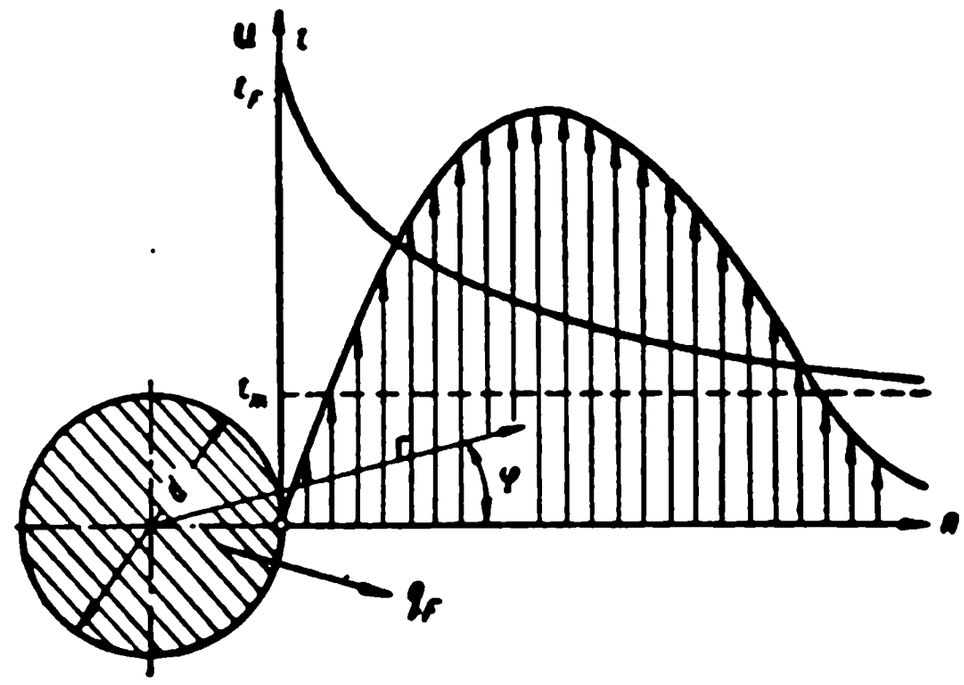

Fig. 50. Diagram of flow around a horizontal cylinder by a natural stream of fluid.

teristic dimension of the body is equal to the cylinder diameter $l_1 = d$, while the coordinates of any point on its surface are determined by the radius d/2 and by the angle φ. Under these conditions the similarity equation (14.11) for the first heat-transmission process (t_F = = const; q_F = ?) simplifies to the following form:

$$\frac{q_F d}{\lambda_m \Delta t} = \varphi_{13}\left(\varphi, \frac{\nu}{a_{p\,m}}, \frac{g\beta_0 \Delta t d^3}{\nu^3}\right) \qquad (15.1)$$

or

$$Nu = \varphi_{13}(\varphi, Pr, Gr).$$

The similarity condition for the heat-transmission coefficient averaged over the cylinder perimeter, $^{\varphi}\bar{\alpha} = {}^{\varphi}\overline{q_F}/{}^{\varphi}\overline{\Delta t}$, assumes a simpler form:

$$^{\bullet}\overline{Nu} = \varphi_{14}(Pr, Gr). \qquad (15.2)$$

For the most frequently encountered first limiting condition of natural convection, when the inertia forces in the fluid stream can be neglected, we obtain an even simpler similarity equation

$$^{\bullet}\overline{Nu} = \varphi_{15}(GrPr). \qquad (15.3)$$

Thus, under these conditions the Nusselt number averaged over the perimeter of the cylinder is determined by the value of the single criterion Gr·Pr.

The simplest experimental procedure for investigating the average

heat transmission in natural convection consists of using as the horizontal cylinder the calorimeter described previously in §13. This calorimeter should be installed in a sufficiently large volume with quiet air, for example in a room, and all the measuring instruments should be located in the neighboring room.

Principal Results of Measurements of Heat Transmission of a Cylinder and the Computation Formula

In the course of the experiments described in [37], the cooling fluids employed were four different kinds of oil, water, and air. The results of the measurements can be represented by means of the following formula:

$$\overline{Nu}_f = 0{,}51\,(GrPr)_f^{0.25}\left(\frac{Pr_f}{Pr_w}\right)^{0.25}. \qquad (15.4)$$

For air this formula simplifies to:

$$\overline{Nu}_f = 0{,}47Gr_f^{0.25}. \qquad (15.5)$$

The similarity equation (15.4) is valid for heat transmission from horizontal cylinders in the case of external natural convection by various fluids (without change in their aggregate state), having Prandtl numbers from 0.6 to $3\cdot10^4$. In this case the defining criterion $(Gr\cdot Pr)$ is in the interval between 500 and $2\cdot10^8$. The factor $(Pr_f/Pr_w)^{0.25}$ takes into account the dependence of the thermodynamic parameters of the fluid on the temperature. The values of the critical Prandtl number for the investigated liquids decreases with increasing temperature. The indices w and f have the same meaning as in §13.

The character of the influence of different factors on the heat transmission from the cylinder in natural convection can be established from the dimensionality formula for $\overline{\alpha}$

$$\overline{\alpha} \sim \frac{\lambda_ж\,(g\beta_0\Delta t a_{ж\,w})^{0.25}}{(d\nu_w)^{0.25}}\left(\frac{\nu_f}{a_{ж\,f}}\right)^{0.5}.$$

Recommendations on the calculation of heat transmission in natu-

ral convection under other conditions are contained in References [31, 35b].

Manu-
script
Page
No.

[List of Transliterated Symbols]

460 ж = zh = zhidkost' = liquid

Chapter 8

FUNDAMENTALS OF BOUNDARY LAYER THEORY

§16. FUNDAMENTALS OF HYDRODYNAMIC BOUNDARY LAYER THEORY

Concept of the Hydrodynamic Boundary Layer

The hydrodynamic boundary layer is called such a region in a stream of a viscous fluid, directly adjacent to the surface of a solid over which the fluid flows; in this region is concentrated almost the entire change in velocity from u_0 away from the body to zero on its surface. The thickness δ of this region is small compared with the distance from the forward edge of the body on which the stream is incident. The viscosity and inertia forces in this region are of the same order of magnitude. The foregoing properties of the hydrodynamic boundary layer are in many cases the consequence of the self-similarity of the velocity field along the stream.

The idea of the boundary layer, proposed by Prandtl in 1903, turned out to be quite fruitful in the mechanics of viscous fluids and in heat transfer.

Formulation of the Problem of Flow Around a Plate

As the simplest example let us consider the flow around a plate by an unbounded stream of a viscous homogeneous fluid (Fig. 51).

It is required to find the velocity distribution in this stream, if the geometrical conditions specify a semi-infinite ($x \geq 0$), smooth, and thin plate. The fluid stream is directed parallel to the plate. The physical conditions specify a homogeneous isotropic fluid, with kinematic viscosity ν (m^2/sec) and a mass density ρ ($kg\text{-}sec^2/m^4$),

while the time conditions specify a stationary stream, i.e., $\partial\vec{u}/\partial\tau = 0$. The boundary conditions indicate that

$$\text{for } y \to \infty\ \vec{u} \to \vec{u}_0 \text{ and } p \to p_0 = \text{const},$$
$$\text{and for } y = 0\ \vec{u} = 0. \tag{16.1}$$

Under these uniqueness conditions the distribution of the velocities in the fluid stream obeys the Navier-Stokes equations

$$\left.\begin{aligned} u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} &= \nu \left[\frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{2x^2}\right] - \frac{1}{\varrho} \cdot \frac{\partial p}{\partial x}; \\ u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} &= \nu \left[\frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_y}{\partial x^2}\right] - \frac{1}{\varrho} \cdot \frac{\partial p}{\partial y} \end{aligned}\right\} \tag{16.2}$$

and the continuity equation

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0. \tag{16.3}$$

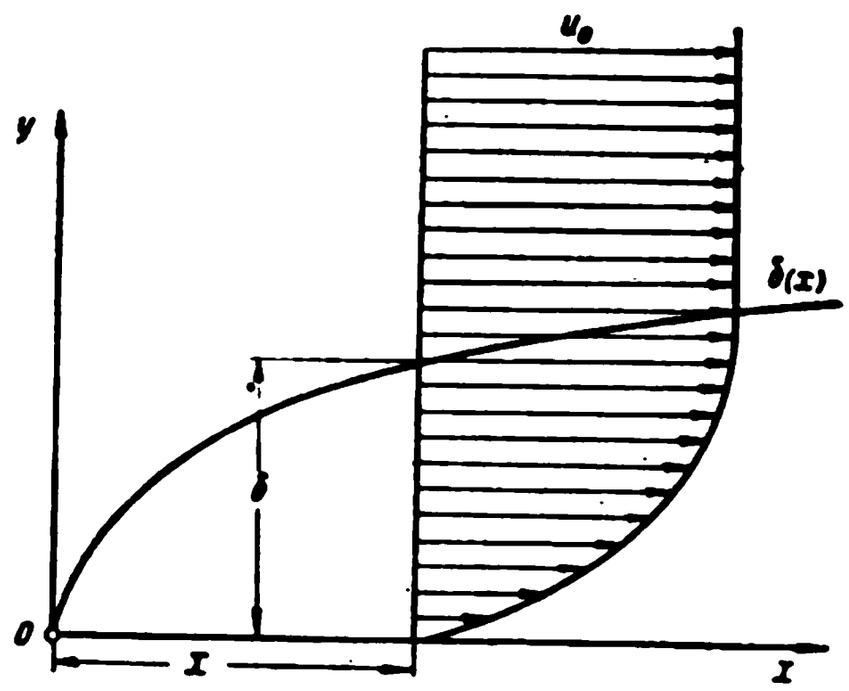

Fig. 51. Diagram of hydrodynamic boundary layer on a plate.

The system (16.2) and (16.3) consists of three equations and contains three unknowns u_x, u_y, and p.

Differential Equation of Asymptotic Hydrodynamic Boundary Layer

The solution of Eqs. (16.2) and (16.3) for boundary conditions (16.1) cannot be obtained analytically. The problem simplifies appreciably if we use the concept of the asymptotic hydrodynamic boundary layer. For this purpose we estimate the order of magnitude of the individual terms in these equations.

Assume that the freely specified quantities u_0, p_0, $\underline{x}$, and ρ are

of the order of unity, i.e.,

$$\Pi(u_0) = \Pi(\rho_0) = \Pi(x) = \Pi(\varrho) = 1.$$

Since $0 \leq u_x \leq u_0$, we have $\Pi(u_x) = 1$, $\Pi(p) = 1$. From the definition of the concept of the boundary layer, it follows that if

$$\Pi(\delta) \ll 1 \text{ and } \Pi(x) = 1, \text{ then } \Pi(y) \ll 1.$$

The order of thickness of the layer is designated by $\Pi(\delta) \equiv \varepsilon \ll 1$.

Since within the limits of the boundary layer $0 \leq y \leq \delta$, we have here $\Pi(y) = \varepsilon$. It follows further from the continuity equation that

$$\frac{\partial u_y}{\partial y} = -\frac{\partial u_x}{\partial x}\ ;\quad \frac{\Pi(u_y)}{\varepsilon} = 1.$$

Consequently, $\Pi(u_y) = \varepsilon$. Let us estimate the order of the terms of the first equation

$$\Pi\left(u_x \frac{\partial u_x}{\partial x}\right) = 1;\ \Pi\left(u_y \frac{\partial u_x}{\partial y}\right) = 1;\ \Pi\left(\frac{1}{\varrho} \cdot \frac{\partial p}{\partial x}\right) = 1;$$

$$\Pi\left(\frac{\partial^2 u_x}{\partial x^2}\right) = 1;\ \Pi\left(\frac{\partial^2 u_x}{\partial y^2}\right) = \frac{1}{\varepsilon^2}.$$

Since

$$\frac{1}{\varepsilon^2} \gg 1, \text{ then } \frac{\partial^2 u_x}{\partial y^2} \gg \frac{\partial^2 u_x}{\partial x^2}. \qquad (16.3a)$$

Consequently, in the approximation of the boundary layer theory the quantity $\partial^2 u_x/\partial x^2$ can be neglected with the quantity $\partial^2 u_x/\partial y^2$. Then the order of magnitude of the viscosity force is $\Pi(\nu)/\varepsilon^2$, and the order of magnitude of the inertia force is equal to unity.

From the definition of the concept of the boundary layer it follows that the inertia and viscosity forces within the limits of the boundary layer are of the same order of magnitude. This requirement will be satisfied if we assume

$$\Pi(\nu) = \varepsilon^2.$$

Thus, the first equation of (16.2) assumes the following form:

$$u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} = -\frac{1}{\varrho} \cdot \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u_x}{\partial y^2}. \qquad (16.4)$$

Let us estimate the order of the terms of the second equation of (16.2):

$$\Pi\left(u_x\frac{\partial u_y}{\partial x}\right)=\varepsilon;\ \Pi\left(u_y\frac{\partial u_y}{\partial y}\right)=\varepsilon;\ \Pi\left(\frac{1}{\varrho}\cdot\frac{\partial p}{\partial y}\right)=\frac{1}{\varepsilon};$$

$$\Pi\left(\nu\frac{\partial^2 u_y}{\partial x^2}\right)=\varepsilon^2;\ \Pi\left(\nu\frac{\partial^2 u_y}{\partial y^2}\right)=\varepsilon.$$

Consequently, all the terms of this equation can be neglected compared with the term $1/\rho\cdot\partial p/\partial y$. Then $\partial p/\partial y \approx 0$. From this condition it follows also that $\partial p/\partial x \approx 0$, since we have for the problem under consideration

$$p|_{y\to\infty}\to p_0=\text{const}.$$

Thus, in the approximation of the boundary-layer theory the calculation of the flow around a plate reduces to the integration of the differential equation of the Prandtl asymptotic boundary layer and of the continuity equation for the indicated types of boundary conditions:

$$\left.\begin{aligned}&u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y}=\nu\frac{\partial^2 u_x}{\partial y^2};\\&\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}=0;\\&u_x|_{y=0;\,x>0}=0;\ u_x|_{y\to\infty;\,x\to u_0};\\&u|_{y=0;\,x=0};\ u_y|_{y\to\infty}\to 0.\end{aligned}\right\}\qquad(16.5)$$

Theory of Hydrodynamic Boundary Layer of Finite Thickness (the Karman Method)

The Karman method of solving the Prandtl problem consists of introducing in the mathematical description of the boundary layer its finite thickness $\delta = \delta(x)$. The value of $\delta(x)$ satisfies the condition $\delta(x) \ll x$, as follows from the definition of the boundary-layer concept.

Let us integrate the Prandtl differential equation from zero to $\delta(x)$:

$$\int_0^{\delta(x)}u_x\frac{\partial u_x}{\partial x}\,dy+\int_0^{\delta(x)}u_y\frac{\partial u_x}{\partial y}\,dy=\nu\int_0^{\delta(x)}\frac{\partial^2 u_x}{\partial y^2}\,dy.$$

After transformation we obtain the so-called Karman integral relation [33]

$$\frac{d}{dx}\int_0^{\delta(x)}\left(1-\frac{u_x}{u_0}\right)\frac{u_x}{u_0}\,dy=\frac{\nu}{u_0^2}\cdot\frac{\partial u}{\partial y}\Bigg|_{y=+0}. \qquad (16.6)$$

To determine the velocity field u_x, Karman proposed to express the quantity u_x only in terms of the parameters u_0 and δ and of the ordinate y, i.e.,

$$u_x=u_x(y,\ u_0,\ \delta).$$

Let us reduce this equation to dimensionless form

$$\frac{u_x}{u_0}=F\left(\frac{y}{\delta}\right). \qquad (16.7)$$

The Karman method consists of choosing as the function $F(y/\delta)$ a "suitable" monotonically increasing function, which satisfies the boundary conditions. This function is substituted into the integral equation (16.6), which is then solved with respect to the thickness $\delta(x)$ of the layer. The expression obtained for $\delta(x)$ is substituted into Eq. (16.7) and the velocity distribution u_x (x, y) is determined.

Let us carry out these transformations in general form. Substituting Eq. (16.7) in Eq. (16.6), we obtain

$$\frac{d}{dx}\int_0^{\delta}\left[1-F\left(\frac{y}{\delta}\right)\right]F\left(\frac{y}{\delta}\right)dy=\frac{\nu}{u_0^2}\cdot\frac{u_0}{\delta}\cdot\frac{dF\left(\frac{y}{\delta}\right)}{dy}\Bigg|_{y=+0}.$$

We denote the dimensionless ordinate by

$$y/\delta\equiv Y.$$

Then

$$\frac{d}{dx}\left\{\delta(x)\int_0^1[1-F(Y)]\,F(Y)\,dY\right\}=\frac{\nu F'(0)}{u_0\delta}.$$

The definite integral in the left half of this equation is equal to a constant quantity, since it does not depend on the arguments x and y. We put

$$\mathcal{J}_1\equiv\int_0^1[1-F(Y)]\,F(Y)\,dY.$$

Taking the constant T_1 outside the differentiation sign, we obtain a

first-order differential equation relative to the thickness of the boundary layer $\delta(x)$

$$\frac{d\delta(x)}{dx} = \frac{2\nu F'(0)}{u_0 \delta(x) J_1}.$$

The boundary condition has the form

$$\delta(0) = 0.$$

Integrating the obtained differential equation with account of the boundary condition, we obtain

$$\delta^2(x) = \frac{2F'(0)}{J_1} \cdot \frac{\nu x}{u_0}. \qquad (16.8)$$

To determine the thickness of the layer $\delta(x)$ it is necessary to calculate T_1 and $F'(0)$.

Calculation of Hydrodynamic Boundary Layer on a Plate

Let us represent the function $u_x/u_0 = F(y/\delta)$ in the form of a fourth-degree polynomial

$$\frac{u_x}{u_0} = a_0 + a_1\left(\frac{y}{\delta}\right) + a_2\left(\frac{y}{\delta}\right)^2 + a_3\left(\frac{y}{\delta}\right)^3 + a_4\left(\frac{y}{\delta}\right)^4.$$

The five coefficients a_0, a_1, ..., a_4 will be determined with the aid of approximate boundary conditions, which are satisfied by the velocity field in the boundary layer of finite thickness:

for Y = 0 $\quad u_x = 0$ and $\frac{\partial^2 u_x}{\partial y^2} \approx 0$;

for Y = δ $\quad u_x \approx u_0$, $\frac{\partial u_x}{\partial y} \approx 0$ and $\frac{\partial^2 u_x}{\partial y^2} \approx 0$.

The last two conditions follow from Eq. (16.5). By determining the coefficients of the polynomial with the aid of these boundary conditions, we obtain

$$\frac{u_x}{u_0} = 2Y - 2Y^3 + Y^4. \qquad (16.9)$$

Relation (16.9) contains two unknowns $u_x(x, y)$ and $\delta(x)$. To determine the thickness of the boundary layer $\delta(x)$ we substitute Eq. (16.9) into Relation (16.8):

$$\delta^2(x) = \frac{2\frac{d}{dY}[2Y - 2Y^3 + Y^4]_{Y=+0}\frac{\nu x}{u_0}}{\int_0^1 (1 - 2Y + 2Y^3 - Y^4)(2Y - 2Y^3 + Y^4)\,dY}.$$

After calculations we obtain an expression for the thickness of the boundary layer

$$\delta \approx 5{,}83\sqrt{\frac{\nu x}{u_0}}. \qquad (16.10)$$

Equations (16.9) and (16.10) describe approximately the velocity distribution $u_x(x, y)$ around the plate in the case of laminar flow of a homogeneous liquid in the boundary layer. These equations make it possible to calculate the tangential viscous-friction stress τ_0 on the plate

$$\tau_0 = \mu\left.\frac{\partial u_x}{\partial y}\right|_{y=+0} = \frac{2\mu u_0}{\delta(x)} = 0{,}343\sqrt{\frac{\mu\rho u_0^3}{x}}. \qquad (16.11)$$

Region of Application of Results Obtained

Equations (16.9)-(16.11) for the distribution of the velocities of the fluid in the stream around the plate and for the tangential friction stresses on its surface are approximate in character and have a limited region of application.

When $Re_x \equiv u_0 x/\nu \geq Re_{x\ kr}$ the stream around the plate loses stability with respect to perturbations, and the laminar boundary layer becomes a turbulent one. Therefore the quantity $Re_{x\ kr}$ should be regarded as the upper limit of the region of applicability of Formulas (16.9)-(16.11). The lower limit of this region is also determined by a certain value of the Reynolds number $Re_{x\ min}$.

The concept of the hydrodynamic boundary layer is based on the inequality $\delta(x) \ll x$. In the region of the plate close to its forward edge, this condition is not satisfied. Consequently, the fundamental approximation of the boundary-layer theory is not applicable here. Let us determine the minimum distance x_{min}, starting with which this theory

yields sufficiently accurate results. For this distance we obtain from Eq. (16.3a)

$$\delta^2(x_{\text{мин}}) \ll x^2_{\text{мин}}.$$

Multiplying this inequality by $(u_0/\nu)^2$ and taking Relation (16.10) into account, we obtain

$$(Re_{x\,\text{мин}})^2 \equiv \left(\frac{x_{\text{мин}} u_0}{\nu}\right)^2 \gg (5{,}83\sqrt{Re_{x\,\text{мин}}})^2.$$

Consequently, $Re_{x\ min} \gg 5.83^2 = 34$.

Thus, the region of applicability of the results of the hydrodynamic boundary layer theory in the case of flow around a plate consists of the following region of variation of the "longitudinal" Reynolds number:

$$34 \ll Re_x < Re_{x\,\text{кр}}.$$

If we assume that the stronger inequality is satisfied at a ratio 1:50, we obtain $1700 \leq Re_x \leq Re_{x\ kr}$.

The critical value of the Reynolds number depends on the degree and character of the perturbations of the incoming fluid stream. If we assume the average value of $Re_{x\ kr}$ to be $4 \cdot 10^5$, we obtain for the applicability of Formulas (16.9)-(16.11) a range $1700 \leq Re_x \leq 4 \cdot 10^5$.

§17. FUNDAMENTALS OF THE THERMAL BOUNDARY LAYER THEORY

Concept of the Thermal Boundary Layer

The thermal boundary layer is that region in the fluid stream, directly adjacent to the solid surface F, through which heat is transferred. In this region is concentrated almost the total change in temperature from t_{zh} away from the surface to t_F on the surface. The thickness $\underline{k}$ of this region is small compared with the distance $\underline{x}$ from the frontal point of the body on which the fluid stream is incident:

$$k(x) \ll x.$$

The intensity of the heat transport via convection and heat conductiv-

ity in the boundary layer has the same order of magnitude.*

Formulation of the Problem of Heat Transmission of a Plate

As the simplest example of the calculation of a thermal boundary layer let us consider the heat transmission from a plate around which

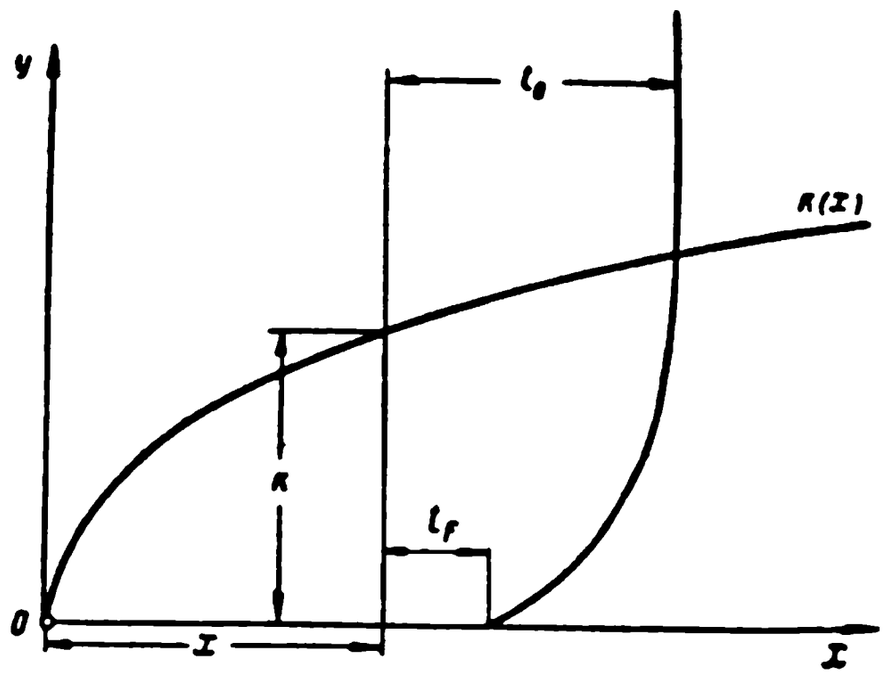

Fig. 52. Diagram of thermal boundary layer on a plate.

an unbounded liquid stream flows (Fig. 52). It is required to find the temperature field in the liquid stream around the plate for the following uniqueness conditions.

The geometrical conditions specify a semi-infinite ($x \geq 0$) smooth and thin plate, parallel to the liquid stream.

The physical conditions specify a homogeneous and isotropic fluid and its thermodynamic parameters a_{pzh}, λ_{zh}, ρ.

The time conditions indicate that the flux is stationary, i.e., $\partial\vec{u}/\partial\tau = 0$ and $\partial t/\partial\tau = 0$.

The boundary conditions determine the temperature of the plate t_F and the temperature of the fluid $t_{zh} > t_F$ away from the plate, and also specify the velocity distribution in the boundary layer.

Under these uniqueness conditions the temperature distribution in the stream is planar and obeys the Fourier-Ostrogradskiy differential equation

$$u_x \frac{\partial\vartheta}{\partial x} + u_y \frac{\partial\vartheta}{\partial y} = a_{p\varkappa}\left(\frac{\partial^2\vartheta}{\partial x^2} + \frac{\partial^2\vartheta}{\partial y^2}\right). \qquad (17.1)$$

where $\vartheta \equiv t - t_f$.

Differential Equation of Asymptotic Thermal Boundary Layer

The analytic solution of Eq. (17.1) for specified boundary conditions simplifies greatly if we use the concept of the asymptotic thermal boundary layer. For this purpose we estimate the order of magnitude of the individual terms of Eq. (17.1). Let $\Pi(x) = \Pi(\vartheta_0) = \Pi(\vartheta) = 1$ ($\vartheta_0 \equiv t_F - t_{zh}$). From the theory of the hydrodynamic boundary layer we know that

$$\Pi(u_x) = 1; \quad \Pi(u_y) = e \ll 1.$$

From the definition of the thermal boundary layer it follows that $k(x) \ll x$, and therefore $\Pi(k) = \varepsilon_1 \ll 1$, with $\varepsilon_1 \sim \varepsilon$. Let us write down the order of magnitude of the individual terms of Eq. (16.1):

$$\Pi\left(u_x \frac{\partial \vartheta}{\partial x}\right) = 1; \ \Pi\left(u_y \frac{\partial \vartheta}{\partial y}\right) = 1; \ \Pi\left(\frac{\partial^2 \vartheta}{\partial x^2}\right) = 1;$$

$$\Pi\left(\frac{\partial^2 \vartheta}{\partial y^2}\right) = \frac{1}{\varepsilon_1^2}.$$

Inasmuch as $1/\varepsilon_1^2 \gg 1$, we have $\partial^2\vartheta/\partial y^2 \gg \partial^2\vartheta/\partial x^2$ and the quantity $\partial^2\vartheta/\partial x^2$ in Eq. (17.1) can be neglected. Consequently, the order of magnitude of the right half of this equation is $\Pi(a_{pzh})/\varepsilon_1^2$.

As indicated above, the order of magnitude of the intensities of the heat transports in the boundary layer by heat conduction and by convection are assumed to be the same. Therefore

$$\frac{\Pi(a_{p\infty})}{\varepsilon_1^2} = 1; \quad \Pi(a_{p\infty}) = \varepsilon_1^2.$$

Thus, the differential equation for convective heat exchange in the approximation of the theory of the asymptotic thermal boundary layer assumes the following form:

$$u_x \frac{\partial \vartheta}{\partial x} + u_y \frac{\partial \vartheta}{\partial y} = a_{p\infty} \frac{\partial^2 \vartheta}{\partial y^2}. \qquad (17.2)$$

This equation must be integrated for the following boundary conditions:

$$\left.\begin{array}{ll} \text{for } y = 0 & \vartheta = 0 \end{array}\right\} \qquad (17.3)$$

The velocity distribution around the plate is known from the calculation of the hydrodynamic boundary layer.

Theory of Thermal Boundary Layer of Finite Thickness

The theory of the thermal boundary layer of finite thickness (the Kruzhilin method) [26a] is analogous to the Karman method for the hydrodynamic layer. It consists of introducing the finite thickness of the thermal layer k, for the determination of which an integral equation is set up.

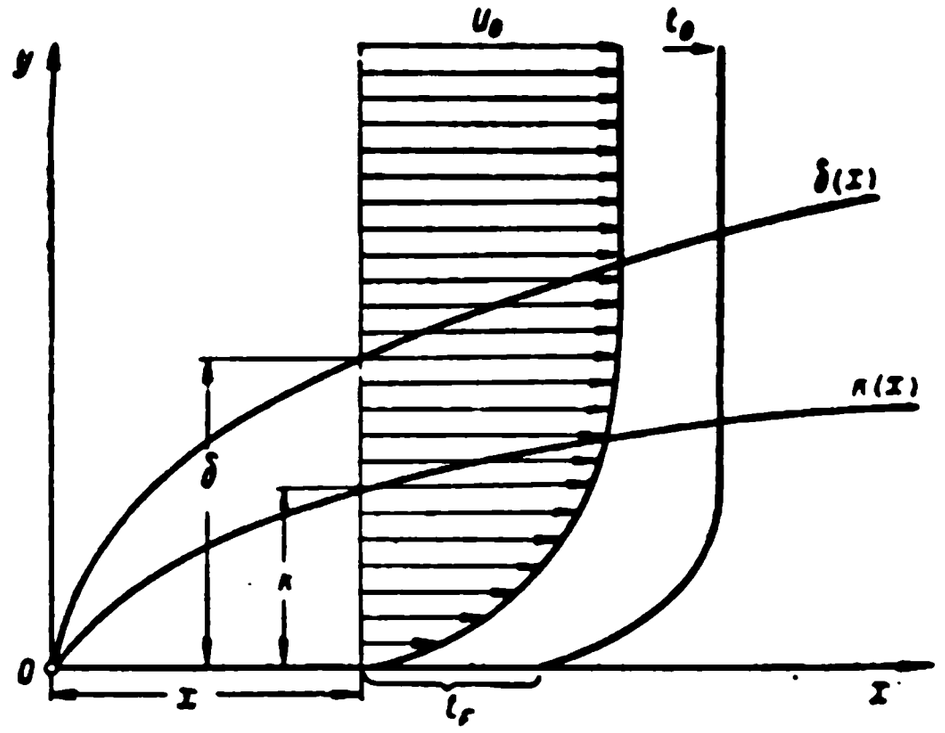

Fig. 53. Case when the thermal layer is thinner than the hydrodynamic layer.

Let us integrate the differential equation of the thermal boundary layer from zero to k(x):

$$\int_0^{k(x)} \left(u_x \frac{\partial \vartheta}{\partial x} + u_y \frac{\partial \vartheta}{\partial y} - a_{p\varkappa} \frac{\partial^2 \vartheta}{\partial y^2}\right) dy = 0.$$

After transformations we obtain the Kruzhilin integral relation

$$\frac{d}{dx} \int_0^{k(x)} \left(1 - \frac{\vartheta}{\vartheta_0}\right) \frac{u_x}{u_0} dy = \frac{a_{p\varkappa}}{u_0 \vartheta_0} \cdot \frac{\partial \vartheta}{\partial y}\Big|_{y=+0}. \qquad (17.4)$$

The integral relation (17.4) is valid when the thermal layer is thinner than the hydrodynamic, $k \leq \delta$ (Fig. 53). If the thermal layer is thicker than the hydrodynamic one (Fig. 54), then there is a section $k - \delta$ inside the thermal layer, where the velocity of the liquid

is constant and equal to u_0. In this case the integral relation (17.4) assumes the form

$$\frac{d}{dx}\left[\int_0^{\delta(x)}\left(1-\frac{\vartheta}{\vartheta_0}\right)\cdot\frac{u}{u_0}\cdot dy+\int_{\delta(x)}^{k(x)}\left(1-\frac{\vartheta}{\vartheta_0}\right)\cdot dy\right]=\frac{a_{p\,m}}{u_0\vartheta_0}\left.\frac{\partial\vartheta}{\partial y}\right|_{y=+0}. \qquad (17.5)$$

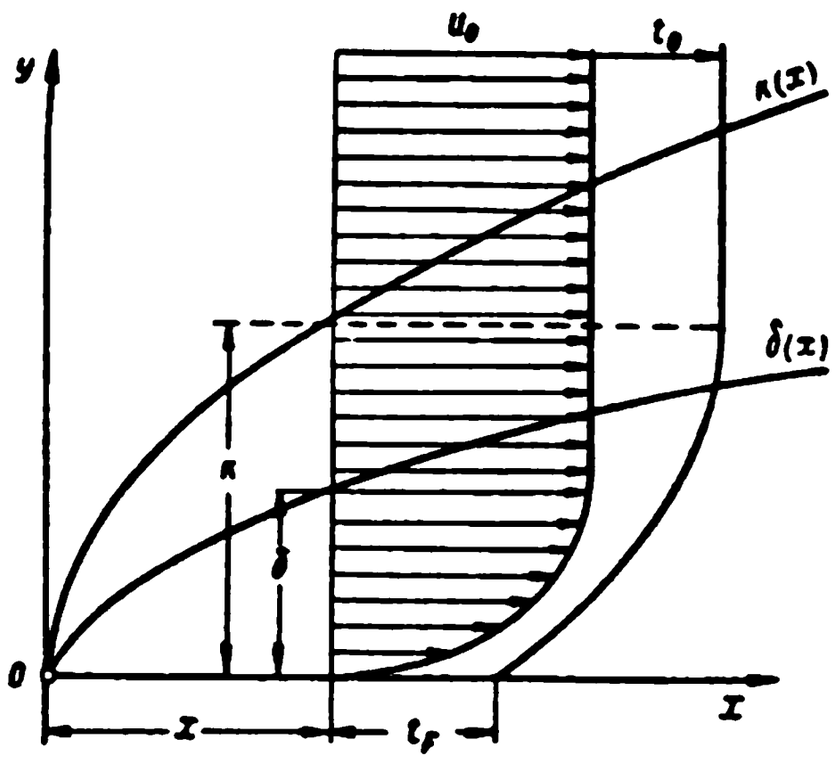

Fig. 54. Case when the thermal layer is thicker than the hydrodynamic layer.

To determine the temperature field we assume that the temperature ϑ at any point of the layer is determined by the ordinate y and by the parameters ϑ_0 and k, i.e.,

$$\vartheta=\vartheta(y,\ \vartheta_0,\ k).$$

Reducing this relation to dimensionless form, we obtain

$$\frac{\vartheta}{\vartheta_0}=\Phi\left(\frac{y}{k}\right) \quad \text{or}$$

$$\vartheta=\vartheta_0\cdot\Phi\left(\frac{y}{k}\right). \qquad (17.6)$$

For the function Φ we choose a "suitable" monotonically increasing function satisfying the approximate conditions on the boundary of the thermal layer of finite thickness. This function is then substituted into the integral relation (17.4) or (17.5) and the latter is solved with respect to the layer thickness $k = k(x)$. The expression obtained for $k(x)$ in conjunction with Eq. (17.6) characterizes the distribution of the temperature in the layer around the plate. Having these quantities, it is easy to calculate the heat-transmission coefficient on the surface of the plate.

Let us carry out these transformations in general form. We consider first a case when $k \leq \delta$. Substituting (16.7) and (17.6) into (17.4), we obtain

$$\frac{d}{dx}\int_0^{k(x)}\left[1-\Phi\left(\frac{y}{k}\right)\right]\cdot F\left(\frac{y}{\delta}\right)\cdot dy=\frac{a_{p\,m}}{u_0}\cdot\frac{\Phi'(+0)}{u_0\cdot k(x)},$$

where

$$\Phi'(+0) \equiv \left.\frac{d\Phi\left(\frac{y}{k}\right)}{d\frac{y}{k}}\right|_{\frac{y}{k}=+0}.$$

Putting $y/k \equiv Y$ and recognizing that $y/\delta = kY/\delta$, we obtain

$$\frac{d}{dx}\left\{k(x)\cdot\int_0^1 [1-\Phi(Y)]\cdot F\left(\frac{k}{\delta}Y\right)\cdot dY\right\} = \frac{a_p\varkappa\cdot\Phi'(+0)}{u_0\cdot k(x)}. \qquad (17.7)$$

The definite integral in the left half of Eq. (17.7) depends only on the ratio k/δ. Therefore, putting

$$\int_0^1 [1-\Phi(Y)]\cdot F\left(\frac{k}{\delta}\cdot Y\right)\cdot dY \equiv \mathcal{J}_2\left(\frac{k}{\delta}\right), \qquad (17.8)$$

we obtain in lieu of (17.7)

$$\frac{d}{dx}\left\{k(x)\cdot\mathcal{J}_2\left(\frac{k}{\delta}\right)\right\} = \frac{a_p\varkappa\cdot\Phi'(+0)}{u_0\cdot k(x)}. \qquad (17.9)$$

We then change over to a new independent variable δ. Since

$$\frac{d}{dx} = \frac{d}{d\delta}\cdot\frac{d\delta}{dx} = \frac{5{,}83^2}{2}\cdot\frac{\nu}{u_0\delta}\cdot\frac{d}{d\delta} = 17\frac{\nu}{u_0\delta}\cdot\frac{d}{d\delta},$$

we have in place of (17.9)

$$\frac{d}{d\delta}\left[k(x)\cdot\mathcal{J}_2\left(\frac{k}{\delta}\right)\right] = \frac{\Phi'(+0)}{17\cdot Pr\cdot\left(\frac{k}{\delta}\right)}. \qquad (17.10)$$

From Eq. (17.10) it follows that $k = k(\delta, Pr)$. Reducing this equation to dimensionless form, we obtain

$$\frac{k}{\delta} = f(Pr) = f_1(\delta). \qquad (17.11)$$

Taking Eq. (17.11) into account, we take the quantity $T_2(k/\delta)$ in Eq. (17.10) outside the sign of differentiation with respect to δ. This equation then assumes the form

$$\mathcal{J}_2\left(\frac{k}{\delta}\right)\cdot\frac{dk}{d\delta} = \frac{\Phi'(+0)\cdot\delta(x)}{17\,Pr\cdot k(x)}$$

or

$$dk^2(x) = \frac{\Phi'(+0)\cdot d\delta^2(x)}{17Pr\cdot\mathcal{J}_2\left(\frac{k}{\delta}\right)}.$$

Integrating, we obtain a computation equation valid for the condition $k \leq \delta$

$$\left(\frac{k}{\delta}\right)^2 \cdot \mathcal{J}_2\left(\frac{k}{\delta}\right) = \frac{\Phi'(+0)}{17Pr}. \qquad (17.12)$$

Let us consider the second case, when $k \geq \delta$. Substituting Eqs. (16.7) and (17.6) into Eq. (17.4) and carrying out analogous transformations, we obtain a second computation relation, which is valid for $k \geq \delta$

$$\left(\frac{k}{\delta}\right)^2 \cdot \mathcal{J}_2\left(\frac{k}{\delta}\right) = \frac{\Phi'(+0)}{17Pr}. \qquad (17.13)$$

where

$$\mathcal{J}_2\left(\frac{k}{\delta}\right) \equiv \int_0^{\frac{\delta}{k}} [1 - \Phi(Y)] \cdot F\left(\frac{k}{\delta} Y\right) dY + \int_{\frac{\delta}{k}}^{1} [1 - \Phi(Y)]\, dY. \qquad (17.14)$$

Calculation of Heat Transmission from a Plate when the Hydrodynamic Layer is Thicker than the Thermal Layer, $\delta \geq k$

Let us consider heat transmission from a plate under conditions such that the hydrodynamic layer is thicker than the thermal one, i.e., $\delta(x) \geq k(x)$. We represent the function $\vartheta/\vartheta_0 = \Phi(y/k)$ in the form of a fourth-degree polynomial

$$\frac{\vartheta}{\vartheta_0} = b_0 + b_1\left(\frac{y}{k}\right) + b_2\left(\frac{y}{k}\right)^2 + b_3\left(\frac{y}{k}\right)^3 + b_4\left(\frac{y}{k}\right)^4.$$

The coefficients b_0, b_1, ..., b_4 will be determined with the aid of the approximate boundary conditions, which are satisfied by the temperature field in the thermal layer of finite thickness:

$$\text{for } y = 0,\ \vartheta = 0 \text{ and } \partial^2\vartheta/\partial y^2 \approx 0;$$

$$\text{for } y = k,\ \vartheta \approx \vartheta_0,\ \partial\vartheta/\partial y \approx 0, \text{ and } \partial^2\vartheta/\partial y^2 \approx 0.$$

The last two conditions follow from Eq. (17.2). After determining the coefficients b_i with the aid of these boundary conditions, we obtain

$$\frac{\vartheta}{\vartheta_0} = 2\left(\frac{y}{k}\right) - 2\left(\frac{y}{k}\right)^3 + \left(\frac{y}{k}\right)^4. \qquad (17.15)$$

We substitute the polynomial (17.15) into the computation relations (17.8) and (17.12):

$$\left(\frac{k}{\delta}\right)^2 \cdot \int_0^1 [1 - 2Y + 2Y^3 - Y^4] \cdot \left[2\left(\frac{k}{\delta} \cdot Y\right) - 2\left(\frac{k}{\delta} \cdot Y\right)^3 + \right.$$

$$\left. + \left(\frac{k}{\delta} \cdot Y\right)^4\right] \cdot dY = \frac{1}{17Pr} \cdot \frac{d}{dY} [2Y - 2Y^3 + Y^4]_{Y=+0}.$$

After transformation we obtain

$$Pr = \frac{\frac{15}{17}\left(\frac{\delta}{k}\right)^3}{1 + \frac{9}{56}\left(\frac{k}{\delta}\right)^2 + \frac{1}{24}\left(\frac{k}{\delta}\right)^3}. \qquad (17.16)$$

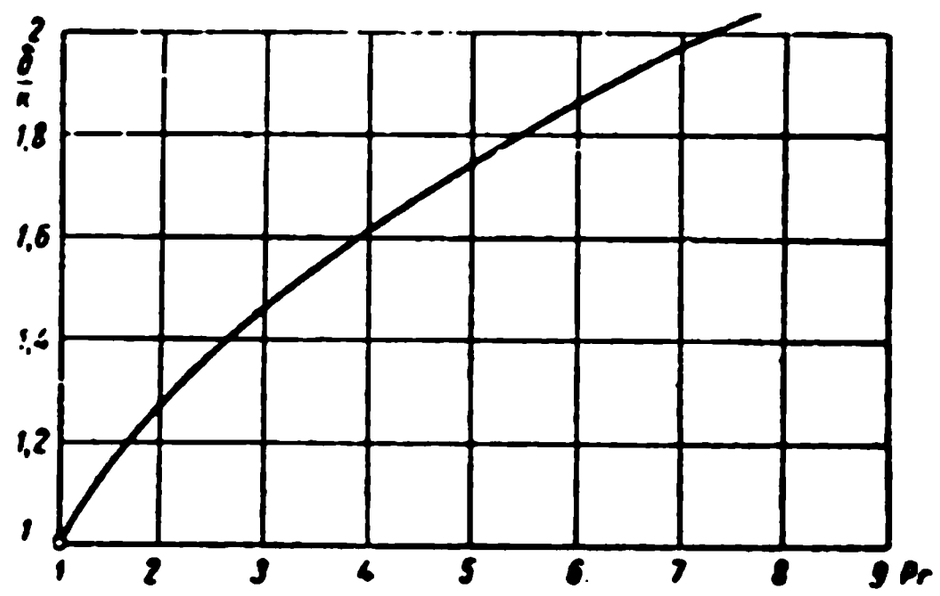

Fig. 55. Dependence of the ratio of the thickness of the hydrodynamic to the thermal layers on the Prandtl number.

Equation (17.16) establishes the dependence of the ratio of the thickness of the hydrodynamic boundary layer to the thickness of the thermal layer on the Prandtl number, i.e., the function $\delta/k = f(Pr)$. This dependence is shown in Fig. 55, which shows that when $Pr = 1$ the thermal and hydrodynamic layers have the same thickness. In this case one observes a similarity or, more accurately, an analogy between the distributions of the excess temperature ϑ and of the velocity u_x in the stream near the plate. When $Pr > 1$ the thickness of the thermal layer becomes smaller than the thickness of the hydrodynamic one, i.e., $\delta > k$, and when $Pr \gg 1$ we have $\delta \gg k$.

Relation (17.16) enables us to determine the local Nusselt number on the plate:

$$Nu_x \equiv \frac{q_F \cdot x}{\lambda_\infty \cdot \Delta t} = \frac{\left|\frac{\partial \vartheta}{\partial y}\right|_{y=+0} \cdot x}{\Delta t} = 2\frac{x}{\delta}\frac{\delta}{k}.$$

In turn we have

$$\frac{x}{\delta} = \frac{x}{5{,}83\sqrt{\frac{\nu \cdot x}{u_0}}} = \frac{1}{5{,}83}\sqrt{\frac{u_0 \cdot x}{\nu}} = \frac{Re_x^{0.5}}{5{,}83}.$$

and therefore

$$Nu_x = \frac{.2}{5{,}83} Re_x^{0.5} \cdot \frac{\delta}{k} = 0{,}343\, Re_x^{0.5} \cdot f(Pr). \qquad (17.17)$$

For liquids having Pr = 1, the ratio is $\delta/k = 1$, i.e.,

$$Nu_x = 0{,}343\, Re_x^{0.5}. \qquad (17.18)$$

For liquids having Pr >> 1 (oil, organic heat carriers) we have $k/\delta \ll 1$. In this case Eq. (17.16) simplifies to

$$Pr \approx \frac{15}{17}\left(\frac{\delta}{k}\right)^3.$$

Consequently,

$$\frac{\delta}{k} \approx \sqrt[3]{\frac{17}{15} Pr} = 1{,}04 Pr^{\frac{1}{3}}, \qquad (17.19)$$

and the Nusselt number is

$$Nu_x \approx 0{,}343 \cdot 1{,}04 Pr^{\frac{1}{3}} \cdot Re_x^{0.5} = 0{,}357 Pr^{\frac{1}{3}} \cdot Re^{0.5}. \qquad (17.20)$$

For liquids having an intermediate value of the Prandtl number, calculation of the heat transmission must be carried out by Formula (17.17), using the plot of the function $\delta/k = f(Pr)$ (see Fig. 55).

Calculation of Heat Transmission from a Plate when the Thermal Layer is Thicker than the Hydrodynamic One ($k \geq \delta$)

Let us consider heat transmission from a plate under conditions such that the thermal layer is thicker than the hydrodynamic one. For this purpose we substitute the polynomial (17.15) into the computation relations (17.13) and (17.14):

$$\left(\frac{k}{\delta}\right)^2 \left\{ \int_0^{\frac{\delta}{k}} [1 - 2Y + Y^3 - Y^4] \cdot \left[2\left(\frac{k}{\delta} Y\right) - 2\left(\frac{k}{\delta} Y\right)^3 + \right.\right.$$

$$\left.\left. + \left(\frac{k}{\delta} Y\right)^4\right] \cdot dY + \int_{\frac{\delta}{k}}^1 [1 - 2Y + 2Y^3 - Y^4] \cdot dY =\right.$$

$$= \frac{1}{17Pr} \frac{d}{dY} [2 \cdot Y - 2 \cdot Y^3 + Y^4]_{Y=+0} .$$

After transformation we obtain

$$Pr = \frac{\frac{20}{51}\left(\frac{\delta}{k}\right)^2}{1 - \left(\frac{\delta}{k}\right) + \frac{4}{9}\left(\frac{\delta}{k}\right)^2 - \frac{1}{14}\left(\frac{\delta}{k}\right)^4 + \frac{1}{54}\left(\frac{\delta}{k}\right)^5} . \qquad (17.21)$$

Equation (17.21) establishes the dependence $\delta/k = f(Pr)$ in the region $k \geq \delta$ (Fig. 56). When $Pr = 1$ we have $\delta/k = 1$. For gases, in which $Pr \leq 1$, the thickness of the thermal layer is somewhat larger than the thickness of the hydrodynamic layer. For liquid metals $Pr \ll 1$. Therefore the thermal layer is here appreciably thicker than the hydrodynamic one. In this case Eq. (17.21) simplifies and assumes the following form:

$$Pr \approx \frac{20}{51}\left(\frac{\delta}{k}\right)^2$$

or

$$\frac{\delta}{k} \approx \sqrt{\frac{51}{20} Pr} = 1{,}6\sqrt{Pr} . \qquad (17.22)$$

Substituting Eq. (17.22) into Eq. (17.17) we obtain

$$Nu_x = 0{,}343 \cdot 1{,}6 Pr^{0.5} \cdot Re_x^{0.5} = 0{,}55 (Re_x \cdot Pr)^{0.5} = 0{,}55 (Pe)^{0.5} . \qquad (17.23)$$

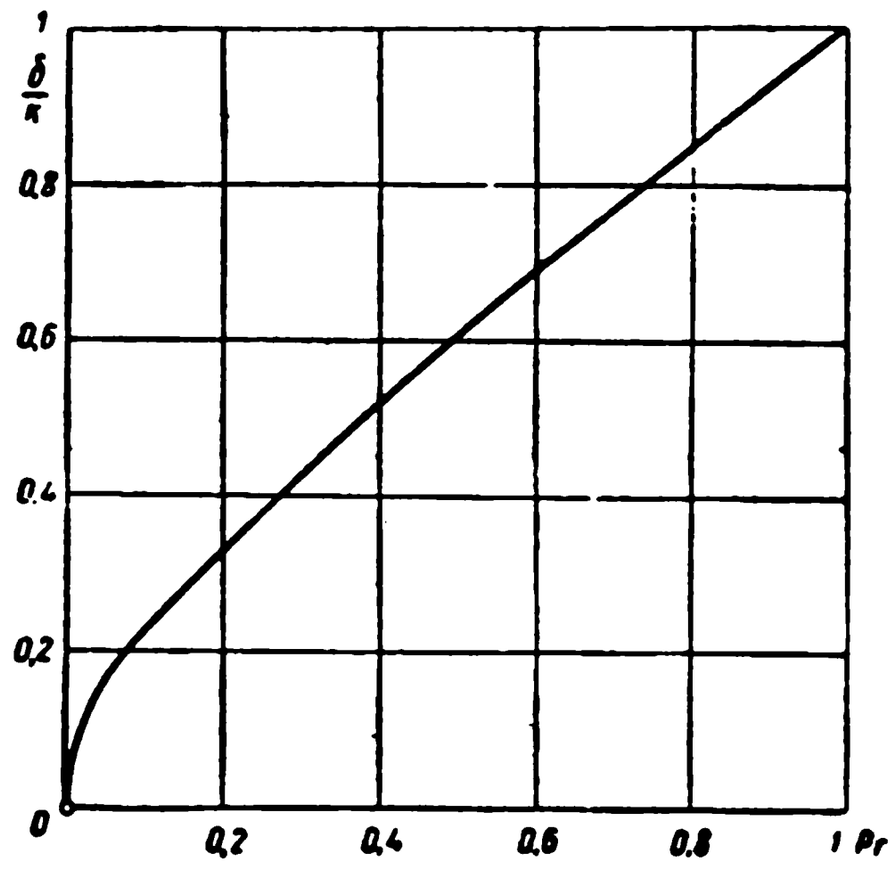

Fig. 56. Dependence of the ratio of the thickness of the hydrodynamic and thermal layers on the Prandtl number.

Consequently, in the case of laminar flow around a plate, the heat transmission to liquid metals, having $Pr \ll 1$, is determined by the Peclet number only, i.e., it is independent of the viscosity of the metal. Figure 56 shows that in the interval $0 < Pr \leq 10^{-2}$ the results of the calculations are satisfactorily interpolated by Formula (17.23), and that in the intermediate region $10^{-2} \leq Pr \leq 1$ the following equations are valid:

$$\frac{\delta}{k} = Pr^{0.4}; \quad \frac{Nu_x}{Re_x^{0.5}} \approx 0{,}343 Pr^{0.4}, \tag{17.24}$$

which are also obtained by interpolation.

Region of Applicability of the Results Obtained

The results of the application of the theory of the hydrodynamic boundary layer for a laminar stream are valid, as shown in §16, in the interval of variation of Reynolds numbers $1700 \leq Re_x \leq 4 \cdot 10^5$. The lower limit is due here to the requirement $\delta(x)^2 \ll x^2$, while the upper limit is due to the loss of stability of the laminar flow.

To determine the boundary of the region in which the results of the calculation for the laminar thermal boundary layer can be employed we must use the inequality $k^2(x) \ll x^2$ and carry out calculations analogous to those given in §16.

After doing this, we find that the region of application of Formula (17.22) $\delta/k \approx 1.6\ Pr^{0.5}$ is determined by the inequalities

$$670 < Re_x \cdot Pr < 4000;$$
$$0{,}0017 < Pr < 0{,}01;$$
$$67\,000 < Re_x < 4 \cdot 10^5.$$

For the formula (17.24) $\delta/k \approx Pr^{0.4}$ we have

$$1700 < Re_x \cdot Pr < 4 \cdot 10^5;$$
$$0{,}06 < Pr < 1;$$
$$1700 < Re_x < 4 \cdot 10^5.$$

Finally, for the formula (17.21) $\delta/k \approx 1.04\ Pr^{1/3}$ the limits of the applicability region have the following form:

$$1700 \leq Re_x \leq 4 \cdot 10^5 \text{ and } Pr \geq 1.$$

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[Footnotes]

478 This equation is the self-similarity condition with respect to $\underline{x}$ for the velocity distribution u_x.

482 The foregoing properties of the thermal boundary layers are in many cases the consequence of the self-similarity of the temperature field $\vartheta \equiv t - t_{zh}$ along the stream.

485 Equation (17.6) is the self-similarity condition for the temperature field ϑ along the stream.

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[List of Transliterated Symbols]

480 кр = kr = kriticheskiy = critical

480 мин = min = minimum = minimum

481 ж = zh = zhidkost' = liquid

Chapter 9

HEAT TRANSFER IN TUBES AND IN CHANNELS WITH FORCED FLOW OF FLUID

§18. SIMILARITY CONDITIONS FOR THE HEAT-TRANSMISSION PROCESSES IN TUBES

Setup and Mathematical Formulation of the Problem

Let us consider the similarity conditions of the heat-transmission process on the internal smooth surface F of a tube of length l and inside diameter d. A forced stream of homogeneous fluid flows through

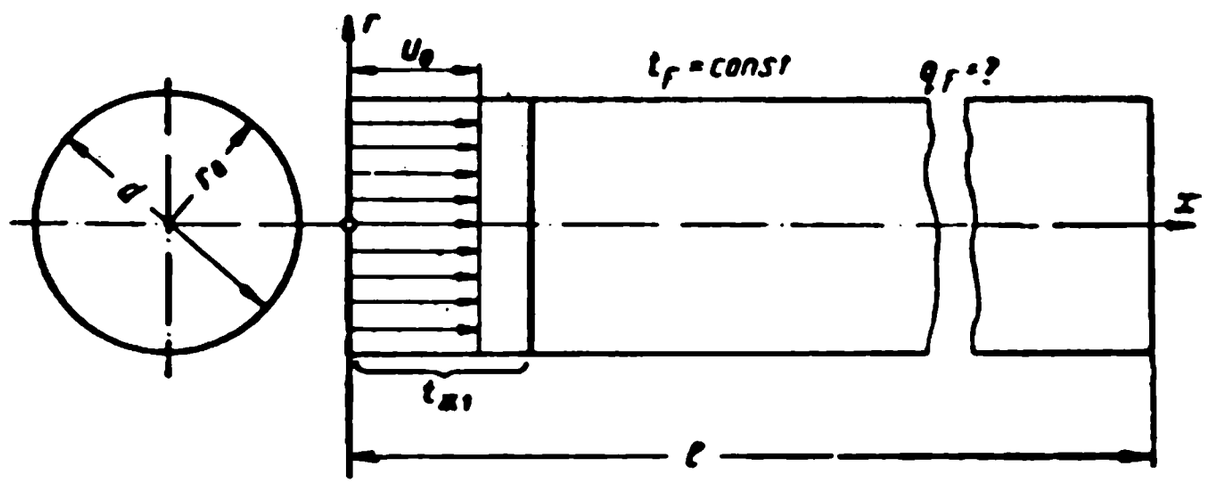

Fig. 57. Determination of the similarity conditions of the first heat-transmission process in a tube.

the tube and has at the entrance a temperature t_{zh1} and a velocity u_1 (m/sec). A constant temperature t_F = const is maintained on the surface F (Fig. 57), or else a constant heat rate q_F = const (Fig. 58). In the former case it is required to determine the distribution of the heat rate $q_F(x)$, while in the latter it is required to determine the temperature distribution $t_F(x)$ along the wall of the tube.

The uniqueness conditions for both characteristic processes of heat transmission are the following.

The geometrical conditions specify a straight, round, smooth tube of diameter d and length l.

The physical conditions indicate that the values of the heat conductivity λ_{zh}, the thermal diffusivity a_{pzh} and the kinematic viscosity ν of the fluid are constant, and that there are no internal heat sources in the stream ($q_V = 0$).

The time conditions indicate that the heat-transmission process considered is stationary.

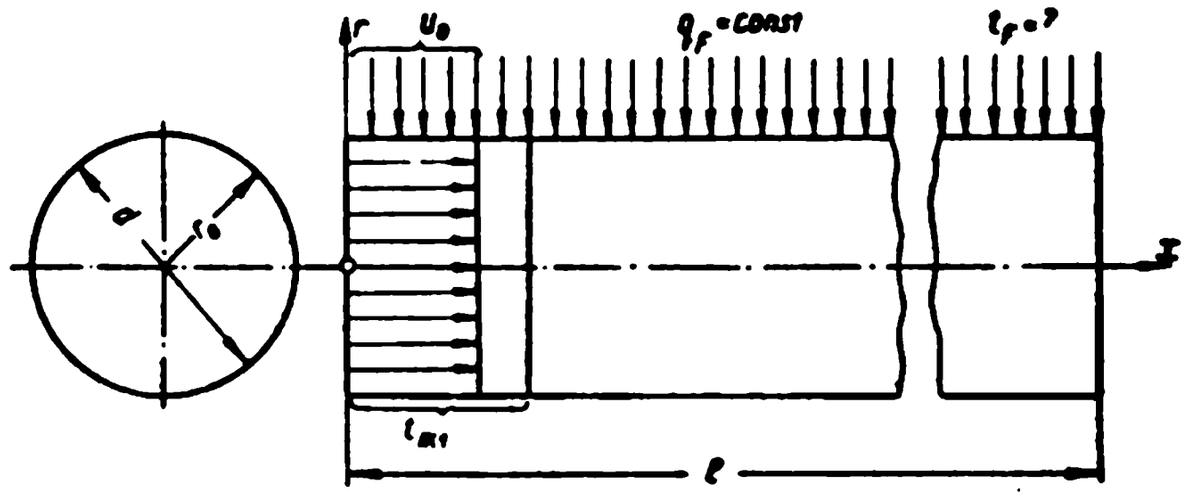

Fig. 58. Determination of the similarity conditions of the second heat-transmission process in a tube.

The boundary conditions specify the distribution of the temperature, the velocity, and heat rate on the solid and fluid boundaries of the stream.

On the fluid boundaries:

a) at the entrance to the tube

$$|u|_{x=0;\ 0<r<r_0} = u_1;\ t|_{x=0;\ 0<r<r_0} = t_{in1};$$

b) at the exit from the tube ($x = l$; $0 \leq r \leq r_0$) the distribution of the temperature and velocity in the fluid stream has the same character as in an arbitrarily long tube, but at a distance l from the entrance.

On the solid boundaries, i.e., on the surface of the tube F, one of the following conditions holds true:

$$t\Big|_{r=\frac{d}{2};\ 0<x<l} = t_F = \text{const} > t_{in1};\ u\Big|_{r=\frac{d}{2};\ 0<x<l} = 0.$$

Then

$$q_F(x) = ?$$

or

$$-q_F = -\lambda_{ж}\left.\frac{\partial t}{\partial r}\right|_{r=\frac{d}{2}-0;\ 0<x<l};\ t\Big|_{x=0;\ 0<r<\frac{d}{2}} = t_0;\ u\Big|_{r=\frac{d}{2};\ 0<x<l} = 0.$$

Then

$$t_F(x) = ?$$

If we use the excess temperature $\vartheta \equiv t - t_{zhl}$, then the differential equations of convective heat exchange and the boundary conditions for the first characteristic heat-transmission process with account of the uniqueness conditions assume the following form:

$$\left.\begin{aligned} &q_F = \lambda_{ж}\cdot\left.\frac{\partial\vartheta}{\partial r}\right|_{r=\frac{d}{2}-0};\ (u,\ \operatorname{grad}\vartheta) - a_{p\,ж}\cdot\nabla^2\vartheta = 0;\\ &\operatorname{rot}\left[(u,\ \operatorname{grad})\,u - \nu\cdot\nabla^2 u\right] = 0;\ \operatorname{div} u = 0;\\ &u\Big|_{r\,\frac{d}{2};\ 0<x<l} = 0;\ \vartheta\Big|_{r=\frac{d}{2};\ 0<x<l} = t_F - t_{ж1} \equiv \Delta t_0;\\ &u\Big|_{x=0;\ 0<r<\frac{d}{2}} = |u_1| = u_1;\ \vartheta\Big|_{x=0;\ 0<r\,\frac{d}{2}} = 0. \end{aligned}\right\} \qquad (18.1)$$

The system (18.1) contains four unknowns q_F, ϑ, u_r, u_x and consists of four equations. The quantities λ_{zh}, a_{pzh}, ν, $\underline{d}$, l, u_1 are parameters whose values are specified by the uniqueness conditions.

The mathematical formulation of the problem for the second characteristic heat-transmission process can be written in the form of the following equations:

$$\left.\begin{aligned} &\vartheta_F = \vartheta\Big|_{r=\frac{d}{2}};\ \ (u,\ \operatorname{grad}\vartheta) - a_{p\,ж}\cdot\nabla^2\vartheta = 0;\\ &\operatorname{rot}\left[(u,\ \operatorname{grad})\ u - \nu\cdot\nabla^2\ u\right] = 0;\ \operatorname{div}\ u = 0;\\ &u\Big|_{r=\frac{d}{2};\ 0<x<l} = 0;\ \lambda_{ж}\left.\frac{\partial\vartheta}{\partial r}\right|_{r=\frac{d}{2}-0} = q_F;\\ &\left|u\right|_{x=0;\ 0<x<\frac{d}{2}} = u_1. \end{aligned}\right\} \qquad (18.2)$$

Here we have four unknowns ϑ, u_r, u_x, ϑ_F and the values of the parameters λ_{zh}, a_{pzh}, ν, $\underline{d}$, l, q_F, u_1 are specified by the uniqueness conditions.

Similarity Criteria for Heat-Transmission Processes in Tubes

To transform Eqs. (18.1) into dimensionless form we introduce the dimensionless temperature $\Theta \equiv \vartheta/\Delta t_0$, velocity $\vec{U} \equiv \vec{u}/u_1$, radius $R \equiv r/r_0$, and abscissa $X \equiv x/r_0$. From these relations we obtain

$$\vartheta = \Delta t_0 \cdot \Theta;\ u = u_0 \cdot U;\ r = r_0 R;\ x = r_0 \cdot X. \tag{18.3}$$

Substituting Eq. (18.3) into Eqs. (18.1) and transforming them, we obtain

$$\left.\begin{gathered}
\frac{\vartheta_F \cdot d}{\lambda_\infty \cdot \Delta t_0} = 2\frac{\partial \Theta}{\partial R}\Big|_{R=1-0};\quad \frac{u_1 \cdot d}{a_{p\infty}} \cdot (U,\ \operatorname{grad}\Theta) - \nabla^2\Theta = 0; \\
\operatorname{Rot}\left[\frac{u_0 \cdot d}{\nu}(U,\ \operatorname{grad})\,U - \nabla^2 U\right] = 0;\quad \operatorname{Div} U = 0; \\
U\Big|_{R=1;\ 0<X<\frac{l}{d}} = 0;\ \Theta\Big|_{R=1;\ 0<X<\frac{l}{d}} = 1;\ U_1 = 1; \\
\Theta_{x1} = 0.
\end{gathered}\right\} \tag{18.4}$$

The solution of the problem (18.4) consists of the following criteria:

$$\frac{q_F \cdot d}{\lambda_\infty \cdot \Delta t} = f\left(X,\ \frac{u_0 \cdot d}{\nu},\ \frac{u_1 \cdot d}{a_{p\infty}}\right) \tag{18.5}$$

or

$$Nu_0 = f\,(X,\ Re,\ Pe).$$

The derivation of Eq. (18.5) is a proof of the Newton-Rikhman law for heat transmission in tubes. Indeed, it follows from this equation that the local value of the heat rate q_F is directly proportional to the initial temperature difference $\Delta t_0 \equiv t_F - t_{zhl}$

$$q_F = \frac{\lambda_\infty}{d} \cdot f\left(X,\ \frac{u_1 \cdot d}{\nu},\ \frac{u_1 \cdot d}{a_{p\infty}}\right) \cdot (t_F - t_\infty). \tag{18.6}$$

The value of the proportionality coefficient should be regarded as a heat-transmission coefficient α_0 referred to the initial temperature difference

$$q_F = \alpha_0 (t_F - t_\infty).$$

The first characteristic heat-transmission process in tubes is employed principally in the calculation of heat transmission in heat-

exchange apparatus. These calculations are best carried out with the aid of the heat-transmission coefficient α, referred to the local temperature difference at any section of the tube at a distance $\underline{x}$ from the entrance

$$\Delta t\,(x) = t_F\,(x) - \bar{t}^{f}_{ж}\,(x).$$

Here $\bar{t}^{f}_{zh}$ is the calorimetric temperature of the fluid stream, averaged over the cross section of the tube, at any distance $\underline{x}$ from the entrance

$$\bar{t}_{ж}\,(x) \equiv \frac{2}{u_1 \cdot r_0^2} \cdot \int^{1} u_x \cdot (r,\, x) \cdot r \cdot t\,(r,\, x) \cdot dr.$$

The similarity equation (18.5) is therefore best transformed in such a way that it contains the local temperature difference Δt. Multiplying and dividing the left half of (18.5) by Δt, we obtain

$$Nu\,\frac{\Delta t}{\Delta t_0} = f\,(X,\ Re,\ Pe), \qquad (18.7)$$

where $Nu \equiv q_F \cdot d/\lambda_{zh} \cdot \Delta t$ is the Nusselt number, calculated for the local temperature drop Δt. The ratio $\Delta t/\Delta t_0$, as follows from (18.4), depends on the criteria X, Pe, and Re:

$$\frac{\Delta t}{\Delta t_0} = f_1\,(X,\ Re,\ Pe). \qquad (18.8)$$

Substituting (18.8) in (18.7) we obtain

$$Nu = f_2\,(X,\ Re,\ Pr). \qquad (18.9)$$

The derivation of Eq. (18.9) can be regarded as a proof of the Newton-Rikhman law for the local temperature difference. Indeed, it follows from (18.9) that

$$q_F = \alpha \cdot \left(t_F - \bar{t}^{f}_{ж}\right), \qquad (18.10)$$

where $\alpha \equiv (\lambda_{zh}/d) f_1(X, Re, Pe)$ is the heat-transmission coefficient referred to the local temperature difference Δt.

In calculation of heat transfer in heat-exchange apparatus, one

usually employs the value of the heat-transmission coefficient $\overline{\alpha}^{l}$ averaged over the length of the tube. This quantity is introduced in such a way, that for average values $\overline{q_F}^{l}$ and $\overline{\Delta t}^{l}$ the Newton-Rikhman law is satisfied:

$$\overline{q_F}^{l} = \overline{\alpha}^{l} \cdot \overline{\Delta t}^{l}. \qquad (18.11)$$

Here

$$\overline{q_F}^{l} \equiv \frac{1}{l} \cdot \int_0^l q_F(x) \cdot dx \; ; \; \overline{\Delta t}^{l} \equiv \frac{1}{l} \cdot \int_0^l \Delta t(x) \cdot dx. \qquad (18.12)$$

It must be emphasized that the average value of the heat-transmission coefficient $\overline{\alpha}^{l} = \overline{q_F}^{l} / \overline{\Delta t}^{l}$ is generally speaking not equal to the mean-integral value $\overline{\alpha}^{l} \neq \frac{1}{l} \int_0^l \alpha dx$. Let us determine expressions for the averages $\overline{q_F}^{l}$ and $\overline{\Delta t}^{l}$. For this purpose we average Eq. (18.6) over l

$$\overline{q_F}^{l} = \frac{\lambda_\infty}{d} \Delta t_0 \frac{1}{l} \cdot \int_0^l f\left(\frac{x}{d}, Re, Pr\right) \cdot dx =$$

$$= \frac{\lambda_\infty \Delta t_0}{d} \cdot f_3(L, Re, Pr). \qquad (18.13)$$

We carry out analogous averaging over l for the local temperature drop (18.8):

$$\overline{\Delta t}^{l} = \Delta t_0 \cdot f_4(L, Re, Pr).$$

Dividing Eq. (18.13) by the last equation and transforming the former, we obtain an expression for the Newton-Rikhman law for the average values of the heat rate and the temperature difference

$$\overline{q}^{l} = \frac{\lambda_\infty}{d} \cdot f_5(L, Re, Pr) \cdot \overline{\Delta t}^{l}.$$

Thus, the heat-transmission coefficient averaged over l is

$$\overline{\alpha}^{l} = \frac{\lambda_\infty}{d} \cdot f_5(L, Re, Pr),$$

and the similarity condition has in this case the form

$$\overline{Nu}^{L} \equiv \frac{\overline{q}^{l} \cdot d}{\overline{\Delta t}^{l} \cdot \lambda_\infty} = f_5(L, Re, Pr). \qquad (18.14)$$

Let us consider the similarity criteria for the second heat-transmission process, when a constant heat rate is maintained on the wall of the tube and it is required to determine $t_F(x)$.

To transform Eqs. (18.2) to a dimensionless form we introduce as the temperature scale a certain finite positive quantity ϑ_0. Then $\vartheta = \vartheta_0 \cdot \Theta$. The transformations of the quantities $\vec{u}$, $\underline{r}$, $\underline{x}$ remain unchanged [see Eq. (18.3)]. After transforming Eqs. (18.2) we obtain

$$\left.\begin{gathered}\frac{\vartheta_F}{\vartheta_0} = \Theta|_{R=1}; \qquad \frac{u_1 \cdot d}{a_{p\infty}} \cdot (\mathbf{U}, \operatorname{grad}\Theta) - \nabla^2\Theta = 0; \\ \operatorname{Rot}\left|(\mathbf{U}, \operatorname{grad})\,\mathbf{U}\frac{u_0 \cdot d}{\nu} - \nabla^2\mathbf{U}\right| = 0; \ \operatorname{Div}\mathbf{U} = 0; \\ \mathbf{U}|_{R=1;\ 0<X<L} = 0; \ 2\frac{\partial\Theta}{\partial R}\Big|_{R=1-0} = \frac{q_F \cdot d}{\lambda_\infty \cdot \vartheta_0}; \\ \mathbf{U}|_{X=0;\ 0<R<1} = 1; \ \ \Theta|_{X=0;\ 0<R<1} = 0.\end{gathered}\right\} \qquad (18.15)$$

The solution of Problem (18.15) contains the following dimensionless quantities:

$$\frac{\vartheta_F}{\vartheta_0} = \varphi_1\left(X,\ Re,\ Pe,\ \frac{q_F \cdot d}{\lambda_\infty \cdot \vartheta_0}\right). \qquad (18.16)$$

However, the surface temperature ϑ_F of the tube cannot depend on the arbitrary scale ϑ_0. Therefore the dimensionless quantities containing ϑ_0 should enter into the solution of the problem in the form of the ratio

$$\frac{\dfrac{\vartheta_F}{\vartheta_0}}{\dfrac{q_F \cdot d}{\lambda_\infty \cdot \vartheta_0}} = \frac{\vartheta_F \cdot \lambda_\infty}{q_F \cdot d},$$

which does not contain ϑ_0. Consequently, Eq. (18.16) assumes the form

$$\frac{\vartheta_F \cdot \lambda_\infty}{q_F \cdot d} = \varphi_2(X,\ Re,\ Pe). \qquad (18.17)$$

The left half of this equation is the reciprocal of the Nusselt number Nu_0, in which the heat-transmission coefficient $\alpha_0 = q_F/\vartheta_F$ is referred to the initial temperature head $\vartheta_F = t_F - t_0$. In contrast to Eq. (18.5), however, the initial temperature head here contains the sought quantity

t_F, which varies along the tube. Here, the heat rate q_F is an assigned quantity.

It is not advisable to change to the local temperature difference $t_F - \bar{t}_{zh}$ in Eq. (18.17), for it is necessary to determine the wall temperature t_F for a specified heat rate on the wall of the tube.

The temperature is best determined directly from Eq. (18.17), bypassing the heat-transmission coefficient. Therefore it is necessary to regard the left half of (18.17) as a dimensionless temperature

$$\Theta_q = \frac{t_F \cdot \lambda_{ж}}{q_F \cdot d}.$$

It is not advantageous to average Eq. (18.17) over the length, since for reliable operation of the reactor channel the value of the average wall temperature is of no significance. It is most important to find under these conditions the greatest local temperature, which can cause destruction of the wall. Along with the quantity Θ_q, it is convenient to introduce the dimensionless temperature $\Theta_F =$ $= (t_F - t_{zh1})/(t_{zh2} - t_{zh1})$ [10b]. Here t_{zh2} is the fluid temperature, averaged over the cross section, at the exit from the tube ($x = l$):

$$t_{ж2} = \frac{2}{u_1 \cdot r_0^2} \cdot \int_0^{r_0} u_x(l, r) \cdot r \cdot t(l, r) \cdot dr.$$

To determine the connection between Θ_q and Θ_F we use the heat-balance equation for the flow of liquid in the tube

$$\frac{\pi d^2}{4} \cdot u_1 \cdot C_{p\,ж} \cdot (t_{ж2} - t_{ж1}) = q_F \cdot \pi \cdot d \cdot l.$$

If we determine $q_F d$ from the last relation and substitute this quantity into the expression for Θ_q, we obtain after transformation

$$\Theta_q = \Theta_F L/4Pe = \varphi_2(X, Re, Pe).$$

Consequently, the similarity equation for Θ_F will have the form

$$\Theta_F \cdot L = \varphi_3(X, Re, Pe). \qquad (18.18)$$

The use of the dimensionless temperature Θ_F has many advantages

as compared with Θ_q.

Indeed, the criterion Θ_F contains the longitudinal temperature drop in the liquid stream $\delta t_{zh} = t_{zh2} - t_{zh1}$.

The quantity δt_{zh} is determined by the values of the velocity u_1, the specific heat of the fluid C_{pzh}, and the diameter of the tube $\underline{d}$ for specified q_F and l. Consequently, the criterion Θ_F already takes into account the influence of these quantities on the wall temperature $t_F(x)$, and therefore the dependence of Θ_F on the Reynolds and Peclet number in Eqs. (18.18) is weaker than the dependence of Θ_q on the same numbers in Eqs. (18.17). This circumstance enables us to decrease somewhat the accuracy of the measurement of the quantities u_1, ν, and a_{pzh}, of which the Reynolds and Peclet numbers are made up.

We note that the dimensionless quantity Θ_F can be obtained also directly from Eqs. (18.2), if one assumes as the temperature scale the longitudinal temperature drop δt_{zh} in the liquid stream.

Similarity Criteria for Heat Transmission of Liquids Having Pr << 1, Pr ~ 1, and Pr >> 1

Let us consider the similarity conditions for the process of heat transmission in tubes for the flow of liquid metals, which have rather small Prandtl numbers (Pr << 1). It was shown above that in this case the thickness of the thermal boundary layer greatly exceeds the thickness of the hydrodynamic layer $k >> \delta$ (see §17).

As applied to a turbulent stream of a liquid having Pr << 1, this means that Pe << Re and that the uneven distribution of the temperature extends over practically all sections of the stream, while the uneven velocity distribution is concentrated principally in a comparatively thin layer near the wall. In this layer the forces of viscous friction in the liquid are much larger than the inertia forces. However, the fundamental change in temperature occurs far away from the

tube walls, in the core of the stream. Here the action of the viscosity forces is much smaller than the action of the inertia forces. Therefore in the region of the principal variation of the temperature in the stream it is possible to neglect the action of the viscosity forces. Consequently, when Pr << 1 the Reynolds number drops out of consideration, and the similarity equations obtained for the heat-transmission process in tubes assume the following form:

$$\left.\begin{aligned} Nu_0 &= f_1(X,\ Pe);\\ Nu &= f_2(X, Pe);\\ \Theta_q &= f_3(X, Pe);\\ \Theta_F &= f_4(X, Pe);\\ {}^{L}\overline{Nu} &= f_5(L,\ Pe). \end{aligned}\right\} \qquad (18.19)$$

Experimental investigations of the heat transmission of molten metals confirm these considerations (see §20).

When a liquid with Prandtl number on the order of unity (Pr ~ 1) flows through the tube, then $\nu \sim a$. Therefore both thermodynamic parameters influence equally the heat transmission, and the defining similarity criteria Re and Pe enter into the similarity equations in the form of the product $Re \cdot Pe$ or $Re \cdot Pr^{0.5}$.

Finally, for the limiting case connected with very large values of the Prandtl number (Pr >> 1), we have $\nu >> a$. In this case the temperature conductivity a_{pzh}, which characterizes convective heat exchange in the stream, drops out of consideration and the similarity is determined by the two criteria X and Re or X and $Re \cdot Pr^{0}$.

Thus, the defining criteria can be written in the following approximate form:

$$X \text{ and } Re \cdot Pr^{n},$$

where $n \to 1$ when $Pr \to 0$; $n \sim 0.5$ when $Pr \sim 1$, and $n \to 0$ when $Pr \to \infty$. Consequently, the approximate similarity equation assumes the form

$$Nu \approx f(X,\ Re \cdot Pr^{n}).$$

Similarity Criteria for Heat Transmission of Gases at High Values of Variable Heat Rate on the Tube Wall [10b]

Under real cooling conditions of the active zone of nuclear power reactors, a relatively high rate q_F is maintained on the channel walls, and its value varies along the length of the channel (Fig. 59). Under these conditions the thermodynamic parameters of the fluid may change in magnitude appreciably over the working interval of temperature variation. Let us consider the similarity conditions for the simplest case of cooling of the active zone, when gas with low subsonic speed flows through a channel with round cross section. Let the thermodynamic parameters of the gas depend only on the temperature t and be independent of the pressure

$$\lambda_ж = \lambda_ж(t);\ \mu = \mu(t);\ C_p = C_p(t);\ \varrho = \varrho(t).$$

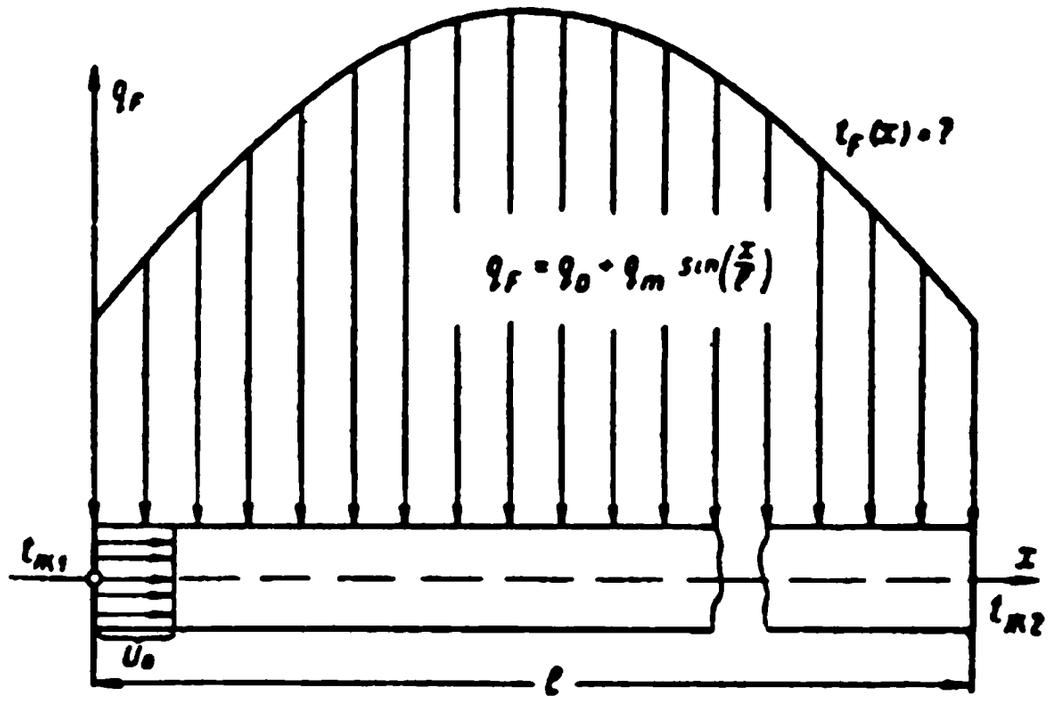

Fig. 59. Distribution of heat rate on the cooled surface of a channel in the active zone of a power reactor.

The gas flows sufficiently slowly so that the change in pressure in the stream is insignificant and the heat-transmission process can be regarded as being approximately isobaric, p ≈ const. The mathematical description of this heat-transmission process has the following form

$$\left.\begin{gathered} t_F = t\big|_{r=\frac{d}{2}};\ C_{p\,\text{ж}}(t)\cdot(\mathbf{u},\ \operatorname{grad} t) - \operatorname{div}\lambda_{\text{ж}}\cdot(t)\operatorname{grad} t = 0; \\ \operatorname{rot}\left[\varrho(t)\cdot(\mathbf{u},\ \operatorname{grad})\mathbf{u} - \operatorname{div}\mu\cdot(t)\operatorname{grad}\mathbf{u}\right] = 0;\ \operatorname{div}\varrho\cdot\mathbf{u} = 0; \\ \mathbf{u}\big|_{r=\frac{d}{2};\ 0<x<l} = 0;\ \lambda_{\text{ж}}\cdot(t)\cdot\frac{\partial t}{\partial r}\Big|_{\frac{d}{2}-0} = q_F(x); \\ \left|\mathbf{u}\right|_{x=0;\ 0<r<\frac{d}{2}} = \mathbf{u}_1;\ t\big|_{x=0;\ 0<r<\frac{d}{2}} = t_{\text{ж}1}. \end{gathered}\right\} \qquad (18.20)$$

The system of equations (18.20) contains four unknowns t_F, $t(x, r)$, u_x, u_r and consists of four equations. The values of the parameters $\underline{d}$, l, u_1, t_{zhl}, $\lambda_{zh}(t)$, $\mu(t)$, $\rho(t)$, and $C_p(t)$ are specified.

To determine the similarity conditions we introduce the following dimensionless quantities:

$$\Theta \equiv \frac{t - t_{\text{ж}1}}{t_{\text{ж}2} - t_{\text{ж}1}};\ R \equiv \frac{r}{r_0} = \frac{2r}{d};\ X = \frac{x}{d};\ \mathbf{U} = \frac{\mathbf{u}}{u_1};$$

$$\Lambda \equiv \frac{\lambda_{\text{ж}} - \lambda_{\text{ж}1}}{\lambda_{\text{ж}2} - \lambda_{\text{ж}1}};\ C_{p\,\text{ж}} \equiv \frac{C_{p\,\text{ж}} - C_{p\,\text{ж}1}}{C_{p\,\text{ж}2} - C_{p\,\text{ж}1}};\ M \equiv \frac{\mu - \mu_1}{\mu_2 - \mu_1};$$

$$\mathrm{P} \equiv \frac{\varrho_{\text{ж}} - \varrho_{\text{ж}1}}{\varrho_{\text{ж}2} - \varrho_{\text{ж}1}}.$$

From these relations we obtain

$$\left.\begin{gathered} t = t_{\text{ж}1} + \delta t_{\text{ж}}\cdot\Theta;\ x = d\cdot X;\ \mathbf{u} = u_1\cdot\mathbf{U};\ r = \frac{d}{2}\cdot R; \\ \lambda_{\text{ж}} = \lambda_{\text{ж}1} + \Delta\lambda_{\text{ж}}\cdot\Lambda;\ C_{p\,\text{ж}} = C_{p\,\text{ж}1} + \Delta C_{p\,\text{ж}};\ \mu = \mu_1 + \Delta\mu\cdot M; \\ \varrho = \varrho_1 + \Delta\varrho\cdot\varphi. \end{gathered}\right\} \qquad (18.21)$$

It is assumed further that each of the dimensionless thermodynamic parameters Λ, C_p, M, and ρ are single-valued functions of the dimensionless temperature:

$$\Lambda = \Lambda(\Theta);\ C_{p\,\text{ж}} = C_p(\Theta);\ M = M(\Theta);\ \varrho = \varrho(\Theta). \qquad (18.22)$$

Let us reduce the mathematical description of the process (18.20) to dimensionless form. For this purpose we substitute Relations (18.21) and (18.22) into Eqs. (18.20). After the transformation we obtain

$$\left.\begin{gathered} \Theta_F = \Theta\big|_{R=\frac{1}{2}};\ \left[1 + \left(\frac{C_{p\,\text{ж}2}}{C_{p\,\text{ж}1}} - 1\right)\cdot C_{p\,\text{ж}}(\Theta)\right]\times \\ \times\, Pe_1\cdot(\mathbf{U},\ \operatorname{Grad}\Theta) - \operatorname{Div}\left[1 + \left(\frac{\lambda_{\text{ж}2}}{\lambda_{\text{ж}2}} - 1\right)\times\right. \\ \left.\times\,\Lambda(\Theta)\right]\cdot\operatorname{Grad}\Theta = 0; \end{gathered}\right\} \qquad (18.23)$$

$$\left.\begin{aligned}&\mathrm{Rot}\left\{\left[1+\left(\frac{\rho_{ж2}}{\rho_{ж1}}-1\right)\right]\cdot\varphi(\Theta)\cdot Re_1\cdot(\mathbf{U},\ \mathrm{Grad})\,\mathbf{U}-\right.\\&\left.-\mathrm{Div}\left[1+\left(\frac{\mu_2}{\mu_1}-1\right)\cdot M(\Theta)\cdot\mathrm{Grad}\,\Theta\right]\right\}=0;\end{aligned}\right| \tag{18.23}$$

$$\left.\begin{aligned}&|\mathbf{U}|_{X=0;\ 0<R<0.5}=1;\ \Theta|_{X=0;\ 0<R<0.5}=0;\\&|\mathbf{U}|_{R=0.5;\ 0<X<L}=0;\ \left[1+\left(\frac{\lambda_{ж2}}{\lambda_{ж1}}-1\right)\right]\times\\&\times\Lambda(\Theta)\cdot\frac{\partial\Theta}{\partial R}\Big|_{R=\frac{1}{2}-0}=\frac{q_F(x)\cdot d}{\lambda_{ж1}\cdot(t_{ж2}-t_{ж1})}\cdot\end{aligned}\right\} \tag{18.24}$$

The solution of the problem (18.24) consists of the following dimensionless quantities:

$$\Theta_F=F\left(X,\ Re_1,\ Pe_1,\ \frac{q_F(x)\cdot d}{\lambda_{ж1}\cdot(t_{ж2}-t_{ж1})},\ \frac{\lambda_{ж2}}{\lambda_{ж1}},\ \frac{\mu_2}{\mu_1},\ \frac{C_{p\,ж2}}{C_{p\,ж1}},\ \frac{\rho_{ж2}}{\rho_{ж1}}\right). \tag{18.25}$$

The similarity equation (18.25) is valid for any liquid whose thermodynamic parameters depend only on the temperature and do not depend on the pressure. For gases the equation simplifies, for in this case the thermodynamic parameters can be regarded approximately as being power functions of the temperature. Then

$$\frac{\lambda_{ж2}}{\lambda_{ж1}}\approx\left(\frac{T_{ж2}}{T_{ж1}}\right)^{n_\lambda};\ \frac{\mu_2}{\mu_1}\approx\left(\frac{T_{ж2}}{T_{ж1}}\right)^{n_\mu};\ \frac{C_{p\,ж2}}{C_{p\,ж1}}\approx\left(\frac{T_{ж2}}{T_{ж1}}\right)^{n_C};$$

$$\frac{\rho_{ж2}}{\rho_{ж1}}\approx\left(\frac{T_{ж2}}{T_{ж1}}\right)^{n_\rho},$$

where T_{zh1} and T_{zh2} are the temperatures of the fluid entering and leaving the channel (^{o}K). Substituting these relations into (18.25) we obtain

$$\Theta_F=F_1\left(X,\ Re_1,\ Pe_1,\ \frac{q_F(x)\cdot d}{\lambda_{ж1}\cdot(t_{ж2}-t_{ж1})},\ \frac{T_{ж2}}{T_{ж1}}\right). \tag{18.26}$$

It should be noted that in the last equations we left out the exponents n_λ, n_μ, n_C, and n_ρ, which should be regarded as similarity criteria. Consequently, Eq. (18.26) extends over heat-transmission processes only in the group of gases in which n_λ = idem; n_C = idem; n_μ = = idem; n_ρ = idem.

Equation (18.26) can be simplified if the criterion

$q_F(x)\cdot d/\lambda_{zh1}(t_{zh2} - t_{zh1})$ is calculated from the heat-balance equation for the channel. After the calculations we obtain

$$\frac{q_F(x)\cdot d}{\lambda_{ж1}\cdot(t_{ж2}-t_{ж1})} = Q_F(X)\cdot\frac{Pe_1}{4L}\cdot\int_0^1 C_p(\Theta)\cdot d\Theta, \qquad (18.27)$$

where $Q_F(X) \equiv q_F(x)/\overline{q}_F^{\,l}$ is the dimensionless distribution of the heat rate along the channel and on its cooled surface. Substitution of Eq. (18.27) in (18.26) yields

$$\Theta_F = F_2\left(X,\ Re_1,\ Re_1,\ L,\ \frac{T_{ж2}}{T_{ж1}}\right)$$

or

$$\Theta_F = F_2\left(X,\ Re_1,\ Pr_1,\ L,\ \frac{T_{ж2}}{T_{ж1}}\right). \qquad (18.28)$$

This example shows clearly how the analysis of the similarity conditions of heat-transmission processes in tubes becomes more complicated if the thermodynamic parameters of the fluid depend on the temperature. The similarity criteria obtained, which enter in Eq. (18.28), have a whole series of singularities.

The Reynolds number $Re_1 = u_1 d/\nu_1$ and the Peclet number $Pe_1 \equiv \equiv u_1 d/a_{zh1}$ are defined for the thermal and hydrodynamic conditions of entrance into the channel, since they contain the quantities a_{pzh1}, u_1, and ν_1. The dependence of the thermodynamic parameters on the temperature is taken into account here by the ratio of the fluid temperatures at the entrance and exit of the channel, T_{zh2}/T_{zh1}.

The criterion $\Theta_F \equiv (t_F - t_{zh1})/(t_{zh2} - t_{zh1})$ contains the longitudinal temperature drop $t_{zh2} - t_{zh1}$, the size of which depends on the velocity of the fluid. Consequently, Θ_F already takes partial account of the influence of the velocity on the temperature t_F of the cooled surface. Therefore the dependence of Θ_F on Re_1 and Pe_1 should be weaker than the dependence of the Nusselt number on these criteria.

§19. PRINCIPLES OF HEAT-TRANSMISSION THEORY IN TUBES AND CHANNELS FOR TURBULENT FLOW OF THE FLUIDS

The Fourier-Ostrogradskiy Differential Equation for Turbulent Flow of Fluid in a Tube

The turbulent flow of liquid differs from the laminar flow in the presence of undamped pulsations of the velocity $\vec{u}'$ and of the temperature t', which have a distorted and random character (Fig. 60). If the

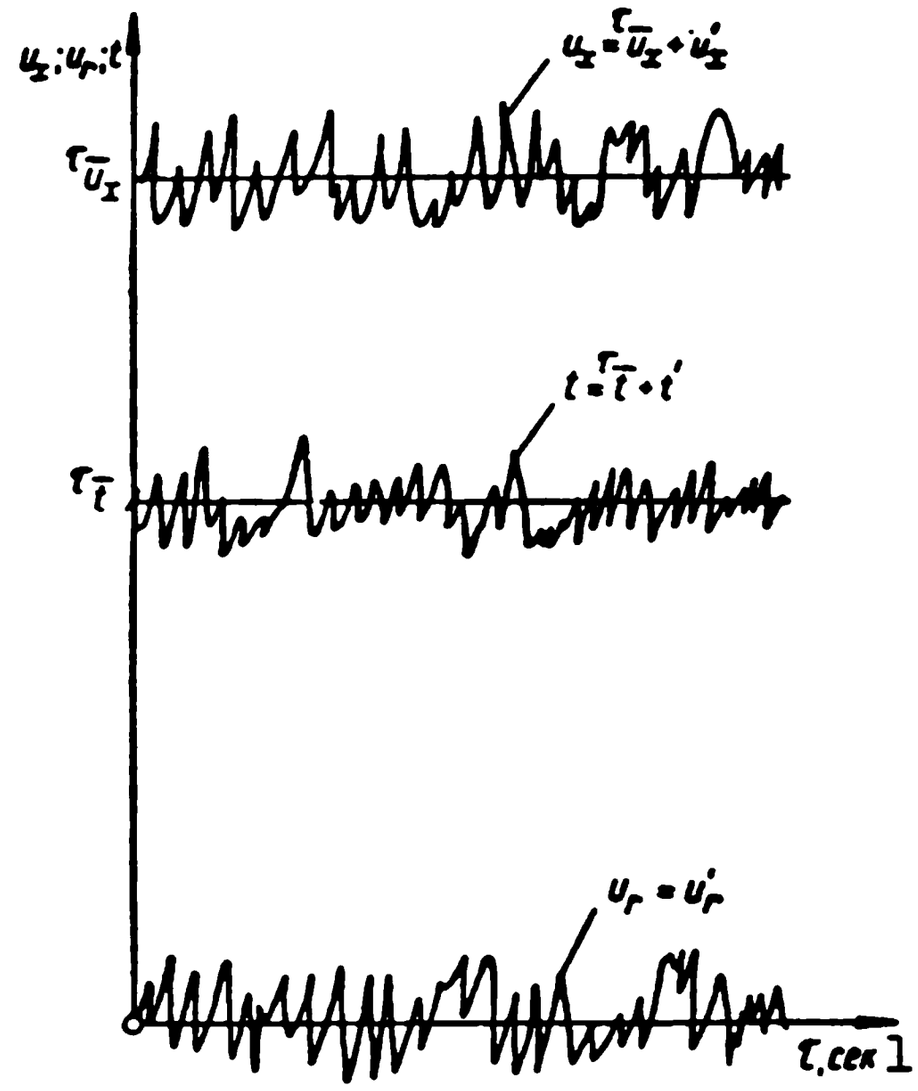

Fig. 60. Temperature and velocity pulsations in a turbulent isothermal fluid stream. 1) sec.

pulsations do not attenuate in time, and the average velocity maintains a constant value, then the turbulent flow is called stationary in the mean.

The turbulent pulsations of the velocity and the temperature give rise to additional momentum and heat transport processes. The turbulent transport of momentum transversely to the fluid stream causes the appearance of additional friction stresses. They are called Reynolds stresses. The action of these stresses is taken into account by introducing the coefficient of turbulent viscosity ν_t. This coefficient depends on the velocity of the stream, on its dimensions, on the viscos-

ity of the fluid, on the coordinates of the point, and on the direction to it, and is therefore not a physical parameter.

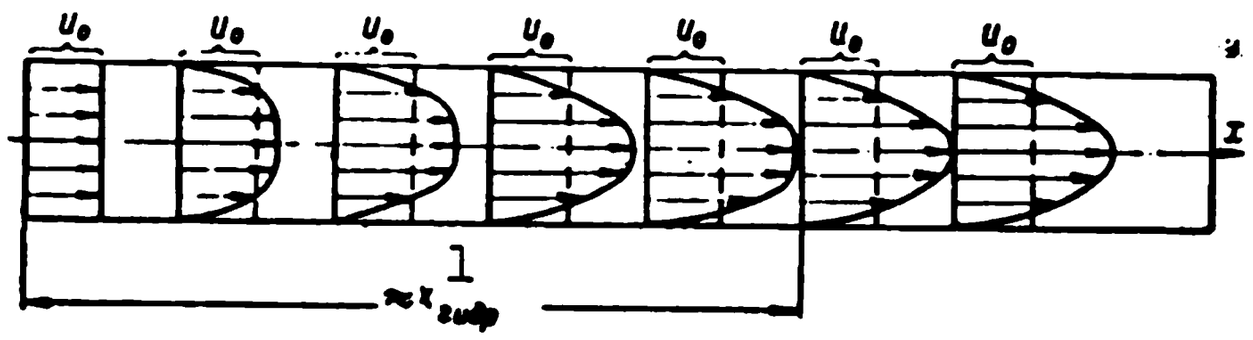

Fig. 61. Hydrodynamic stabilization of a fluid stream in a tube. 1) Hydrodynamic.

Turbulent transport of heat transversely to the fluid stream improves heat transmission. This effect is taken into account by reducing a turbulent temperature conductivity coefficient a_t, which likewise is not a physical parameter, because it depends on the same quantities as ν_t, and also on the thermal conductivity and the specific heat of the fluid.

With the aid of the coefficient of turbulent temperature conductivity a_t it is possible to write down the Fourier-Ostrogradskiy differential equation for the temperature distribution in a stationary turbulent stream, stabilized in the mean, of a homogeneous fluid in a tube

$$\overline{u}_x^{\tau}(r)\cdot\frac{\partial\overline{t}^{\tau}(x_0,r)}{\partial x}=\frac{1}{r}\cdot\frac{\partial}{\partial r}\left[r\cdot(a_{pm}+a_{rt})\cdot\frac{\partial\overline{t}^{\tau}(x,r)}{\partial r}\right]+$$
$$+\frac{\partial}{\partial x}\left[(a_{pm}+a_{xt})\cdot\frac{\partial\overline{t}^{\tau}(x,r)}{\partial x}\right]. \qquad (19.1)$$

In this equation $\overline{t}^{\tau}(x, r)$ is the time-averaged fluid temperature at the point with coordinates $\underline{x}$ and $\underline{r}$; a_{rt} and a_{xt} are the coefficients of turbulent temperature conductivity transverse to and along the stream, respectively; $\overline{u}_x^{\tau}(r)$ is the time-averaged value of the axial velocity at a point located a distance $\underline{r}$ from the tube axis.

The quantity $\overline{u}_x^{\tau}(r)$ depends on $\underline{r}$ and does not depend on $\underline{x}$. Such a velocity distribution is established at a certain distance $\underline{x}$ from the entrance into the tube (Fig. 61). The quantity x_{gidr} is called the in-

itial or entrance portion of the hydrodynamic stabilization, while the fluid stream in the sections $x \geq x_{gidr}$ is called stabilized. Consequently, Eq. (19.1) is written out for a stabilized stream. It contains four unknown quantities $\overline{t}^{\tau}$, $\overline{u}_x^{\tau}$, a_{rt}, and a_{xt}.

The distribution of the velocity $\overline{u}_x^{\tau}$ over the section of the stabilized stream of a homogeneous liquid in a tube has by now been investigated quite thoroughly [71, 72]. Therefore the function $\overline{u}_x^{\tau}(r)$ should be regarded as specified. The second term in the right half of (19.1) is usually neglected (for more details see below). Therefore the value of the longitudinal turbulent temperature conductivity a_{xt} drops out of consideration.

To determine the transverse turbulent temperature conductivity a_{rt} we use the Fourier-Ostrogradskiy law for the vector of convective heat exchange in a homogeneous fluid

$$q = -\lambda_{ж} \operatorname{grad} t + C_{p\,ж} t_{ж} u. \qquad (19.2)$$

In a turbulent stream, the instantaneous values of each of the quantities $\vec{q}$, t, and $\vec{u}$ can be represented in the form of a sum consisting of the time-averaged values $\overline{\vec{q}}^{\tau}$, $\overline{t}^{\tau}$, and $\overline{\vec{u}}^{\tau}$ and of the turbulent pulsations $\vec{q}'$, t', and u'_x. Consequently,

$$q = \overline{q}^{\tau} + q'; \; t = \overline{t}^{\tau} + t'; \; u = \overline{u}^{\tau} + u'.$$

Substituting these expressions into (9.27) and making suitable transformations, we obtain

$$\overline{q}^{\tau} + q' = -\lambda_{ж} \cdot \operatorname{grad}\left(\overline{t}^{\tau} + t'\right) + C_{p ж} \cdot \left(\overline{t}^{\tau} \cdot \overline{u}^{\tau} + \overline{t}^{\tau} \cdot u' + t' \cdot \overline{u} + t' \cdot u'\right).$$

Let us average the last equation with respect to the time τ. The average values of the pulsating components vanish, and we obtain

$$\overline{q}^{\tau} = -\lambda_{ж} \cdot \operatorname{grad} \overline{t}^{\tau} + C_{p ж} \cdot \overline{t}^{\tau} \cdot \overline{u}^{\tau} + C_{p ж} \overline{t' \cdot u'}^{\tau}. \qquad (19.3)$$

Comparing Eqs. (19.2) and (19.3) we reach the conclusion that the

equation for the averaged vector of convective heat exchange (19.3) contains a term $C_{pzh}\ \overline{t' \cdot \vec{u}'}$, which takes into account the additional heat transport arising in the turbulent stream as a result of the action of the turbulent pulsations of velocity and of temperature.

Consequently, the average value of the turbulent heat-transport vector is

$$\bar{q}_\tau = C_{p\rho} \cdot \overline{t' \cdot u'}. \qquad (19.4)$$

Let us write this equation for turbulent heat transport transverse to the fluid stream in the tube:

$$\bar{q}_{r,\tau} = C_{p\rho} \cdot \overline{t' \cdot u'_r}. \qquad (19.5)$$

In analogy with the Biot-Fourier law we introduce into consideration the coefficient of turbulent heat conductivity λ_{rt} in the radial direction:

$$q_{r,\tau} = -\lambda_{r,\tau} \cdot \frac{\partial \bar{t}}{\partial r}. \qquad (19.6)$$

It must be noted that Eq. (19.6) is not a law, and the quantity λ_{rt} is not a physical parameter. Comparing Eqs. (19.5) and (19.6) we obtain the following expression for the turbulent temperature conductivity transverse to the stream:

$$a_{r,\tau} = -\frac{\overline{t' \cdot u'_r}}{\frac{\partial \bar{t}}{\partial r}}. \qquad (19.7)$$

In accordance with the assumption of Prandtl, the magnitude of the turbulent pulsation t' is directly proportional to the gradient of the averaged temperature transverse to the stream:

$$t' \approx l_r \frac{\partial \bar{t}}{\partial r}. \qquad (19.8)$$

The quantity l_r is a distance between the two layers of liquid r and $r + l$, over the extent of which the final volumes of the fluid

(the "turbulent batches") transfer the heat or momentum transversely to the stream. It is assumed here that the reserves of heat or momentum in the "turbulent batches" remain constant as they are displaced by a distance l_r.

Substituting Eq. (19.8) into Eq. (19.7) and making suitable transformations, we obtain

$$a_{r,\tau} = -\overline{l_r u'_r}. \qquad (19.9)$$

Let us carry out an analogous calculation for turbulent friction in the stream. We average over the time the Navier-Stokes differential equations, obtaining additional terms that characterize the additional stresses caused by the turbulent momentum transfer across the stream. In particular, the turbulent friction stress τ_t has the following form:

$$\tau_\tau = -\rho_m \cdot \overline{u'_x \cdot u'_r}. \qquad (19.10)$$

In analogy with Newton's law for viscous friction of fluids, we introduce the coefficient of turbulent viscosity

$$\tau_\tau \equiv \mu_{r,\tau} \cdot \frac{\overline{d u_x(r)}}{dr}. \qquad (19.11)$$

The latter equation is not a law, and the quantity μ_{rt} is not a physical parameter.

Comparing Eqs. (19.11) and (19.10) we obtain

$$\nu_{r,\tau} = -\frac{\overline{u'_x \cdot u'_r}}{\overline{d u_x}}. \qquad (19.12)$$

In accordance with the Prandtl assumption, the magnitude of the turbulent pulsation u'_x is directly proportional to the gradient of the averaged velocity transverse to the stream:

$$u'_x \approx l_r \cdot \frac{\overline{d u_x}}{dr}, \qquad (19.13)$$

where l_r has the same meaning as in Formula (19.8). Substituting For-

mula (19.13) in Formula (19.12) we obtain

$$\nu_{r,t} = -\overline{l_r \cdot u_r'}. \qquad (19.14)$$

From a comparison of Eqs. (19.14) and (19.10) it follows that the coefficients of turbulent temperature conductivity and viscosity have identical values:

$$a_{r,t} = \nu_{r,t} \text{ or } Pr_t = 1. \qquad (19.15)$$

It is necessary to emphasize that the equality $Pr_t = 1$ is the consequence of the assumption that the reserves of heat and momentum in the "turbulent batches" remain unchanged as they are displaced through a distance l_r.

Thus, the determination of the turbulent temperature conductivity a_{rt} reduces to a calculation of the turbulent viscosity ν_{rt}. The latter is easy to do if one knows the velocity distribution $\bar{u}_x(r)$ over the cross section of the tube.

For a stabilized fluid stream in a tube, the pressure drop Δp is consumed in overcoming the friction forces in the stream. Therefore for a cylindrical stream region with radius r we have

$$\Delta p \cdot \pi \cdot r^2 = 2\pi \cdot r \cdot \tau_r.$$

If $r = r_0$, then

$$\Delta p \cdot \pi \cdot r_0^2 = 2\pi \cdot r_0 \cdot \tau_p.$$

Dividing the first equation by the second we obtain an equation for the linear distribution of the tangential stresses in the stabilized stream (Fig. 62)

$$\frac{\tau_r}{\tau_p} = \frac{r}{r_0}.$$

In turn

$$\tau = -\rho \cdot (\nu + \nu_{r,t}) \cdot \frac{d\bar{u}_x}{dr} = \rho \cdot \nu \left(1 + \frac{\nu_{r,t}}{\nu}\right) \cdot \frac{d\bar{u}}{dy},$$

where $y \equiv r_0 - r$ is the distance from the wall of the tube. Substitut-

ing the last equation in the preceding one, we obtain an expression for the dimensionless turbulent viscosity due to the radial turbulent momentum transfer,

$$\frac{\nu_{r\tau}}{\nu} = \frac{R}{\frac{d\varphi}{d\eta}} - 1, \qquad (19.16)$$

where

$R \equiv \frac{r}{r_0}$ — is the dimensionless radius;

$\varphi \equiv \bar{u}_x \cdot \sqrt{\frac{\varrho}{\tau_F}}$ — is the dimensionless velocity;

$\eta \equiv \frac{y}{\nu} \cdot \sqrt{\frac{\tau_F}{\varrho}}$ — is the dimensionless distance from the wall of the tube.

Experimental investigations [61, 72] of the velocity distribution in turbulent streams in tubes indicate that the dimensionless velocity φ is a function of the dimensionless distance η from the tube wall, i.e., $\varphi = \varphi(\eta)$.

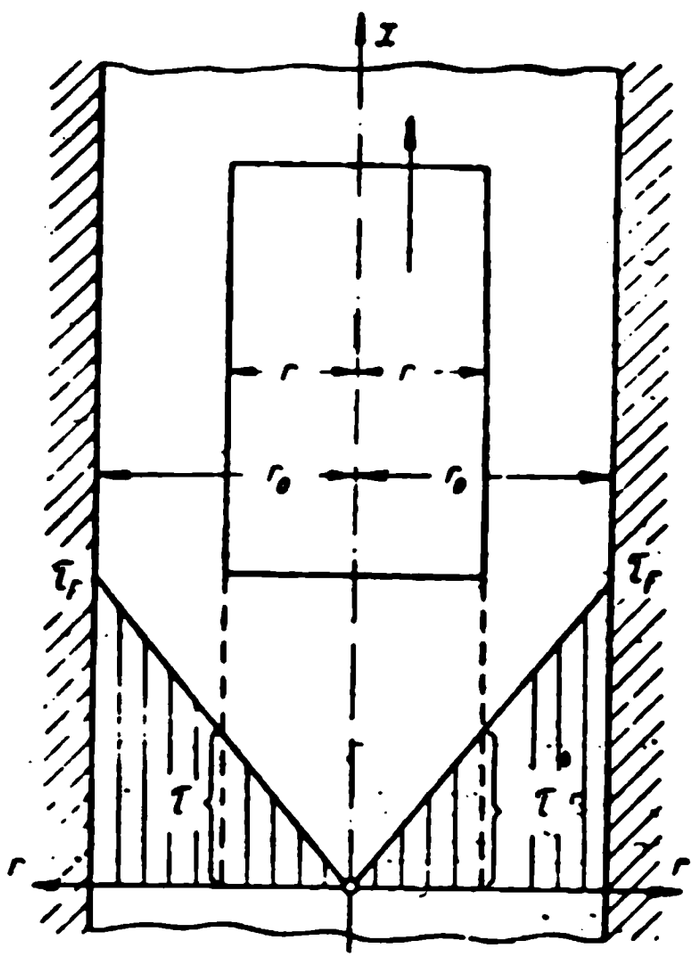

Fig. 62. Distribution of tangential stresses in a stabilized stream of fluid in a tube.

By way of an example of the dependence $\varphi(\eta)$ we present the following formula [72]:

$$\varphi = 5{,}75 \cdot \lg\left\{(1 + 0{,}4\eta) \times \frac{1{,}5(1+R)}{1+2R^3}\right\} + 7{,}8\left(1 - e^{-\frac{\eta}{11}} - \frac{\eta}{11} \cdot e^{-0.33\eta}\right). \qquad (19.17)$$

It must be noted that the dependence $\varphi(\eta)$, obtained by Nikuradze [38], is based on falsified experimental data [68].

The dimensionless turbulent temperature conductivity depends on the dimensionless radius and also on the Reynolds and Prandtl numbers:

$$\frac{a_{r\tau}}{a_{p\infty}} = \frac{a_{r\tau}}{\nu_{r\tau}} \cdot \frac{\nu_{r\tau}}{\nu} \cdot \frac{\nu}{a_{p\infty}} = \frac{\frac{\nu_{r\tau}}{\nu} Pr}{P_{r\tau}}. \qquad (19.18)$$

It follows from (19.15) that $P_{rt} = 1$. Measurements of the velocity $\bar{u}_x$ in the turbulent stream indicate that

$$\frac{\bar{u}_x}{u_0} = f_1\left(\frac{u_0 \cdot d}{\nu}, \frac{r}{r_0}\right) = f_1(R, Re).$$

Consequently,

$$\frac{\nu_r}{\nu} = f_2(Re, R).$$

Therefore

$$\frac{a_r}{a_p} = f_3(Re, Pr, R).$$

Thermal Stabilization in a Fluid Stream

Thermal stabilization is observed in a fluid stream if the field of the excess temperature $\vartheta(x, r) = t_f(x) - t(x, r)$ remains similar to itself in different sections of the stream, i.e., if the distribution $\vartheta(x, r)$ is self-similar in x.

Experimental investigations of heat transmission in tubes lead to the conclusion that the phenomenon of thermal stabilization is observed at a certain distance x_{tepl} from the inlet. The quantity x_{tepl} is called the initial portion of thermal stabilization, while the heat transmission in the cross sections $x \geq x_{tepl}$ is called stabilized.

The physical meaning of this phenomenon consists in the fact that when $x \geq x_{tepl}$ the conditions of entrance of the fluid into the tube cease to exert a noticeable influence on the temperature field in the stream, a field formed under the influence of the "nearest" boundary conditions maintained on the tube wall. In this sense the phenomenon of thermal stabilization in a fluid stream is analogous to the regular thermal mode in solids.

Let us consider the self-similarity conditions of the field $\vartheta \equiv \equiv t_F - t$ along a turbulent stream of a homogeneous liquid, when a constant temperature $t_п = \text{const} > t_н$ is maintained on the tube wall. In

this case the differential equation (19.1) and the boundary condition assume the following form:

$$u_x(r)\cdot\frac{\partial\vartheta}{\partial x}=\frac{1}{r}\cdot\frac{\partial}{\partial r}\left[r\cdot(a_m+a_{r,\tau})\cdot\frac{\partial\vartheta}{\partial r}\right]+$$

$$+\frac{\partial}{\partial x}\left[(a_m+a_{x,\tau})\cdot\frac{\partial\vartheta}{\partial x}\right];\quad \vartheta\Big|_{r=r_0}=0. \qquad (19.19)^*$$

The condition for self-similarity of ϑ with respect to $\underline{x}$

$$\frac{\vartheta(r_1,x)}{\vartheta(r_2,x)}=f(x)$$

will be satisfied if the field $\vartheta(r, x)$ can be represented in the form of a product of two functions $\vartheta_x(x)$ and $\vartheta_r(r)$, each of which depends only on one argument

$$\vartheta=\vartheta_x\cdot\vartheta_r. \qquad (19.20)$$

Substituting Eq. (19.20) in Eq. (19.19) and transforming them, we obtain

$$\frac{1}{\vartheta_x}\cdot\frac{d\vartheta_x}{dx}=\frac{1}{ru_x(r)\cdot\vartheta_r}\cdot\frac{d}{dr}\left[r\cdot(a_{p\,m}+a_{x,\tau})\cdot\frac{\partial\vartheta_r}{dr}\right]+$$

$$+\frac{1}{u_x(r)\,\vartheta_x}\cdot\frac{d}{dx}\left[(a_m+a_{x,\tau})\cdot\frac{d\vartheta_x}{dx}\right]. \qquad (19.21)$$

From Eq. (19.21) it follows that the variables ϑ_x and ϑ_r are separable only if the second term in the right half of this equation can be neglected. This consideration is a mathematical expression of the fact that the thermal stabilization in the fluid stream can be observed only if the changes in the total thermal conductivity along the stream are small compared with its changes transversely to the stream. Neglecting this term, we obtain

$$\frac{1}{\vartheta_x}\cdot\frac{d\vartheta_x}{dx}=\frac{1}{ru_x(r)\,\vartheta_r}\cdot\frac{d}{dr}\left[r\cdot(a_{p\,m}+a_{r,\tau})\frac{d\vartheta_r}{dr}\right]=$$

$$=\text{const}\equiv -m. \qquad (19.22)$$

From the last equation we obtain $\vartheta_x \sim e^{-mx}$. Consequently, the temperature field – self-similar with respect to $\underline{x}$ in the fluid stream at t_F = const – has the form

$$\vartheta=\vartheta_r\cdot e^{-m\cdot x}. \qquad (19.23)$$

The function $\vartheta_r(r)$ and the quantity $\underline{m}$ are determined from the equation

$$\frac{1}{r \cdot u_x(r) \cdot \vartheta_r} \cdot \frac{d}{dr}\left[r \cdot (a_{\text{л}} + a_{\text{т}}) \cdot \frac{d\vartheta_r}{dr}\right] = -m$$

and the boundary condition

$$\vartheta\big|_{r=r_0} = 0.$$

It is easy to prove that the stabilized heat-transmission coefficient $\alpha_{0\ stab} = q_F/(t_F - t_{zhl})$ referred to the initial temperature difference depends on the distance from the entrance into the tube:

$$\alpha_{0\ \text{стаб}} \sim e^{-m \cdot x},$$

which is not convenient in carrying out heat-transfer calculations.

The stabilized heat-transmission coefficient

$$\alpha_{\text{стаб}} = \frac{q_F}{t_F - t_{\text{ж}}},$$

referred to the local temperature difference does not depend on $\underline{x}$. Consequently, the similarity equations in long tubes, where thermal stabilization occurs, assume a simpler form:

$$\left.\begin{aligned} Nu_{\text{стаб}} &= \Phi_1(Re, Pr); \\ \overline{Nu}_{\text{стаб}} &= \Phi_2(Re, Pr). \end{aligned}\right\} \qquad (19.24)$$

This property of the stabilized heat-transmission coefficient α_{stab} makes its use more convenient than that of $\alpha_{0\ stab}$. We note that the quantity $\underline{m}$ in Eq. (19.23) can be regarded as the rate of stabilized variation of ϑ along the stream. The quantity $\underline{m}$ is connected with the Nusselt number by the simple relation [14]

$$Nu_{\text{стаб}} = \frac{1}{4} Pe \cdot d \cdot m.$$

Consequently, by measuring the quantity $\underline{m}$ we can calculate the value of the stabilized heat-transmission coefficient α_{stab}. We note also that when t_F = const the value of the heat-transmission coefficient $\overline{\alpha}_{\underline{x}}$ averaged over $\underline{x}$ does not coincide with α_{stab} even in rather long tubes

Let us consider the self-similarity conditions of the field $\vartheta \equiv \equiv t_F - t$ along a turbulent stream of a homogeneous fluid, when a constant heat rate $q_F = \text{const}$ is maintained on the tube wall. Substituting the condition for self-similarity with respect to $\underline{x}$

$$t(x, r) = t_F(x) - \theta_x(x) \cdot \theta_r(r)$$

into the boundary condition

$$\lambda_{ж} \frac{\partial t}{\partial r}\Big|_{r_0=0} = q_F = \text{const},$$

we obtain

$$\lambda_{ж} \cdot \theta_x \cdot \frac{d\theta_r}{dr}\Big|_{r_0=0} = q_F = \text{const}.$$

From the last relation it follows that when $q_F = \text{const}$ the self-similarity condition will be satisfied if one assumes

$$\vartheta_x = 1.$$

Then

$$t(x, r) = t_F(x) - \theta_r(r), \qquad (19.25)$$

Substituting Eq. (19.25) into Eq. (19.19) we obtain

$$\frac{dt_F(x)}{dx} = -\frac{1}{u_x(r) \cdot r} \cdot \frac{d}{dr}\left[r \cdot (a_{p ж} + a_{r т}) \cdot \frac{d\theta_r}{dr}\right] -$$
$$- \frac{1}{u_x(r)} \frac{d}{dx}\left[(a_{p ж} + a_{x т}) \cdot \frac{d\theta_x}{dx}\right].$$

In the last equation the variables separate provided the term $d/dx[(a_{pzh} + a_{xt}) \cdot d\vartheta_x/dx]$, i.e., the change in the heat conductivity along the stream, can be neglected. Doing so, we obtain

$$\frac{dt_F(x)}{dx} = -\frac{1}{r \cdot u_x(r)} \frac{d}{dr}\left[r \cdot (a_{p ж} + a_{r т}) \cdot \frac{d\theta_r}{dr}\right] = \text{const} \equiv m. \qquad (19.26)$$

From Eq. (19.26) we get an equation for the self-similar (with respect to $\underline{x}$) temperature field in the fluid stream in a tube at constant heat rate on the wall

$$t(x, r) = m \cdot x - \theta_r(r). \qquad (19.27)$$

The function $\vartheta(r)$ and the quantity $\underline{m}$ are determined from Eq. (19.26)

and from the boundary condition

$$\lambda_{ж} \cdot \frac{d\Theta_r}{dr}\Big|_{r_0=0} = q_F .$$

The stabilized heat transmission coefficient $\alpha_{0\ stab} = q_F/\Delta t_0$, referred to the initial temperature difference, also depends on $\underline{x}$:

$$\alpha_{0\,стаб} = \frac{q_F}{mx - t_{ж1}} .$$

whereas the stabilized heat-transmission coefficient

$$\alpha_{стаб} = \frac{q_F}{t_F(x) - \overline{t}_{ж}(x)} = \frac{q_F}{\overline{\vartheta}} ,$$

referred to the local temperature difference does not depend on $\underline{x}$.

In the presence of thermal stabilization (q_F = const) the similarity equations for the dimensionless temperatures Θ_q and Θ_F of the wall tube at constant physical parameters of the fluid assume the simpler form

$$\Theta_q = F_1(Re, Pe) + \frac{4}{Pe} X, \quad \Theta_F \cdot L = F_2(Re, Pe) + X. \tag{19.28}$$

Approximate Analytic Method of Calculating Stabilized Heat Transmission in a Tube

Let us consider the expression for the Nusselt number for stabilized heat transmission in a tube

$$Nu_{стаб} = \frac{q_F \cdot 2r_0}{\lambda_{ж} \cdot (t_F - \overline{t}_{ж})} . \tag{19.29}$$

where $\overline{\vartheta} = t_F - \overline{t}_{zh}$ is the local value of the temperature drop at any cross section of the stream. By definition

$$\overline{\vartheta} \equiv \frac{2}{u_0 \cdot r_0^2} \cdot \int_0^{r_0} \vartheta \cdot u_x(r)\, r\, dr = \frac{2}{u_0 \cdot r_0^2} \int_0^{r_0} \frac{\partial \vartheta}{\partial r} \cdot dr \int_0^{r} u_x(r)\, r\, dr . \tag{19.30}$$

From the equation for the radial heat flux in the turbulent stream

$$q_r = -\lambda_{ж} \cdot \left(1 + \frac{\lambda_{r\tau}}{\lambda_{ж}}\right) \frac{\partial \vartheta}{\partial r} = -\lambda_{ж} \cdot \left(1 + \frac{Pr}{Pr_\tau} \cdot \frac{\nu_{r\tau}}{\nu}\right) \cdot \frac{\partial \vartheta}{\partial r}$$

we obtain

$$\frac{\partial\theta}{\partial r} = -\frac{q_r}{1+\frac{Pr}{Pr_T}\cdot\frac{\nu_T}{\nu}}. \qquad (19.31)$$

Substituting Eq. (19.31) into Eq. (19.30) we obtain

$$\overline{\theta} = \frac{2}{\lambda_{ж}\cdot u_0\cdot r_0^2}\int_0^r \frac{q_r\cdot dr}{1+\frac{Pr}{Pr_T}\cdot\frac{\nu_T}{\nu}}\int_0^r r\cdot u_x(r)\,dr. \qquad (19.32)$$

Finally, substituting Eq. (19.32) in Eq. (19.29) we obtain after transformation the following expression for the Nusselt number [76]:

$$Nu_{ст\text{-}б} \equiv \frac{1}{\int_0^1 \frac{Q_r\cdot dR}{1+\frac{Pr}{Pr_T}\cdot\frac{\nu_T}{\nu}}\int_0^R U_x\cdot R\cdot dR}, \qquad (19.33)$$

where

$$R \equiv \frac{r}{r_0};\quad U_x \equiv \frac{u_x}{u_0};\quad Q_r \equiv \frac{q_r}{q_F}.$$

To determine the dimensionless radial heat flux Q_r we use the equation for the energy conservation law, which for a stationary axially symmetrical temperature field in a stream in a tube without heat sources ($q_V = 0$) has the following form:

$$\frac{1}{r}\cdot\frac{\partial}{\partial r}(r\cdot q_r) + \frac{\partial q_x}{\partial x} = 0.$$

Integrating this equation twice within the limits from 0 to $\underline{r}$ and from 0 to r_0, we obtain

$$q_r = -\frac{1}{r}\cdot\int_0^r \frac{\partial q_x}{dx}\cdot r\cdot dr;$$

$$q_F = -\frac{1}{r_0}\cdot\int_0^{r_0} \frac{\partial q_x}{\partial x}\cdot r\cdot dr.$$

Consequently,

$$Q_r \equiv \frac{q_r}{q_F} = \frac{1}{R}\cdot\frac{\int_0^R \frac{\partial q_x}{\partial x}\cdot R\cdot dR}{\int_0^1 \frac{\partial q_x}{\partial x}\cdot R\cdot dR}. \qquad (19.34)$$

In the presence of thermal stabilization along the tube axis in a stream of a homogeneous fluid, it is possible to neglect the variation of the heat conductivity along $\underline{x}$.* Therefore

$$\frac{\partial q_x}{\partial x} = \frac{\partial q_{x\,\text{конв}}}{\partial x} = C_{p\,\text{ж}} \cdot u_x(r) \cdot \frac{\partial t}{\partial x}. \qquad (19.35)$$

Substituting Eq. (19.35) into Eq. (19.34) we obtain

$$Q_r = \frac{1}{R} \cdot \frac{\int_0^R U_x \cdot \frac{\partial t}{\partial x} \cdot R \cdot dR}{\int_0^1 U_x \cdot \frac{\partial t}{\partial x} \cdot R \cdot dR}. \qquad (19.36)$$

Equations (19.33) and (19.36) can be used for an approximate calculation of stabilized heat transmission in tubes for different boundary conditions. By way of the simplest example let us consider stabilized heat transmission in a tube, on the walls of which a constant heat rate q_F = const is maintained. In this case the temperature distribution along the fluid stream obeys the straight-line equation

$$t = m \cdot x - \vartheta_r(r). \qquad (19.37)$$

We therefore have here $\partial t/\partial x = m = \text{const}$ and in place of Eq. (19.36) we get

$$Q_r = \frac{1}{R} \frac{\int_0^R U_x \cdot R \cdot dR}{\int_0^1 U_x \cdot R \cdot dR} = \frac{2}{R} \cdot \int_0^R U_x \cdot R \cdot dR. \qquad (19.38)$$

Substituting (19.38) into (19.33) we obtain the so-called Lyon integral [65]

$$Nu_{\text{стаб}} = \frac{1}{2 \int_0^1 \frac{\left[\int_0^R U_x \cdot R \cdot dR\right]^3 \cdot dR}{R \cdot \left(1 + \frac{Pr}{Pr_T} \cdot \frac{\nu_T}{\nu}\right)}}. \qquad (19.39)$$

which was used by its author to calculate heat transmission of molten metals.

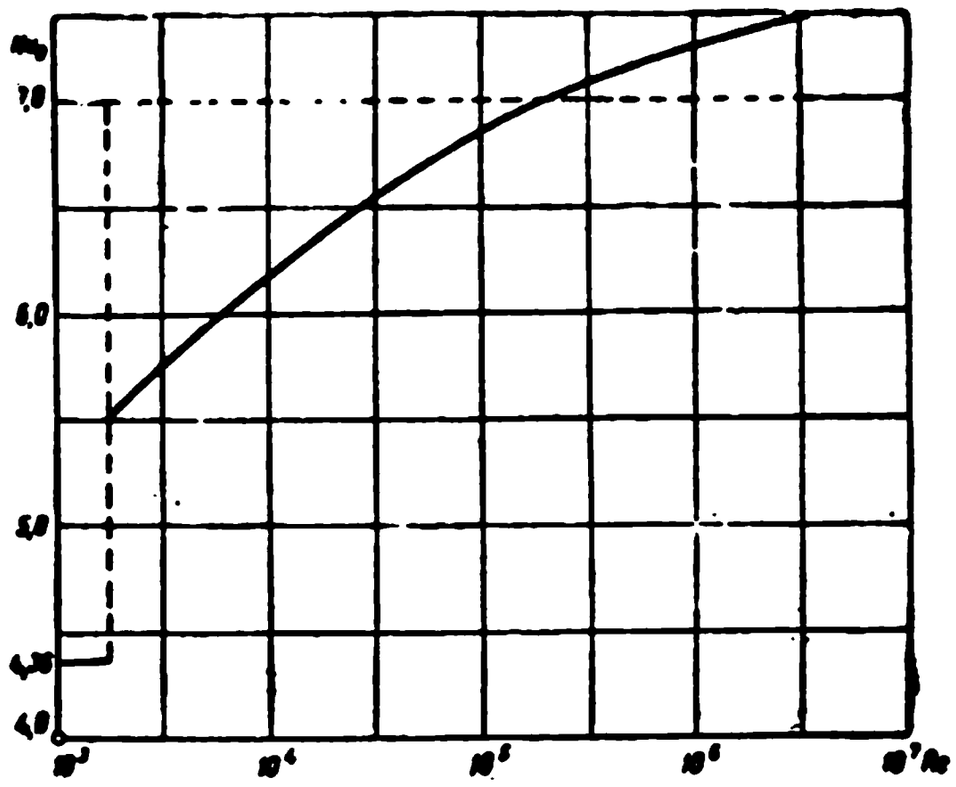

Fig. 63. Dependence of the Nusselt number on the Reynolds number for stabilized turbulent heat transmission of liquids having a vanishingly small Prandtl number.

By way of an example let us determine the Nusselt number for laminar flow of a liquid in a tube. In this case $a_{rm} = 0$ and $U_x = 2(1 - R^2)$. We therefore obtain in place of Eq. (19.39)

$$Nu_{стаб} = \frac{1}{4\int_0^1 \frac{dR}{R}\cdot\left[\int_0^R (1-R^2)\cdot R\cdot dR\right]^2} = \frac{48}{11} \approx 4{,}36. \qquad (19.40)$$

By way of another example let us consider heat transmission at q_F = const in a tube with turbulent flow of a fluid that has a vanishingly small Prandtl number ($Pr \to 0$). In this case $Nu_{stab} = Nu_0 = f(Re)$ and Eq. (19.38) assumes the following form:

$$Nu_0 = \frac{1}{2\int_0^1 \frac{1}{R}\left[\int_0^R U_x\cdot R\cdot dR\right]^2\cdot dR}. \qquad (19.41)$$

Calculation of the dependence of Nu_0 on Re by Formula (19.41) will be carried out using Eq. (19.17) for the distribution of the velocity U_x in a stabilized turbulent fluid stream. It is seen on Fig. 63 that the number Nu_0 can be regarded approximately as constant, equal

$\leq Re \leq 3.3 \cdot 10^6$. However, when $Re < 3 \cdot 10^4$ the value of Nu_0 decreases noticeably and at $Re = 2400$ it reaches 5.48, which amounts to 78% of $Nu_0 = 7$. The results of the calculations of Nu_0 can be represented in the form of the following relationship:

$$Nu_0 \approx 7{,}24 - \frac{49{,}23}{Re^{0.428}}, \qquad (19.42)$$

which is valid in the absence of "thermal contact resistance" between the fluid and the tube wall, and also when $2400 \leq Re \leq 3.3 \cdot 10^6$.

Using the generalized mean-value theorem [5], we can represent Eq. (19.39) in a different form which in some cases is more convenient to use:

$$Nu_{стаб} = \frac{1 + \overline{\frac{Pr}{Pr_T} \cdot \frac{\nu_{rT}}{\nu}}}{2 \int_0^1 \frac{1}{R} \cdot \left[\int_0^R U_x \cdot R \cdot dR \right]^3 \cdot dR} = Nu_0(Re) +$$

$$+ Nu_0(Re) \cdot f_1(Re, Pe) = Nu_0(Re) + f(Re, Pe). \qquad (19.43)$$

By way of a more complicated example let us consider the use of Eqs. (19.31) and (19.36) to calculate stabilized heat transmission in a tube on the walls of which a constant temperature $t_F = const$ is maintained. In this case the temperature distribution along the fluid stream obeys the exponential law (19.23)

$$t = t_F - \vartheta_r(r) \cdot e^{-m \cdot x}.$$

Therefore

$$\frac{\partial t}{\partial x} = m \cdot \vartheta_r \cdot e^{-m \cdot x} = m \cdot (t_F - t) = m \cdot \vartheta$$

and in place of Eq. (19.36) we obtain

$$Q_r = \frac{1}{R} \cdot \frac{\int_0^R \Theta \cdot U_x \cdot dR}{\int_0^1 \Theta \cdot U_x \cdot dR}. \qquad (19.44)$$

Equation (19.44) is suitable only for the considered heat-transmission conditions ($t_F = const$). This equation contains two unknown quantities:

the dimensionless radial heat flux Q_r and the dimensionless temperature $\Theta \equiv \vartheta/\vartheta_0$, where $\vartheta_0 \equiv t_F - t_0$, and t_0 is the temperature of the fluid on the axis of the tube, i.e., at $r = 0$.

We can set up a supplementary equation by using the analog of the Biot-Fourier law for turbulent heat conductivity

$$q_r = -\lambda_{эф} \cdot \left(1 + \frac{Pr}{Pr_т} \cdot \frac{\nu_{rт}}{\nu}\right) \cdot \frac{\partial \theta}{\partial r}$$

and the expression

$$q_F = -\lambda_{эф} \cdot \frac{\partial \theta}{\partial r}\Big|_{r_0 - 0}.$$

Dividing the first equation by the second we obtain

$$Q_r = \left(1 + \frac{Pr}{Pr_т} \cdot \frac{\nu_{rт}}{\nu}\right) \frac{\frac{\partial \theta}{\partial r}}{\frac{\partial \theta}{\partial r}\Big|_{r_0 - 0}}$$

or

$$\frac{\partial \theta}{dr} = \frac{Q_r}{1 + \frac{Pr}{Pr_т} \cdot \frac{\nu_{rт}}{\nu}} \cdot \frac{\partial \theta}{\partial r}\Big|_{r_0 - 0}.$$

Integrating the last equation with respect to $\underline{r}$ from $\underline{r}$ to r_0 and from 0 to r_0, we obtain

$$\theta = -\frac{\partial \theta}{dr}\Big|_{r_0 - 0} \cdot \int_R^1 \frac{Q_r}{\Lambda} \cdot dR; \quad \theta_0 = -\frac{\partial \theta}{\partial r}\Big|_{r_0 - 0} \cdot \int_0^1 \frac{Q_r}{\Lambda} \cdot dR.$$

Dividing the first equation by the second we get

$$\Theta = \frac{\int_R^1 \frac{Q_r}{\Lambda} \cdot dR}{\int_0^1 \frac{Q_r}{\Lambda} \cdot dR}, \tag{19.45}$$

where

$$\Lambda \equiv 1 + \frac{Pr}{Pr_т} \cdot \frac{\nu_{rт}}{\nu}.$$

The two integral equations (19.44) and (19.45) contain two unknown quantities Q_r and Θ. This system of equations is best solved with re-

spect to Q_r by successive approximations. Let us carry out this calculation for a stabilized laminar flow in the tube, when

$$\Lambda = 1 \text{ and } U = 2(1 - R^2).$$

In this particular case Eqs. (19.33), (19.44), and (19.45) assume the following form:

$$Nu_{cross} = \frac{1}{2\int_0^1 Q_r \cdot dR \cdot \int_0^R (1 - R^2) \cdot R \cdot dR}; \qquad (19.46)$$

$$Q_r = \frac{\int_0^R \theta \cdot (1 - R^2) \cdot dR}{\int_0^1 \theta \cdot (1 - R^2) \cdot dR}; \qquad (19.47)$$

$$\theta = \frac{\int_R^1 Q_r \cdot dR}{\int_0^1 Q_r \cdot dR}. \qquad (19.48)$$

By way of the first approximation we assume $Q_{r1} = R$. Then

$$\theta_1 = \frac{\int_R^1 R \cdot dR}{\int_0^1 R \cdot dR} = 1 - R^2. \qquad (19.49)$$

Substituting (19.49) in (19.47) we obtain the second approximation for the distribution Q_r, etc. The process of successive approximations converges rapidly, and the third iterations yield sufficient accuracy

$$Nu_{cross} \approx \frac{1}{2\int_0^1 \left[Q_{r\,III}(R) \cdot dR \cdot \int_0^R (1 - R^2) \cdot dR \right]} = 3{,}67. \qquad (19.50)$$

Thermohydrodynamic Analogy

O. Reynolds advanced the hypothesis that the mechanisms of turbulent transport of heat and momentum transversely to a stream of fluid in a tube are analogous. This hypothesis enables us to establish a connection between the heat transmission and the hydraulic resistance in the case of turbulent flow of a fluid in tubes.

There exist several variants for the use of this analogy. Thus, for example, the equation $a_{nt} = \nu_{nt}$ or $Pr_t = 1$ is essentially the most

general consequence of the Reynolds hypothesis. Let us present one of the simplest modifications of the thermohydrodynamic analogy. The results of the measurements of the time-averaged fields of the temperature $\underline{t}$ and the velocity u_x in stabilized turbulent streams of fluids with Pr ~ 1 lead to the conclusion that the functions $\vartheta(r) \equiv t - t_F$ and $u_x(r)$ are "similar" to each other. Indeed, both functions vary relatively little in the core of the stream and change rapidly near the walls of the channel (Fig. 64). It is therefore natural to assume that these functions are analogous to each other, i.e., they are connected by an analogy transformation

$$\vartheta(r) = A_{\vartheta;u} \cdot u_x(r), \qquad (19.51)$$

where $A_{\vartheta;u}$ is the analogy constant. This dimensional quantity does not depend on the radius $\underline{r}$.

Let us use Relation (19.51) to determine the Nusselt number on the wall of a channel with cross section f_0. Let

$$Nu \equiv \frac{q_F \cdot d_0}{\lambda_m \cdot {}^{f}\overline{\vartheta}}, \qquad (19.52)$$

where

$${}^{f}\overline{\vartheta} = \frac{1}{u_0 \cdot f_0} \cdot \int_0^{f_0} u_x \cdot \vartheta \cdot df. \qquad (19.53)$$

Substituting the equation (19.51) in the equation (19.53) we obtain

$$ {}^{f}\overline{\vartheta} = \frac{A_{\vartheta;u}}{u_0 \cdot f_0} \cdot \int_0^{f_0} u_x^2 \cdot df = $$

$$ = A_{\vartheta,u} \cdot u_0 \cdot \frac{\int_0^{f_0} \left(\frac{u_x}{u_0}\right)^2 \cdot df}{f_0} =$$

$$ = A_{\vartheta,u} \cdot u_0 \cdot {}^{f}\overline{U_x^2}, \qquad (19.54)$$

where ${}^{f}\overline{U_x^2}$ is the value of the square of the

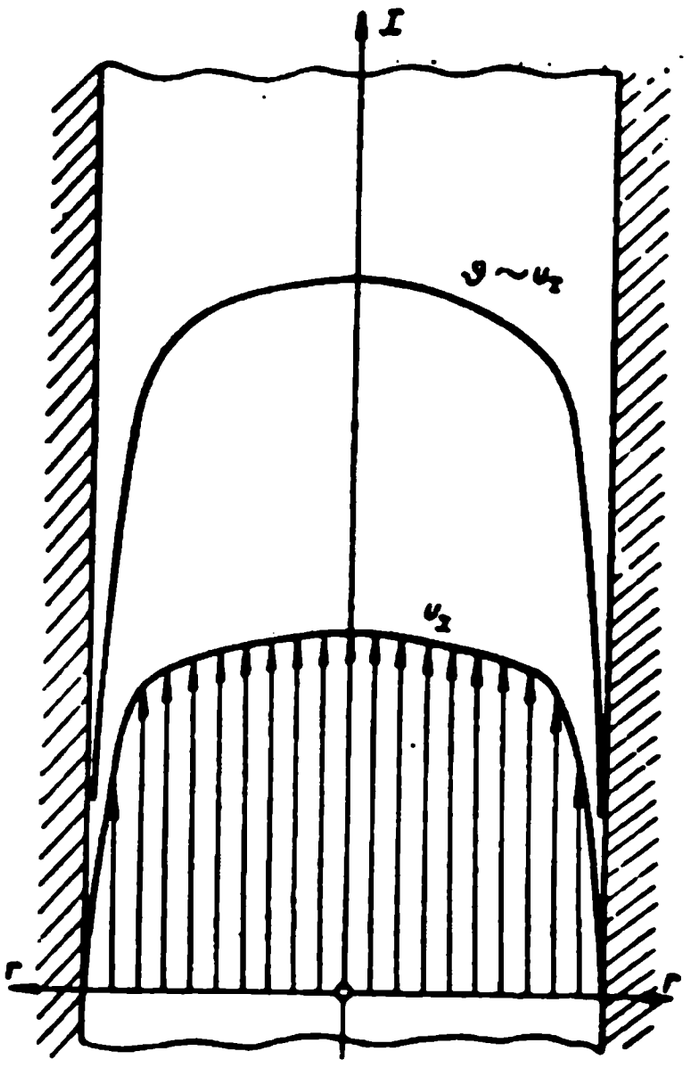

Fig. 64. Distribution of time-averaged temperature and velocity transversely to a turbulent stream of fluid in a channel

dimensionless velocity $U_x \equiv u_x/u_0$ averaged over the cross section of the tube. To determine the heat rate q_F on the wall of the channel we use the Biot-Fourier law

$$q_F = -\lambda_ж \cdot \frac{\partial \theta}{\partial n}\Big|_{n=0}.$$

Substituting Expression (19.51) in the last equation, we obtain

$$q_F = \lambda_ж \cdot A_{\theta\, u} \cdot \frac{\partial u_x}{\partial n}\Big|_{n=0}. \qquad (19.55)$$

To determine the velocity gradient on the channel wall we use Newton's law for viscous friction

$$\tau_F = \mu \cdot \frac{\partial u_x}{\partial n}\Big|_{n=0}.$$

From this relation we obtain

$$\frac{\partial u_x}{\partial n}\Big|_{n=0} = \frac{\tau_F}{\mu}.$$

Let us substitute the expression for the velocity gradient in Eq. (19.55). We then obtain

$$q_F = \frac{\lambda_ж \cdot A_{\theta.\,u} \tau_F}{\mu}. \qquad (19.56)$$

The tangential stress on the channel wall is

$$\tau_F = \frac{\Delta p \cdot f_\theta}{\Pi \cdot l} = \frac{1}{8} \cdot \xi(Re_э) \cdot \varrho \cdot u_0^2,$$

where $\xi(Re_e)$ is the coefficient of friction resistance in the equivalent round tube. Substituting the last relation for τ_F in (19.56) we obtain

$$q_F = \frac{A_{\theta.\,u}}{8} \cdot \frac{\lambda_ж}{\nu} \cdot \xi(Re_э) \cdot u_0^2. \qquad (19.57)$$

Using (19.57) and (19.54) we can readily calculate the Nusselt number

$$Nu_{стаб} = \frac{1}{8} \cdot \xi(Re_э) \cdot Re_э \left[\overline{U_x^2}\right]^{-1}. \qquad (19.58)$$

The dependence

$$\overline{U_x^2} \equiv 2\int_0^1 U_x^2 \cdot R \cdot dR = f(Re_э)$$

is plotted in Fig. 65.

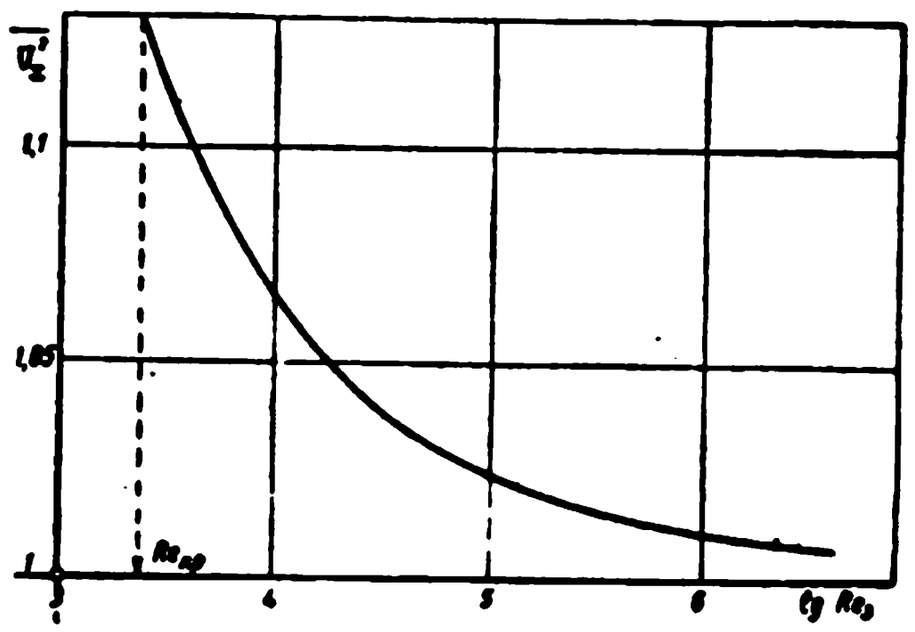

Fig. 65. Dependence of the average value of the square of the dimensionless velocity on the Reynolds number in turbulent stabilized flow in an equivalent tube.

Formula (19.58), which is one of the results of using the thermohydrodynamic analogy, gives approximate values of the Nusselt number for fluids whose Prandtl number differs little from unity.

Calculation of Hydraulic Resistance and Heat Transmission in the Flow of Fluids in Channels with Nonround Transverse Cross Section

At the present time the hydraulic resistance and heat transmission have been investigated most thoroughly for the flow of fluids in tubes. However, in heat-exchange apparatus and in nuclear reactors one frequently employs channels which do not have round cross sections. The calculations of the heat transmission and the hydraulic resistance in such channels usually are reduced to the determination of the same quantities in some equivalent tube, which has a diameter

$$d_e = 4f/P, \qquad (19.59)$$

where f is the active section of the channel and P is the wetted perimeter of its cross section.

In many cases such an approximate calculation gives satisfactory results, and this explains its extensive use. The limits of applicability of this approximation have not been established. An exact solution of this problem is difficult. One can cite, however, a few simple con-

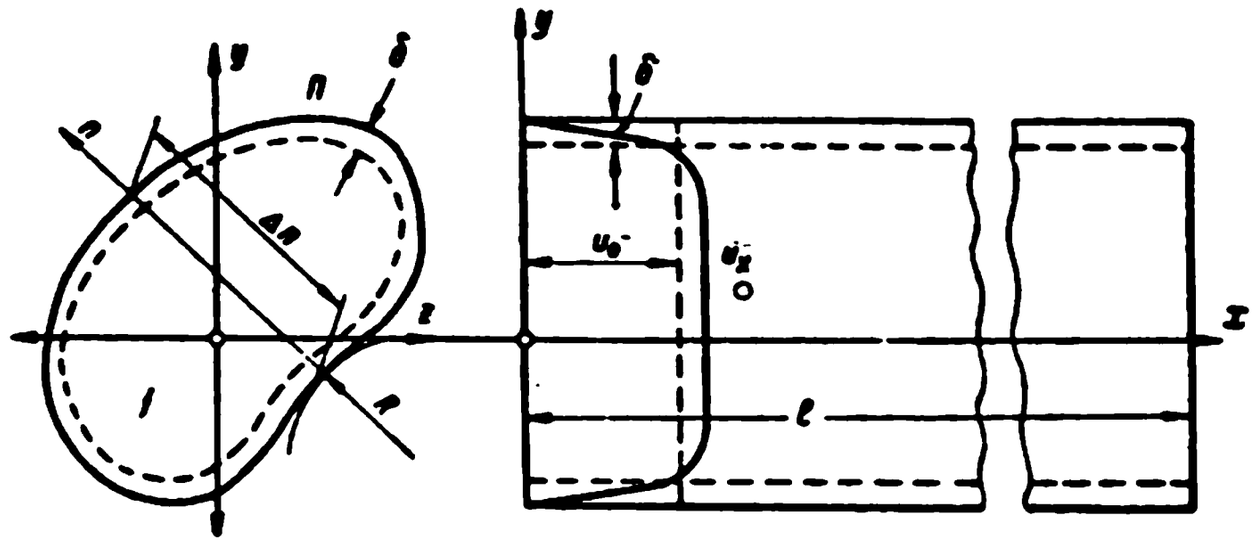

Fig. 66. For the calculation of the hydraulic resistance of a channel with non-round transverse cross section.

siderations which enable us to carry out an approximate qualitative estimate of the possibility of using the "equivalent diameter" in each specific case.

Assume that it is necessary to determine the hydraulic resistance of friction in a channel with cross section $\underline{f}$, perimeter P, and length l (Fig. 66), in which flows a stationary turbulent stream of a homogeneous fluid, stabilized in the mean.

The velocity distribution $u_x(y, z)$ over the cross section of the channel remains unchanged along its $\underline{x}$ axis. The stream velocity averaged over the cross section $\underline{f}$ is equal to u_0, and the fluid viscosity and density are μ and ρ. The velocity distribution $u_x(y, z)$ is such that in the core of the stream the velocity changes insignificantly, and almost all its change from 0 on the wall to $u_{maks} \approx u_0$ is concentrated in a thin boundary layer δ. The shape of the cross section $\underline{f}$ of the channel is such that at all points of the perimeter P the thickness of the layer δ is negligibly small compared with the "length of the normal" Δn and with the radius of curvature R of the perimeter P:

$$\left.\begin{aligned} \delta &\ll \Delta n; \\ \delta &\ll R. \end{aligned}\right\} \qquad (19.60)$$

If these conditions are satisfied the layer thickness δ can be regarded as approximately constant over the entire surface F of the

friction between the stream and the channel wall.

The hydraulic resistance of this channel is equal to

$$\Delta p = \frac{\tau_F \cdot \Pi \cdot l}{f}. \qquad (19.61)$$

To determine the tangential stress τ_F on the wall we use Newton's law

$$\tau_F = \mu \cdot \frac{\partial u_x}{\partial n}\Big|_{n=+0}. \qquad (19.62)$$

The value of the velocity gradient will be expressed in terms of u_0 and δ:

$$\frac{\partial u_x}{\partial n}\Big|_{n=+0} = \frac{u_0}{\delta}. \qquad (19.63)$$

This relation can be regarded as a definition of the thickness of the hydrodynamic layer. Substituting (19.62) and (19.63) in (19.61) we obtain

$$\Delta p = \frac{\Pi \cdot \mu \cdot u_0 \cdot l}{f \cdot \delta}. \qquad (19.64)$$

Thus, the solution of the problem posed reduces to a determination of the thickness of the layer δ. From (19.63) we obtain

$$\delta = \frac{u_0}{\frac{\partial u_x}{\partial n}\Big|_{n=+0}} \qquad (19.65)$$

In addition, from the Helmholz equation and from the uniqueness conditions for the stream under consideration we can readily obtain the following system of defining parameters for the velocity field u_x in the stream:

$$u_x = f_1(z, y, \nu, u_0, l_1, \ldots, l_m), \qquad (19.66)$$

where l_1, ..., l_m are the characteristic dimensions of the cross section $\underline{f}$ of the channel.

If we take Inequalities (19.60) into account, then Eq. (19.66) can be simplified:

$$u_x = f_2(n, \delta, \nu, u_0). \qquad (19.67)$$

Substituting Eq. (19.67) in Eq. (19.65), we get

$$\delta = f_3(u_0, \nu).$$

After reducing this equation to dimensionless form we obtain

$$\delta \sim \frac{\nu}{u_0}. \qquad (19.68)$$

The last relation is satisfied for flow of fluid in channels with rough walls. If the channel has smooth walls, Relation (19.68) must be regarded as a first approximation to the actual conditions.

Let us consider an equivalent tube with diameter $d = d_e$ and length $l = l_e$. The diameter d_e is unknown. Assume that there flows in this tube the same fluid as in the channel ($\mu_e = \mu$; $\rho_e = \rho$). The flow velocities are the same in both cases $u_{0e} = u_0$, and the thickness of the hydrodynamic layer δ_e is small compared with the radius of the equivalent tube:

$$\delta_e \ll \frac{d_e}{2}.$$

Consequently, in the equivalent tube the inequalities (19.60) and Relation (19.68) are satisfied:

$$\delta_e \sim \frac{\nu_e}{u_{0e}}.$$

Inasmuch as $u_{0e} = u_0$ and $\nu_e = \nu$, it follows from (19.68) that the thickness of the hydrodynamic layers in the equivalent tube and in the channel will be approximately the same:

$$\delta \approx \delta_e. \qquad (19.69)$$

Let us define the diameter d_e of the equivalent tube in such a way that the pressure drops in the channel and in the tube are the same, $\Delta p = \Delta p_e$.

The hydraulic resistance of the tube is

$$\Delta p = \frac{\Pi_e \cdot \mu_e \cdot u_{0e} \cdot l_e}{f_e \cdot \delta_e} = \frac{4\mu \cdot u_e \cdot l}{d_e \cdot \delta_e} \approx \frac{4\mu \cdot u_e \cdot l}{d_e \cdot \delta}. \qquad (19.70)$$

Comparing Relation (19.70) with Eq. (19.64), we obtain the diam-

eter of the equivalent tube

$$d_e = 4f/P.$$

Thus, the universally accepted expression (19.59) for d_e is the consequence of Inequalities (19.60), which are therefore the necessary conditions for the applicability of Expression (19.59) for d_e.

In calculating the hydraulic resistance in each specific channel it is necessary to check whether Inequalities (19.60) are satisfied over the entire or almost the entire perimeter P of the cross section f of this channel. For this purpose it is necessary to determine the thickness of the hydrodynamic layer δ from the formula

$$\delta = \frac{8\nu}{u_0 \cdot \xi(Re_e)} \text{ or } \frac{\delta}{d_e} = \frac{8}{Re_e \cdot \xi(Re_e)}, \qquad (19.71)$$

where $Re_e \equiv u_0 \cdot d_e/\nu$. Knowing δ and having a drawing of the cross section of the channel, it is easy to check whether Conditions (19.60) are satisfied.

Formula (19.71) has been obtained from Eq. (19.70) and from the Darcy relation

$$\Delta p_e = \xi(Re_e) \cdot \frac{l_e}{d_e} \cdot \frac{\varrho_e \cdot u_{0e}^2}{2},$$

where $\xi(Re_e)$ is the coefficient of hydraulic resistance to friction in the tube. If the tube is rough, then ξ = const and Relation (19.68) follows from (19.71). For smooth tubes the value of ξ depends on the Reynolds number, $\xi = \xi(Re_e)$. In this case Relation (19.68) is approximate.

In laminar stabilized flow of fluid in a tube, Conditions (19.60) are not satisfied. Indeed, in this case we have

$$\delta = \frac{8 d_e Re_e}{Re_e 64} = \frac{d_e}{8}; \quad \Delta n = d_e; \quad \delta \sim \Delta n.$$

Consequently, one cannot use d_e to calculate the hydraulic resistance of channels for stabilized laminar flow of fluid. It is essential to

note that in the initial portion of the fluid flow, i.e., at the entrance to the tube, the thickness of the layer δ is usually small. Therefore d_e can apparently be used to determine the local values of $\xi(Re)$ at the start of the initial portion also in the case of laminar flow.

The situation is different in turbulent flow. If we use for ξ the Blasius law

$$\xi = \frac{0.316}{Re_e^{0.25}}.$$

then

$$\delta = \frac{8d_e \cdot Re_e^{0.25}}{Re_e \cdot 0.316} \approx \frac{25d_e}{Re_e^{0.75}}.$$

The two conditions (19.60) for the tube reduce to one

$$\delta \ll \frac{d_e}{2}.$$

Therefore

$$\frac{25d_e}{Re_e^{0.75}} \ll \frac{d_e}{2}.$$

Consequently, $Re_e^{0.75} \gg 50$ and $Re_e \gg 190$. The last relations indicate that d_e can be used only in turbulent flow.

Let us consider the necessary conditions for the applicability of d_e to the calculation of heat transmission in channels with nonround cross sections.

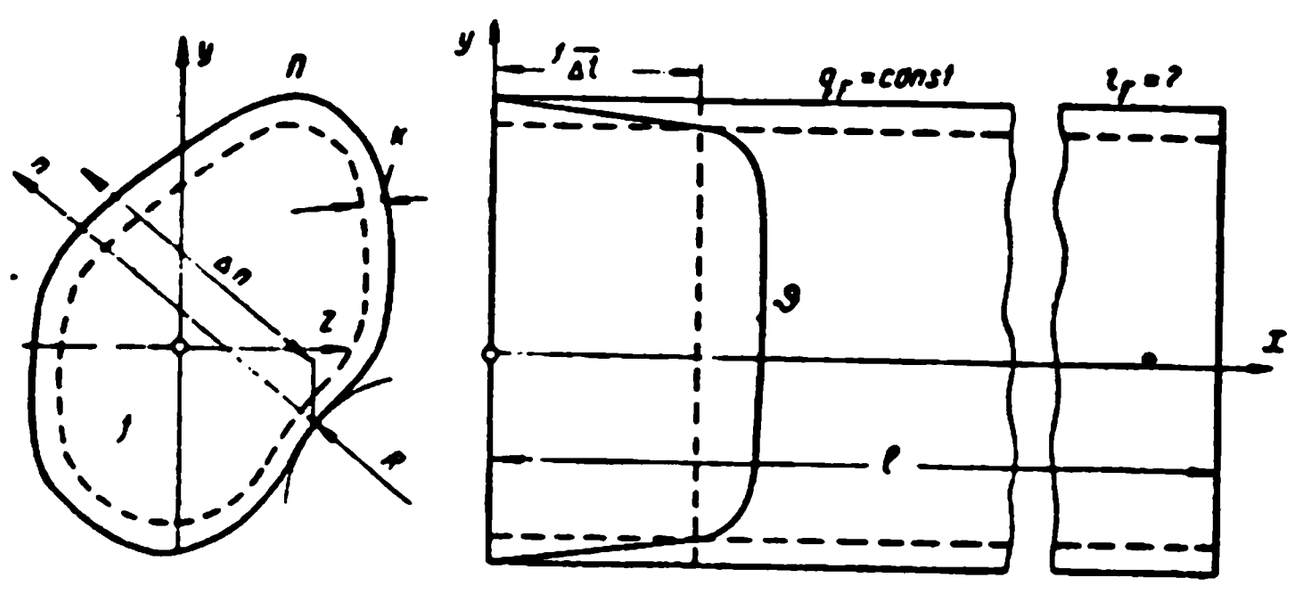

Fig. 67. For the calculation of heat transmission in a channel with nonround cross section.

We first determine the heat-transmission coefficient α and the temperature of the wall t_F in a channel, which has a cross section $\underline{f}$, a perimeter P, and a length l (Fig. 67). Let us consider the simplest case, when on the surface F of the channel a constant heat rate q_F = = const is maintained. Flowing in the channel is a stationary stream of a homogeneous fluid, with a Prandtl number not smaller than unity $Pr \geq 1$, stabilized in the mean. Along with the hydrodynamic stabilization there is also thermal stabilization. Assume that the velocity distribution $u_x(y, z)$ is such that at all points of the perimeter P the conditions (19.60) concerning the smallness of the thickness of the hydrodynamic layer δ are satisfied. Since the fluid has $Pr \geq 1$, the thickness $\underline{k}$ of the thermal layer does not exceed the thickness of the hydrodynamic layer δ. Therefore in all points of the perimeter the conditions for the smallness of the thickness of the thermal boundary layer are satisfied:

$$\left.\begin{aligned} k &\ll \Delta n; \\ k &\ll R. \end{aligned}\right\} \qquad (19.72)$$

Thus, the thickness $\underline{k}$ can be regarded as approximately constant over the entire heating surface of the channel. If we determine $\underline{k}$ using the relation

$$\left.\frac{\partial \theta}{\partial n}\right|_{n=+0} = \frac{\bar{t}_f}{k}. \qquad (19.73)$$

then we obtain for the heat-transmission coefficient α and for the wall temperature t_F the following expressions:

$$\alpha = \frac{\lambda_m}{k};$$

$$t_F = \bar{t}_m + \frac{q_F k}{\lambda_m}. \qquad (19.74)$$

where

$$\bar{t}_m = \frac{1}{u_0 \cdot f}\int_0^f u_x(y, z) \cdot t \cdot df$$

is the average calorimetric temperature of the fluid at any section of

the stream; $\overline{\Delta t}^{f} \equiv t_F - \overline{t}^{f}_{zh}$ is the local temperature drop, also at any section of the stream.

Thus, the solution of our problem reduces to a determination of the thickness of the thermal layer $\underline{k}$ and of the average temperature $\overline{t}^{f}_{zh}$. The latter quantity can be readily determined from the heat balance for a portion of the channel of length $\underline{x}$:

$$\overline{t}^{f}_{x} = t_{x1} + \frac{\Pi \cdot q_F \cdot x}{u_0 \cdot C_{p_m} \cdot f}. \qquad (19.75)$$

Substituting Eq. (19.75) into Eq. (19.74), we obtain

$$t_F = t_{x1} + \frac{\Pi \cdot q_F \cdot x}{u_0 \cdot C_{p\,x} \cdot f} + \frac{q_F k}{\lambda_x}. \qquad (19.76)$$

From Eq. (19.73) we obtain

$$k = \frac{\overline{\Delta t}^{f}}{\left.\frac{\partial \theta}{\partial n}\right|_{n=+0}}. \qquad (19.77)$$

and from the Fourier-Ostrogradskiy and the uniqueness conditions for the problem under consideration we obtain the following system of defining parameters for the stabilized field of excess temperature:

$$t - t_{x1} = \varphi_1(y, z, u_x, a_{p\,x}, l_1, l_2, \ldots, l_m) + \frac{\Pi \cdot q_F \cdot x}{u_0 \cdot C_{p\,x} \cdot f}. \qquad (19.78)$$

From Eq. (19.75) and (19.78) it follows that

$$t - \overline{t}^{f}_{x} = \varphi_1(y, z, u_x, a_{p\,x}, l_1, l_2, \ldots, l_m). \qquad (19.79)$$

We have previously found that

$$u_x = \varphi_2(n, \delta, \nu, u_0), \qquad (19.80)$$

and we therefore have for the conditions $k \ll \Delta n$ and $k \ll R$

$$t - \overline{t}^{f}_{x} = \varphi_3(n, \delta, \nu, u_0, k, a_{p\,x}).$$

The local temperature difference is

$$\overline{\Delta t}^{f} \equiv t_F - \overline{t}^{f}_{x} = \varphi_4(\delta, \nu, u_0, k, a_{p\,x}). \qquad (19.81)$$

The temperature gradient in the stream of fluid on the walls of the channel is

$$\left.\frac{\partial t}{\partial n}\right|_{n=0} = \varphi_6(\delta, \nu, u_0, k, a_{p\,ж}). \qquad (19.82)$$

Substituting Eqs. (19.81) and (19.82) in Eq. (19.77) we obtain

$$k = \varphi_8(\delta, \nu, u_0, k, a_{p\,ж}). \qquad (19.83)$$

Let us reduce this equation to dimensionless form. Then

$$k = \frac{a_{p\,ж}}{u_0} \cdot \varphi_7\left(\frac{\nu}{a_{p\,ж}}, \frac{\delta \cdot u_0}{\nu}\right). \qquad (19.84)$$

We have shown above that for the conditions under consideration $\delta \cdot u_0/\nu \approx$ const. In addition the value of the Prandtl number is specified here ($Pr \geq 1$).

Let the equivalent round tube have a diameter d_e and a length $l_e = l$. On its surface we maintain a constant heat rate $q_{Fe} = q_F$, and in the tube there flows the same liquid as in the channel ($Pr_e = Pr \geq 1$; $a_{pzhe} = a_{pzh}$; $\nu_e = \nu$; $C_{pzhe} = C_{pzh}$); the flow velocities in both cases are the same ($u_{0e} = u_0$).

Let the thickness of the hydrodynamic layer be small compared with the radius of the equivalent tube:

$$\delta_0 \ll \frac{d_0}{2}.$$

inasmuch as $Pr \geq 1$, we also have $k_e \ll d_e/2$. Consequently, the inequalities (19.72) and Relation (19.84) are satisfied in the equivalent tube. Since all the terms in the right half of Eq. (19.84) for the channel and for the equivalent tube have the same value, we get

$$k \approx k_e. \qquad (19.85)$$

Let us define the diameter of the equivalent tube such that the channel wall temperature and the tube temperature will be the same:

$$t_F = t_{Fe}.$$

For the tube we have

$$t_{F0} = t_{ж1} + \frac{4q_{F0}x}{u_{0ж}C_{p\,ж}\rho_{ж}d_0} + \frac{q_{F0}\delta_0}{\lambda_{ж0}}. \qquad (19.86)$$

Comparing the last relation with Eq. (19.76) we obtain

$$d_e = 4f/P.$$

From Relations (19.85) and (19.74) it follows that Inequalities (19.72) and the relation $Pr \geq 1$ are the necessary conditions for the applicability of d_e in the calculation of heat transmission in channels.

Analogous results are obtained also for the other characteristic heat-transmission process, when a constant temperature $t_F = const$ is maintained on the surface F. In this case the quantity d_e must be determined from the condition of equality of the average heat rates $\bar{q}_F = \bar{q}_{Fe}$.

In calculating heat transmission in each specific channel it is necessary to check whether Inequalities (19.72) are satisfied on the entire or almost on the entire wetted perimeter P. For this purpose the thickness of the thermal layer is determined from the formula

$$k = \frac{d_e}{Nu} \quad \text{or} \quad \frac{k}{d_e} = \frac{1}{Nu_e}. \tag{19.87}$$

Since $k \ll d_e/2$, we have $d_e/Nu \ll d_e/2$. Consequently, the condition of applicability of the equivalent diameter for the calculation of heat transmission has the following form:

$$Nu_{stab} \gg 2. \tag{19.88}$$

In laminar stabilized flow of fluid in a tube $Nu_{stab} = 4.37$ (for $q_F = const$) or $Nu_{stab} = 3.67$ (for $t_F = const$). Consequently, it is impossible to use d_e in these cases to calculate the heat transmission in the channels.

In the case of turbulent stabilized flow of nonmetallic liquids

$$Nu_{стаб} = 0{,}021 Re^{0.8} \cdot Pr^{0.43} \gg 2.$$

For $Pr = 1$ we have $0.021\ Re^{0.8} \gg 2$ or $0.021\ Re^{0.8} \geq 40$. Thus, $Re \geq 12{,}500$.

For turbulent stabilized flow of molten metals ($Pr \sim 0.01$ and $k > \delta$) we have

$$Nu_{ст.б} \approx 4{,}5 + 0{,}014\,(Re \cdot Pr)^{0.8}$$

For $Pr = 10^{-2}$ we have $4.5 + 0.014(Re/100)^{0.8} \geq 40$. Consequently, $Re \geq$ $\geq 1.82 \cdot 10^6$. It is therefore impossible to use d_e for the calculation of heat transmission in the flow of molten metals in channels under ordinary conditions, when $Re \sim 10^4$-10^5.

It must be noted that the calculation of the hydraulic resistance in the flow of molten metals in channels must be carried out in the same manner as for the flow of nonmetallic liquids.

The foregoing considerations concerning the limits of applicability of the equivalent diameter can be regarded as approximate conditions for a unique self-similarity of the temperature fields and the velocity fields in a stabilized stream of liquid with respect to the form and dimensions of the transverse cross section of this stream.

§20. FUNDAMENTAL RESULTS OF EXPERIMENTAL INVESTIGATIONS OF HEAT TRANSMISSION IN TUBES AND CHANNELS FOR TURBULENT FLOW OF FLUIDS

Simplest Experimental Procedure for the Investigation of Local Heat Transmission in Tubes

To determine the connection between the criteria in the similarity equations it is necessary to measure all the quantities that contain the criteria. At the present time various methods are known for the measurement of heat transmission in tubes [41a, b].

Figure 68 shows the principal diagram of the simplest experimental setup for the measurement of heat transmission from the wall of a tube to a stream of nonmetallic liquid [60], when an approximately constant heat rate $q_F = \text{const}$ is maintained on the wall of the tube. Pump 12 feeds the working liquid through flow meter 9 into the inlet chamber 2. Here the liquid, after going through a quieting grid, enters into the working region, made in the form of a tube 4, through the walls of which electric current flows. The heat released here heats the liquid.

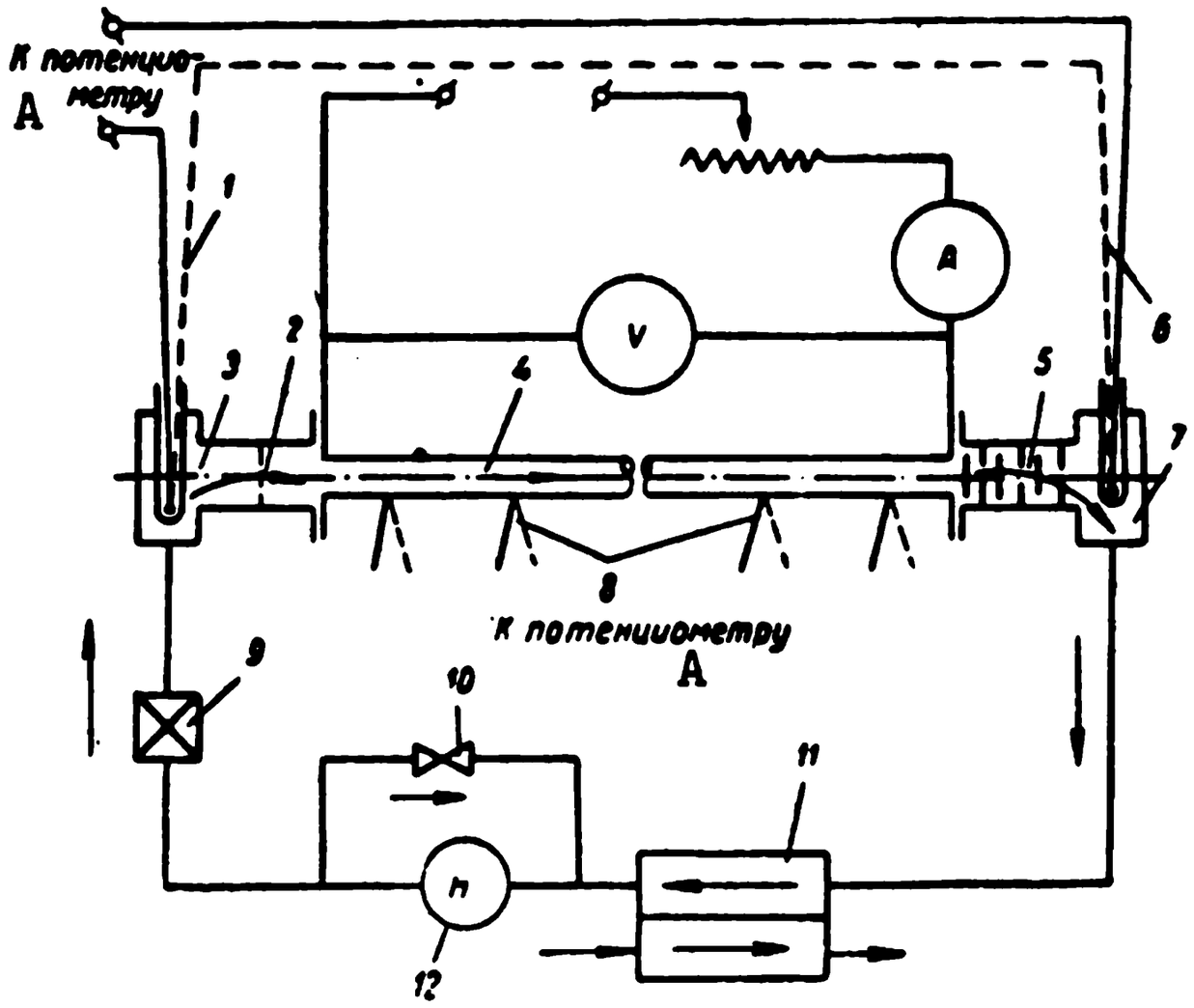

Fig. 68. Schematic diagram of simplest experimental setup for the measurement of heat transmission in a tube with q_F = const. A) To potentiometer.

uid is mixed and its temperature is equalized. Then the liquid passes through a chamber 7 and enters into a cooler 11; there it is cooled to the initial temperature. The speed of the liquid is regulated in tube 4 by means of a bypass valve 10. On the wall of tube 4 are installed thermocouples 8, which measure the temperature $t_F(x)$. Chambers 3 and 7 contain two (or more) thermocouples 1 and 6, which measure the longitudinal temperature drop $\delta t \equiv t_{zh2} - t_{zh1}$ in the stream. In addition, one places in chamber 3 an additional thermocouple which measures the temperature of the liquid t_{zh1} ahead of the tube 4.

To check the correctness of the operation of the installation, the heat consumption, calculated from the readings of the electric instruments connected in the circuit of the working region, is compared with the heat flow calculated from the heating of the liquid in the working region.

The described scheme for the measurement of heat transmission in the tube can be used also to investigate the heat transmission of

molten metals [74]. In this case, however, part of the electric current will flow through the metal stream, which will cause the appearance of heat sources in the working liquid. This circumstance must be taken into account in processing the measurement results.

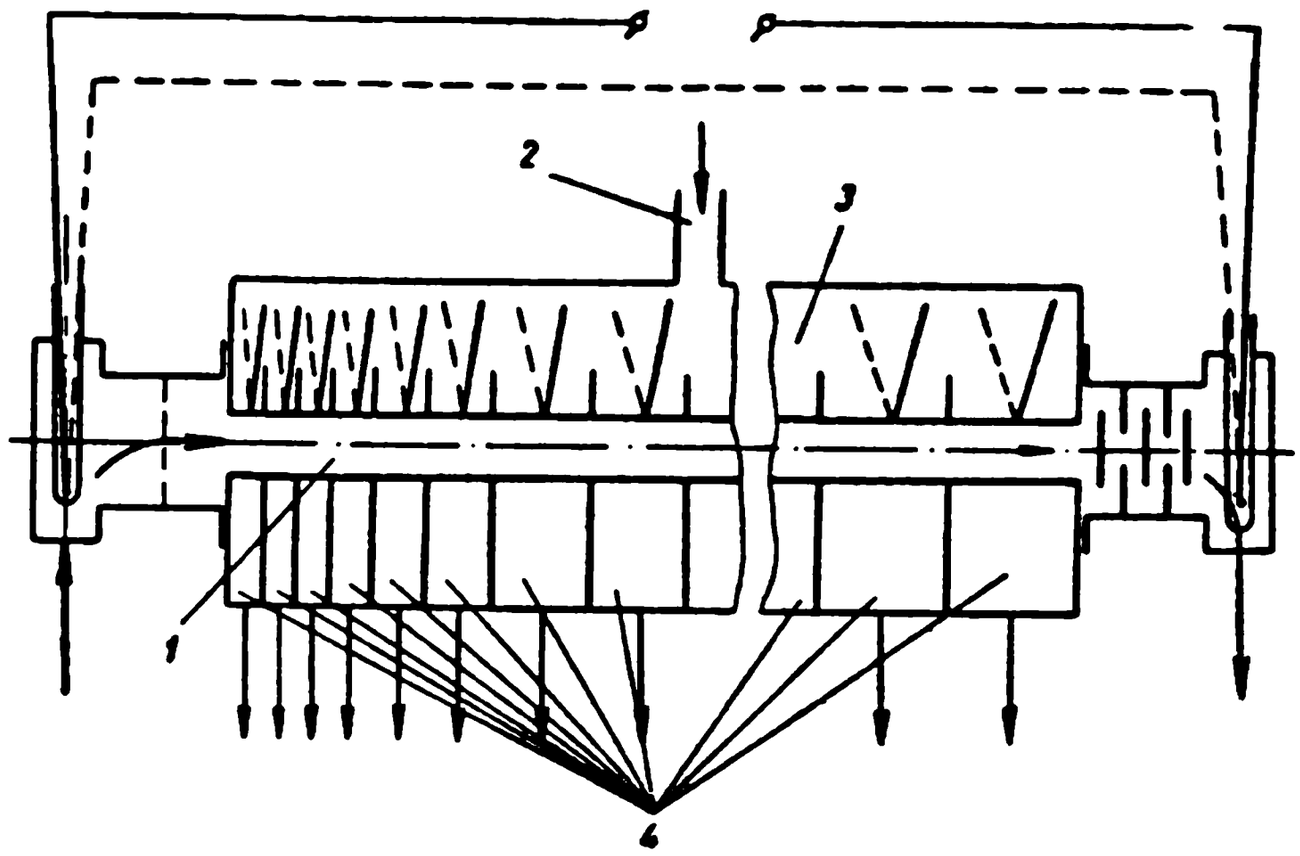

Fig. 69. Schematic diagram of the working region of an installation for the measurement of heat transmission in a tube with t_F = const.

Figure 69 shows the schematic diagram of the working region of the installation for the measurement of heat transmission in a tube, on the wall of which an approximately constant temperature t_F = const is maintained [1]. The working region 1 is surrounded by a jacket 3, into which steam is fed through a fitting 2. The steam space is separated by partition into individual chambers 4. The steam condenses on tube 1, and the condensate flows off into chambers 4. The amount of condensate formed per unit time in each of the chambers 4 is determined by weighing. Knowing the flow of the condensate it is easy to determine the flow of heat through each of the segments into which the surface of tube 1 is divided.

Heat Transmission in Tubes and Channels with Developed Turbulent Flow of Nonmetallic Liquids

To calculate the coefficient of convective heat transmission in a tube or in a channel, averaged over the length, it is possible to use the equation of M.A. Mikheyev [35a, b]

$$\overline{Nu}_f = 0{,}021 Re_f^{0.8} \cdot Pr_f^{0.43} \left(\frac{Pr_f}{Pr_w}\right)^{0.25}. \qquad (20.1)$$

This equation interpolates the results of the measurements of heat transmission, carried out by different authors in developed turbulent flow of liquids, when there is no boiling $t_F < t_n$ and $Re_f \geq 10^4$. Equation (20.1) is valid for heat transmission of nonmetallic liquids, the Prandtl number of which lies in the interval $0.6 \leq Pr_f \leq 700$. In addition, the Prandtl number of these liquids decreases with increasing temperature, i.e., $(\partial Pr/\partial T)_p \leq 0$. It is therefore impossible to use Eq. (20.1) for the calculation of heat transmission in the region of the thermodynamic crisis. The error in the calculation of the heat transmission by means of Eq. (20.1) amounts to approximately $\pm 5\%$, if the heat rate on the heating surface does not exceed $q_F \leq 0.5 \cdot 10^6$ kcal/m^2-hr for liquids and $q_F \leq 0.1 \cdot 10^6$ kcal/m^2-hr for gases. As a result of additional investigations [60] it was established that Formula (20.1) can be used for the calculation of heat transmission to a stream of water also at higher values of heat rate ($q_F \leq 4 \cdot 10^6$ kcal/m^2-hr); in this case the error increases to $\pm 10\%$. The indices f and w indicate that the thermodynamic parameters of the fluid are chosen, respectively, for an average stream temperature

$$\overline{t}_ж = \frac{1}{2}(t_{ж1} - t_{ж2})$$

or for an average wall temperature

$$\overline{t}_F = \frac{1}{l}\int_0^l t_F dx.$$

The dependence of the thermodynamic parameters of the fluid on the temperature is taken into account by the factor $(Pr_f/Pr_w)^{0.25}$, which was determined by experiment. If the thermodynamic parameters of the fluid in the specified temperature interval depend linearly on the temperature, then an analysis of the similarity conditions for the heat-transmission process leads in this case to the appearance of new criteria [30c]

$$\frac{\lambda_{mf}}{\lambda_{mw}};\ \frac{\mu_f}{\mu_w};\ \frac{c_{pmf}}{c_{pmw}};\ \frac{\varrho_{mf}}{\varrho_{mw}}. \qquad (20.2)$$

The factor $(Pr_f/Pr_w)^{0.25}$ indicates that these criteria can enter into the computation equation in the following combination:

$$\frac{\frac{\mu_f}{\mu_w}\cdot\frac{c_{pmf}}{c_{pmw}}}{\frac{\varrho_{mf}}{\varrho_{mw}}\cdot\frac{\lambda_{mf}}{\lambda_{mw}}}.$$

Equation (20.1) is valid in approximately the following range of variation of the criteria (20.2):

$$1 < \frac{\lambda_{mf}}{\lambda_{mw}};\ \frac{\mu_f}{\mu_w};\ \frac{c_{pmf}}{c_{pmw}};\ \frac{\varrho_{mf}}{\varrho_{mw}} < 1{,}5.$$

Formula (20.1) is suitable for the calculation of heat transmission in either tubes or channels with irregular cross sections. In the latter case one introduces as the characteristic dimension the equivalent diameter

$$d_e = 4f/P,$$

where P is the wetted perimeter. The possibility of using d_e is limited by the condition

$$k \ll \Delta n;\ R,$$

which should be satisfied for all or almost all points of the perimeter P. In this case the calculated value of the heat-transmission coefficient does not exceed its true value.

It must also be noted that Eq. (20.1) interpolates the results of

measurement of heat transmission in tubes and channels with smooth walls. Therefore its use for the calculation of heat transmission in the presence of roughness on the walls leads to a certain underestimate of $\bar{\alpha}_L$ compared with its true value. Equation (20.1) makes it possible to determine the stabilized mean value of the heat-transmission coefficient

$$\bar{\alpha}_l = \frac{\bar{q}_F}{\bar{t}_F - \bar{t}_m}.$$

where

$$\bar{q}_F = \frac{1}{l}\int_0^l q_F\,dx;$$

$$\bar{t}_F = \frac{1}{l}\int_0^l t_F\,dx;$$

$$\bar{t}_m = \frac{1}{2}(t_{m1} + t_{m2}).$$

Therefore the formula is valid for long tubes in channels having $l/d_e \geq 50$.

For the calculation of the average heat transmission in short tubes with $l/d_e < 50$, the results of calculations by means of Formula (20.1) must be multiplied by a correction factor $\varepsilon_l = \varepsilon_l(l/d_e, Re_e)$, which takes into account the influence of the length of the short tube on the heat transmission (Fig. 70) [1].

In short tubes the heat transmission depends on the conditions of the entry of the fluid into the tube or the channel. This influence, however, does not extend usually beyond $(15\text{-}20)d_e$.* Therefore for short tubes and channels the given value of the correction coefficient ε_l is valid for different entrance conditions, if $15 \leq l/d_e \leq 50$. For shorter tubes and channels, having $l/d_e \leq 15$, this value of ε_l is valid only if the entrance edge of the channel is sharp. It must also be noted that a smooth entrance into the tube for a fixed heat rate on the wall

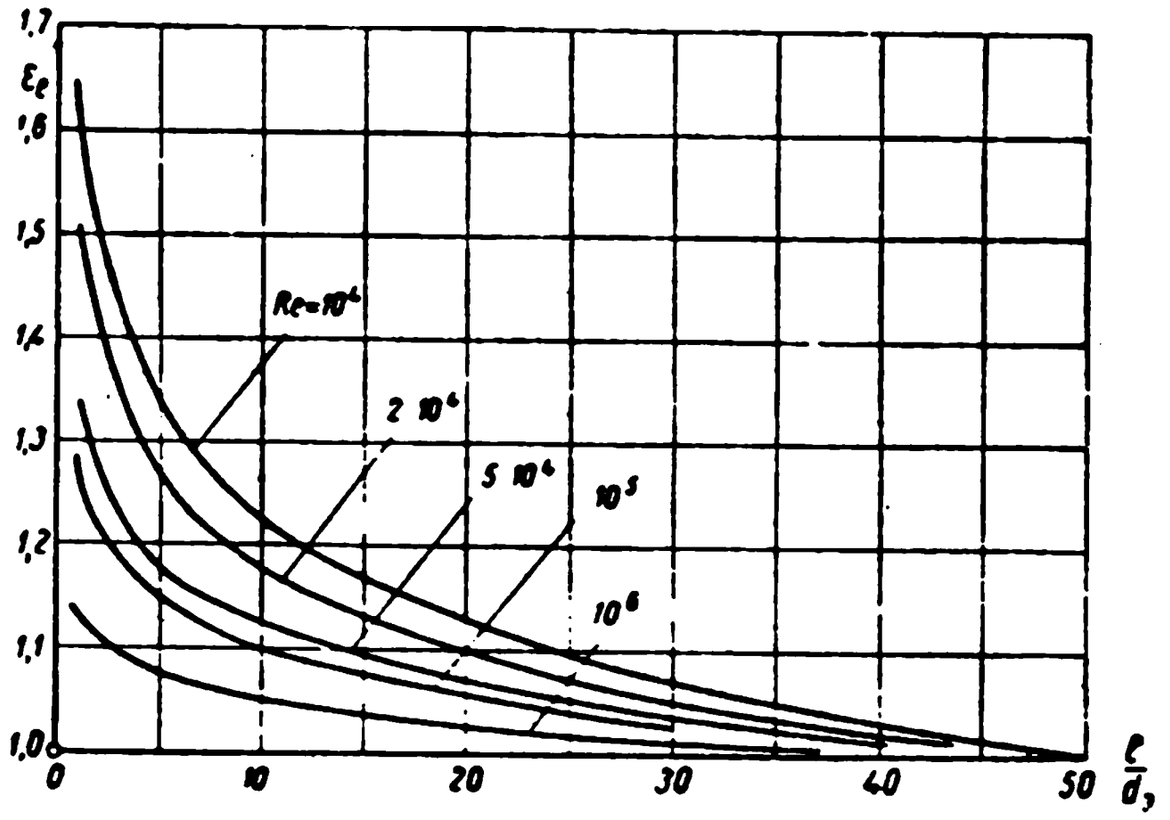

Fig. 70. Correction coefficient which takes into account the influence of the length of the short tube on the heat transmission [1].

q_F = const can cause it to become overheated as a result of formation of a laminar boundary layer (Fig. 71).

To calculate the average heat transmission in bent tubes (Fig. 72), the results of calculations by means of Formula (20.1) must be multiplied by a correction factor

$$\varepsilon_\varphi = 1 + 1{,}77\frac{d}{R}\cdot\frac{\varphi}{2\pi},$$

where R is the radius of the turn of the tube; $0 \leq \varphi \leq 2\pi$ is the angle of the turn; d is the inside diameter of the tube.

To calculate the average heat transmission to a stream of gases it is necessary to assume in (20.1) $(Pr_f/Pr_w) = 1$. For air, which has $Pr \approx 0.72$, the calculation formula assumes the following form:

$$\overline{Nu}_l = 0{,}018\,Re_l^{0.8}. \qquad (20.3)$$

Equation (20.1) can be used both for the determination of the average temperature of the channel wall $\bar{t}_F^{\,l}$, if the thermal load q_F = = const is specified, and for the determination of the average thermal load $\bar{q}_F^{\,l}$, if the average wall temperature $\bar{t}_F^{\,l}$ is specified.

In many cases it is necessary to know the local values of the

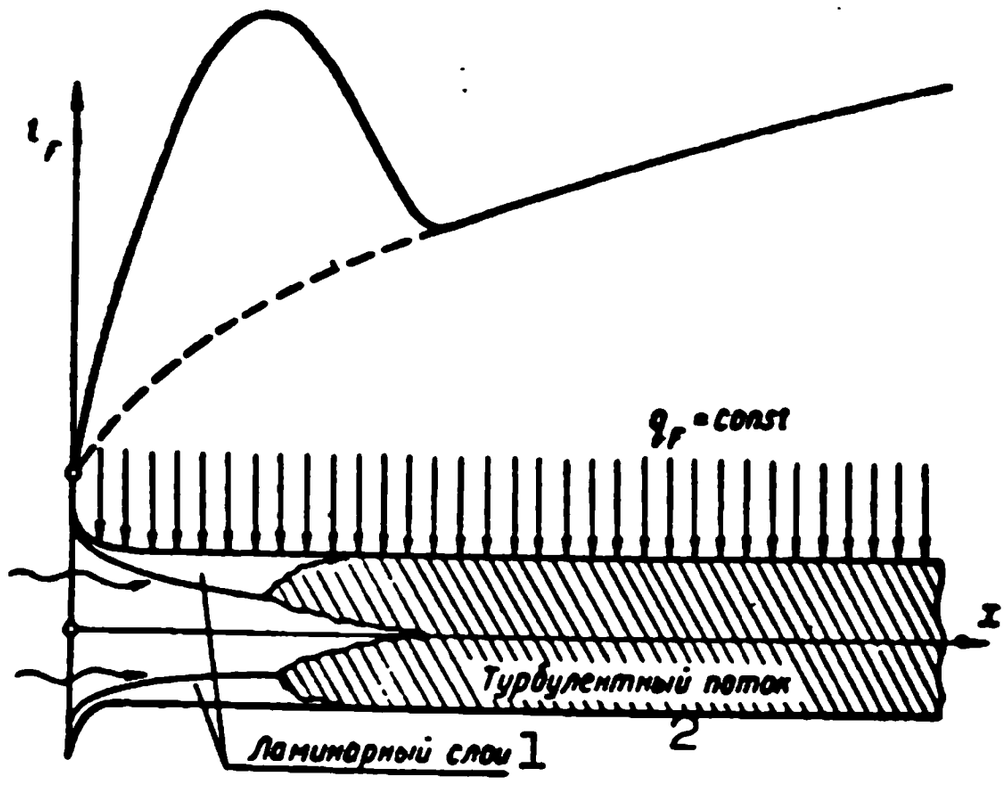

Fig. 71. Distribution of temperature t_F in the initial portion of a tube which has a smooth entrance and a constant heat rate on the wall. 1) Laminar layer; 2) turbulent stream.

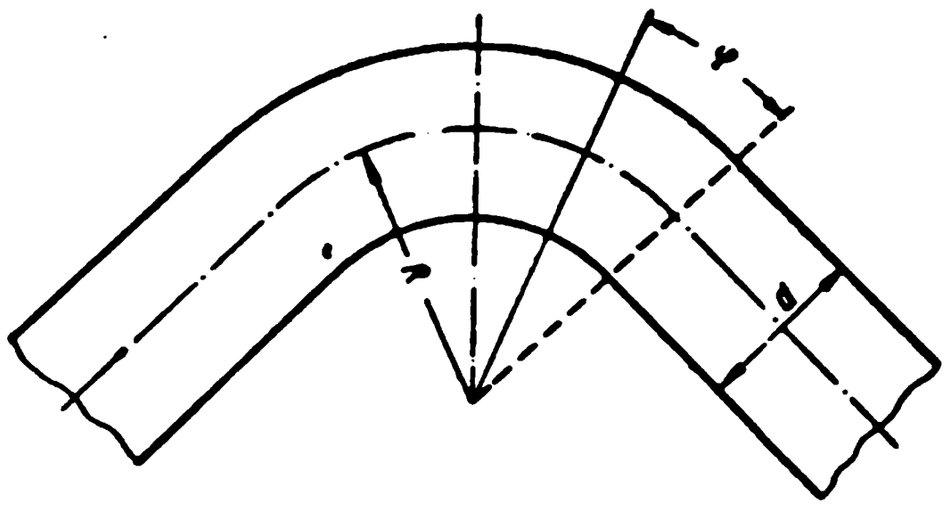

Fig. 72. For the calculation of heat transmission in bent tubes.

heat-transmission coefficients α, the heat rate q_F, and the wall temperature. An approximate calculation of the local characteristics of the heat transmission in tubes can be carried out likewise by using Eq. (20.1).

After determining the distribution of the average value of $\overline{t_F}^x$ or $\overline{q_F}^x$, it is easy to calculate also the distribution of the local values of t_F or q_F. Another variant of the calculation is based on the assumption that Formula (20.1) describes also the local characteristics of the heat exchange. In this case the equation should be written in a different form:

$$Nu_{ж} = 0{,}021 Re_{ж}^{0.8} \cdot Pr_{ж}^{0.4} \left(\frac{Pr_{ж}}{Pr_{ст}}\right)^{0.25}$$

or

$$\frac{q_F(x) \cdot d_э}{\lambda_{ж} \cdot [t_F(x) - t_{ж}(x)]} = 0{,}021 \cdot \left(\frac{u_{0x} d_э}{\nu_{ж}}\right)^{0.8} \cdot Pr_{ж}^{0.43} \cdot \left(\frac{Pr_{ж}}{Pr_{ст}}\right)^{0.25}, \qquad (20.4)$$

where $q_F(x)$, $t_F(x)$, and $t_{zh}(x)$ are the heat rate, the wall temperature,* and the fluid temperature** in the cross section of the tube (or the channel) located at a distance $\underline{x}$ from the entrance, while λ_{zh}, ν_{zh}, Pr_{zh}, and Pr_W refer to $t_{zh}(x)$ and $t_F(x)$, respectively.

It must be noted that Eq. (20.4) is approximate and needs experimental verification. However, until more reliable data are obtained, this equation can be used for the second variant of the calculation of the local characteristics of the heat transmission in tubes and in channels. After comparing the results of the two variants of calculation, the "heavier" one should be chosen.

To calculate the local values of the heat-transmission coefficient one can use also the following formulas [60]:

$$\left.\begin{aligned} Nu_{ж} &= 0{,}0274 C_x \cdot Re_{ж}^{0.8} \cdot Pr_{ж}^{0.36} \left(\frac{Pr_{ж}}{Pr_{ст}}\right)^{0.11}; \\ C_x &\equiv 1 + (2{,}76 - 0{,}44 \lg Re_{ж}) \cdot \left[\left(\frac{d}{x}\right)^{0.6} - \left(\frac{1}{15}\right)^{0.6}\right]. \end{aligned}\right\} \qquad (20.5)$$

These formulas, obtained as a result of an investigation of local heat transmission for flow of water in a round tube, are valid for $2 \cdot 10^4 \leq Re_{zh} \leq 1.3 \cdot 10^5$; $2 \leq Pr_{zh} \leq 12$; $1 \leq Pr_{zh}/Pr_W \leq 6.5$; $0.45 \leq x/d \leq 80$, and $q_F \leq 3.3 \cdot 10^6$ kcal/m^2-hr. The possibility of using this formula for the calculation of heat transmission of other liquids calls for verification. Therefore Eq. (20.1) retains the meaning of a generalized relation, which should be used for the calculation of the heat transmission of any nonmetallic liquid under the conditions indicated above.

The presented calculation formulas are valid only for a turbulent stream that is stationary in the mean. In this connection interest at-

:o the results of an investigation of the heat transmission
: walls of a tube to a nonstationary turbulent stream, which
.odic oscillations of the average velocity over the cross sec-
; a certain frequency of oscillations of this stream, the heat-
:sion coefficient reaches a maximum value, which exceeds by 80%
'esponding value of $\bar{\alpha}_l$ in a quiet nonpulsating turbulent stream.
.vely small increase in the hydraulic resistance of the tube is
l at the same time.
: calculation of heat transmission in tubes for laminar flow of
should be carried out in accordance with the equations obtained

:nsmission of Gases at Large Heat Rates

· calculations of the local heat transmission of gases that
:h subsonic velocity in tubes and channels, the following equa-
recommended [16]:

$$Nu_{\text{ж}} = 0.025 Re_{\text{ж}}^{0.8} \cdot Pr_{\text{ж}}^{0.4} \cdot \left(\frac{T_{\text{ж}}}{T_F}\right)^{P\left(\frac{x}{d}\right)} \qquad (20.6)$$

$$Nu_{x} \equiv \frac{q_F(x) \cdot d}{\lambda_{\text{ж}} \cdot [t_F(x) - t_{\text{ж}}(x)]};$$

$$Re_{\text{ж}} \equiv \frac{u_{\text{ж}} \cdot d}{\nu_{\text{ж}}}; \qquad Pr_{\text{ж}} \equiv \frac{\nu_{\text{ж}}}{a_{p\text{ж}}}.$$

:cript "zh" in these equations indicates that the physical
:r is chosen at the temperature $t_{zh}(x)$.
: exponent P depends on the dimensionless distance from the en-
.nto the tube or the channel. This dependence is shown in Fig.

:ation (20.6) interpolates (with an error of $\pm 5\%$) the results
:rements of the local heat transmission of air as it is heated
)e on whose walls an approximately constant heat rate q_F =

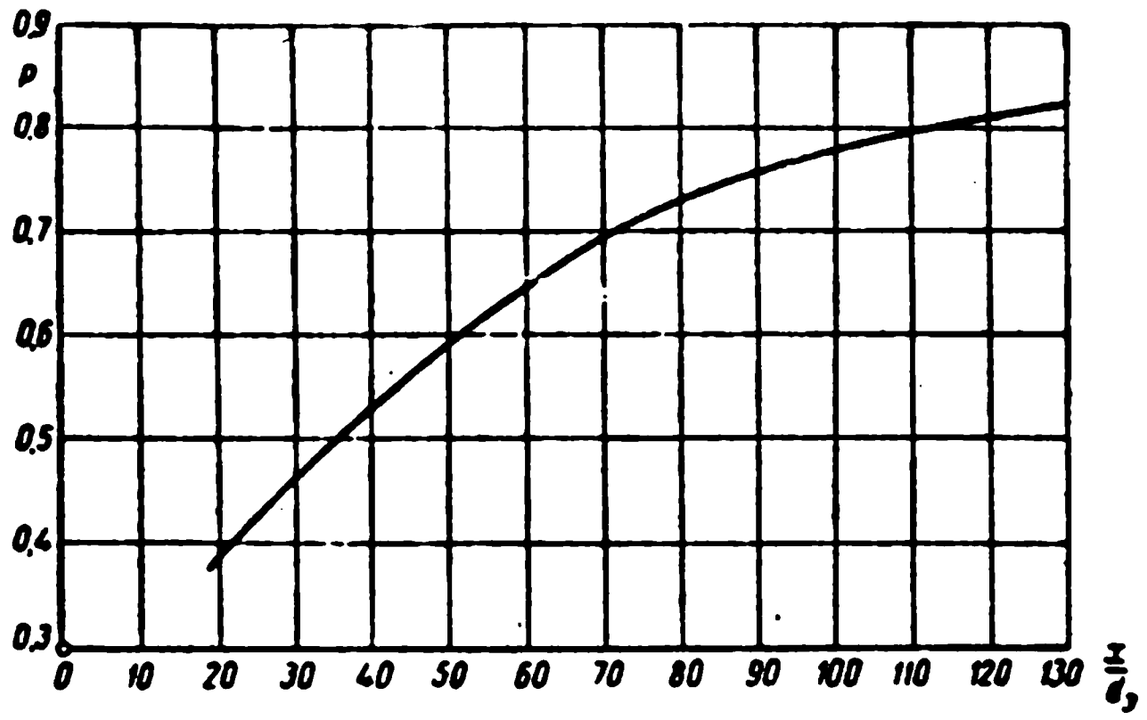

Fig. 73. Dependence of the exponent P on the dimensionless distance x/d_e from the entrance to the tube.

$= \text{const} \leq 3.5 \cdot 10^5$ kcal/m^2-hr is maintained; the wall temperature reached here $t_F \leq 1000^\circ$C, the local temperature difference did not exceed $\Delta t \leq 500^\circ$C, the Reynolds number at the entrance of the tube varied over the range $10^5 \leq Re_1 \leq 6.4 \cdot 10^5$, and the ratio of the wall temperature to the liquid temperature in one and the same cross section varied in the range $1 \leq T_F/T_{zh} \leq 2.5$.

It is impossible to determine directly the local temperature of the wall $t_F(x)$ from Eq. (20.6). To eliminate this shortcoming, the results of the measurement of the heat transmission [16] were represented [10b] in the form of a connection between the dimensionless wall temperature

$$\theta_F \equiv \frac{t_F - t_{ж1}}{t_{ж2} - t_{ж1}}$$

and the criteria*

$$L \equiv \frac{l}{d}; \quad X \equiv \frac{x}{d}; \quad Re_1 \equiv \frac{u_{01} d}{\nu_{ж1}}; \quad \frac{T_{ж2}}{T_{ж1}}.$$

The results of such a processing of the experimental data are shown in Fig. 74 in the coordinates

$$\frac{\theta_F \cdot L^{1.17}}{\left(Re_1 \cdot \frac{T_{ж2}}{T_{ж1}}\right)^{0.17}} = f(x).$$

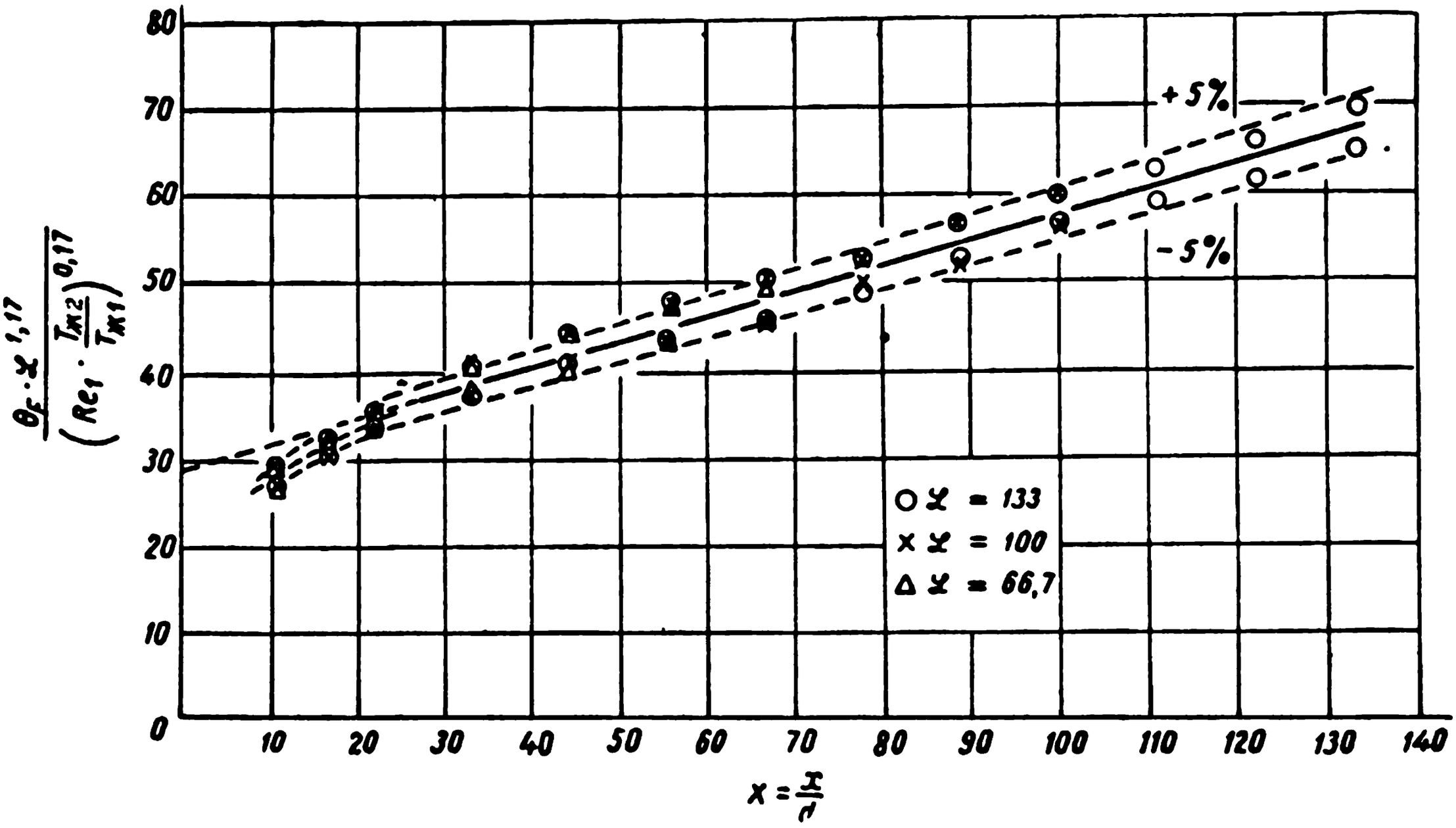

Fig. 74. Dependence of the complex $\frac{\Theta_F L^{1.17}}{\left(Re_1 \cdot \frac{T_{ж2}}{T_{ж1}}\right)^{0.17}}$ on $X \equiv x/d$.

It follows from the plot that the experimental data given in [16] can be represented with an error of $\pm 5\%$ in the form of the following equation, which is valid for $X \geq 30$ and $q_F \approx$ const:

$$\frac{\Theta_F \cdot L^{1.17}}{\left(Re_1 \cdot \frac{T_{ж2}}{T_{ж1}}\right)^{0.17}} = 29\left(1 + \frac{X}{100}\right). \qquad (20.7)$$

Equation (20.7), which makes it possible to determine directly the wall temperature $t_F(x)$, contains merely one thermodynamic parameter (the viscosity ν_{zh1}), which pertains to the lowest air temperature t_{zh} (at the entrance to the tube). The viscosity ν_{zh1} has a relatively small exponent 0.17, which reduces the requirement concerning the accuracy of its determination. In addition, all the criteria in this equation have been obtained theoretically, i.e., they do not entail the use of empirical procedures.

These properties of Eq. (20.7) confirm the general considerations with respect to the convenience of using the dimensionless temperature $\Theta_F \equiv (t_F - t_{zh1})/(t_{zh2} - t_{zh1})$, which were formulated in §19.

Heat Transmission from Molten Metals

Molten metals have found use as coolants because of their thermal stability and relatively high coefficient of heat transmission. In addition, molten metals have two features which distinguish them appreciably from so-called "ordinary" coolants (nonmetallic liquids, gases, and vapors). The first is that molten metals have a very high temperature conductivity compared with kinematic viscosity, i.e., Prandtl numbers that are small compared with unity, $a >> \nu$ and $Pr << 1$. As a result of this feature, the thickness of the thermal boundary layer in a stream of molten metal greatly exceeds the thickness of the hydrodynamic layer, and heat conduction in the turbulent stream of liquid metal makes a noticeable contribution to the transverse transport of heat not only in the layer near the wall, but also in the turbulent core of the stream, where viscosity does not exert a noticeable influence on the structure of the stream. For this reason the Reynolds number in the turbulent stream of the metal has relatively little effect on the heat transmission, and the value of the Nusselt number is determined principally by the Peclet number ($Pe = Re \cdot Pr$). For this reason the turbulent thermal diffusivity is not equal here to the turbulent viscosity, $a_t \neq \nu_t$ and $Pr_t \neq 1$.

The second feature of molten metals, which manifests itself when they are used as coolants, lies in the possibility of appearance of "thermal contact resistance" between the liquid metal and the solid heating surface. The physical nature of this effect is not clear as yet at the present time.

transmission in turbulent flow of molten metals in long tubes with round cross section, one can use the following formula [36]:

$$\overline{Nu_l} = 3{,}3 + 0{,}014\,(Re \cdot Pr)_l^{0.8}. \qquad (20.8)$$

This formula is valid for $Re_f = 10^4$-10^6; $Pr_f = 10^{-2}$-$30 \cdot 10^{-3}$; $Pe_f = 3 \times$ $\times\ 10^2$-$30 \cdot 10^3$, and $L \geq 30$ for a heat rate $q_F \leq 2 \cdot 10^6$ kcal/m^2-hr.

To determine heat transmission in a short tube with $L \leq 30$, the results of the calculations by Formula (20.8) must be multiplied by a correction coefficient ε_l, which takes into account the increase in the heat transmission in the initial portion

$$\varepsilon_l = \frac{1{,}72}{L^{0.16}}.$$

The conditions of entrance of the liquid metal in the channel apparently exert a relatively weak influence on the intensity of heat transmission in the initial portion.

To estimate the largest possible value of the local stabilized heat transmission in turbulent flow of molten metals in long tubes with round cross section, one usually employs the theoretically obtained formula [65]:

$$Nu_{стаб} = 7 + 0{,}025\,(Re \cdot Pr)^{0.8}. \qquad (20.9)$$

This formula is an approximate interpolation (with error $\pm$10%) of results of calculations of the number Nu_{stab} by means of Eq. (19.39). In carrying out these calculations it was assumed that the "thermal contact resistance" between the metal and the heating surface is zero, that the turbulent thermal diffusivity is equal to the turbulent viscosity, i.e., that the turbulent Prandtl is equal to unity, and that the distribution of the time-averaged velocity in the turbulent liquid stream in the tube obeys the laws which were determined experimentally by Nikuradze [38].

The calculations were carried out for $Re = 4 \cdot 10^3$-$3 \cdot 10^6$; $Pr = 10^{-3}$

to 10^{-1}, and Pe = $4\text{-}3\cdot10^{5}$.

Experimental investigations of heat transmission of molten metals were carried out by many authors [29, 36, 66]. The results of most of these investigations lie between the values of Nu obtained from Formulas (20.8) and (20.9), so that the latter can be regarded as the upper and lower limits of the stabilized heat transmission of molten metals in turbulent flow and round tubes. The reasons for the foregoing ambiguity lie apparently in the difficulties involved in investigating this process, and also in the different values of the "thermal contact resistance" in different experimental setups. Therefore the first limitation which was assumed by Lyon with regard to the absence of the "thermal contact resistance" leads to the highest values of the heat transmission.

The second assumption, that $a_{rt} = \nu_{rt}$ and $Pr_t = 1$, exaggerates somewhat the calculated value of Nu and does not correspond to reality. As a result of measurements of the temperature field in turbulent metal streams, carried out by many authors [66], and also as a result of theoretical estimates of the values of the turbulent Prandtl number [10a], it can be suggested that when Pr $\ll$ 1 we have $Pr_t > 1$, i.e., $a_{rt} < \nu_{rt}$. However, sufficiently reliable laws for the calculation of Pr_t have not yet been determined to this day.

The third assumption of Lyon, regarding the use of the Nikuradze data, leads to an erroneous expression for the absolute term in Formula (20.9). It was shown above that the absolute term in the equation

$$Nu = Nu_0(Re) + f(Re, Pe)$$

can be represented in the form of the formula

$$Nu_0 = 7{,}24 - \frac{49}{Re^{0{,}428}},$$

from which it follows that $Nu_0 \approx 7$ only when Re $\geq 3\cdot10^{4}$; when Re < 3 ×

$\times 10^4$ the value of Nu_0 decreases noticeably and when Re = 2400 it reaches 5.48, which amounts to 78% of $Nu_0 = 7$.

Calculation of the Hydraulic Resistance and Heat Transmission in Channels with Annular Cross Section with the Aid of the Equivalent Diameter

The hydraulic resistance and heat transmission in channels with annular cross section, i.e., in so-called annular gaps, can be approximately calculated by means of formulas valid for a round tube having equivalent diameter

$$d_э = \frac{4f}{\Pi} = \frac{4\pi\left(d_2^2 - d_1^2\right)}{4\pi\left(d_2 + d_1\right)} = d_2 - d_1.$$

It is necessary here, however, that Inequalities (19.60) and (19.72) be satisfied at all points of the perimeter of the annular gap, namely

$$\delta,\ k \ll \Delta n;\ R.$$

As applied to the case under consideration, these inequalities assume the following form:

$$\delta;\ k \ll \frac{d_2 - d_1}{2};\quad \frac{d_1}{2}.$$

If some quantity κ is sufficiently large compared with unity, $\kappa \gg 1$, then

$$2\delta;\ 2k < \frac{d_2 - d_1}{\varkappa},\quad \frac{d_1}{\varkappa}. \qquad (20.10)$$

For the thickness of the hydrodynamic layer δ and of the thermal layer $\underline{k}$, for flow of fluid in the channel, we have previously obtained the following expressions:

$$\delta = \frac{8d_э}{Re_э \cdot \xi(Re_э)};$$

$$k = \frac{d_э}{Nu_э}.$$

As applied to the presently considered case they assume the form

$$\delta = \frac{8(d_2 - d_1)}{Re_э \xi(Re_э)} \text{ and } k = \frac{d_2 - d_1}{Nu_э}. \qquad (20.11)$$

Substituting (20.11) in (20.10) and transforming these equations, we obtain approximate expressions for the limits of applicability of the

equivalent diameter in calculations of the hydraulic resistance and heat transmission in annular gaps.

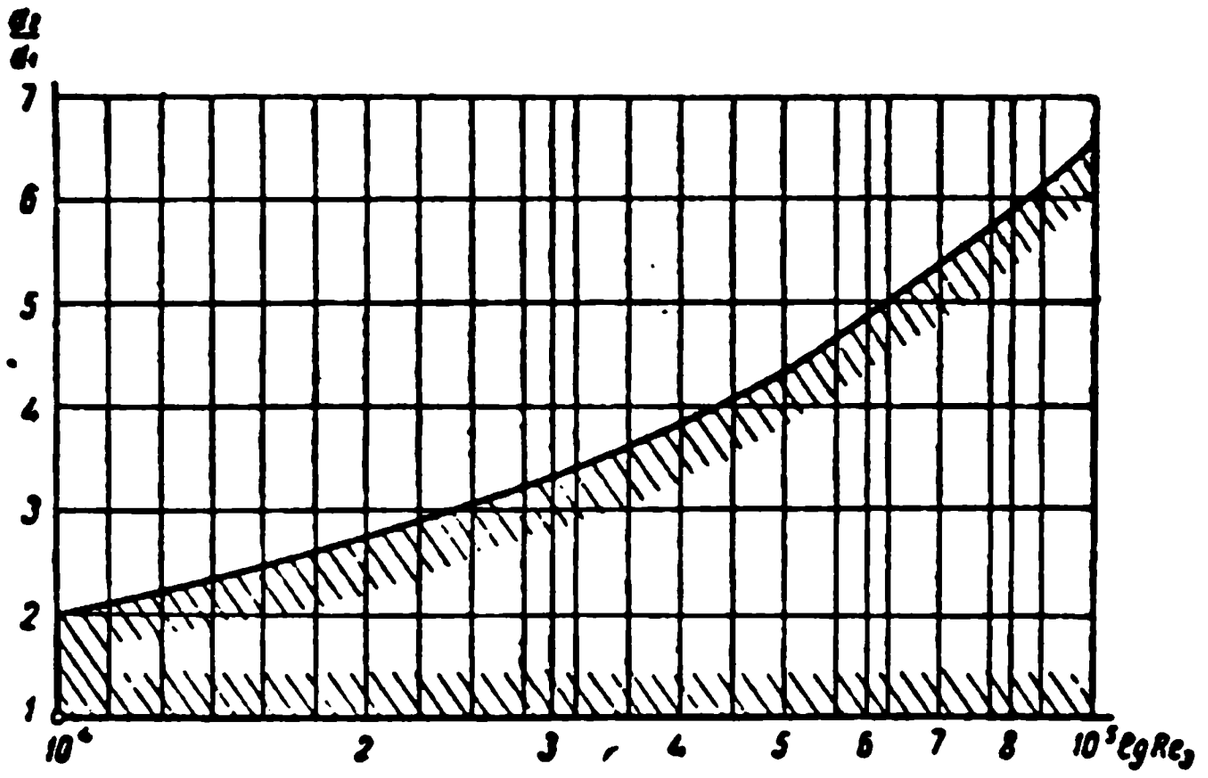

Fig. 75. Region of applicability of the equivalent diameter for the calculation of the hydraulic resistance of a channel with annular cross section.

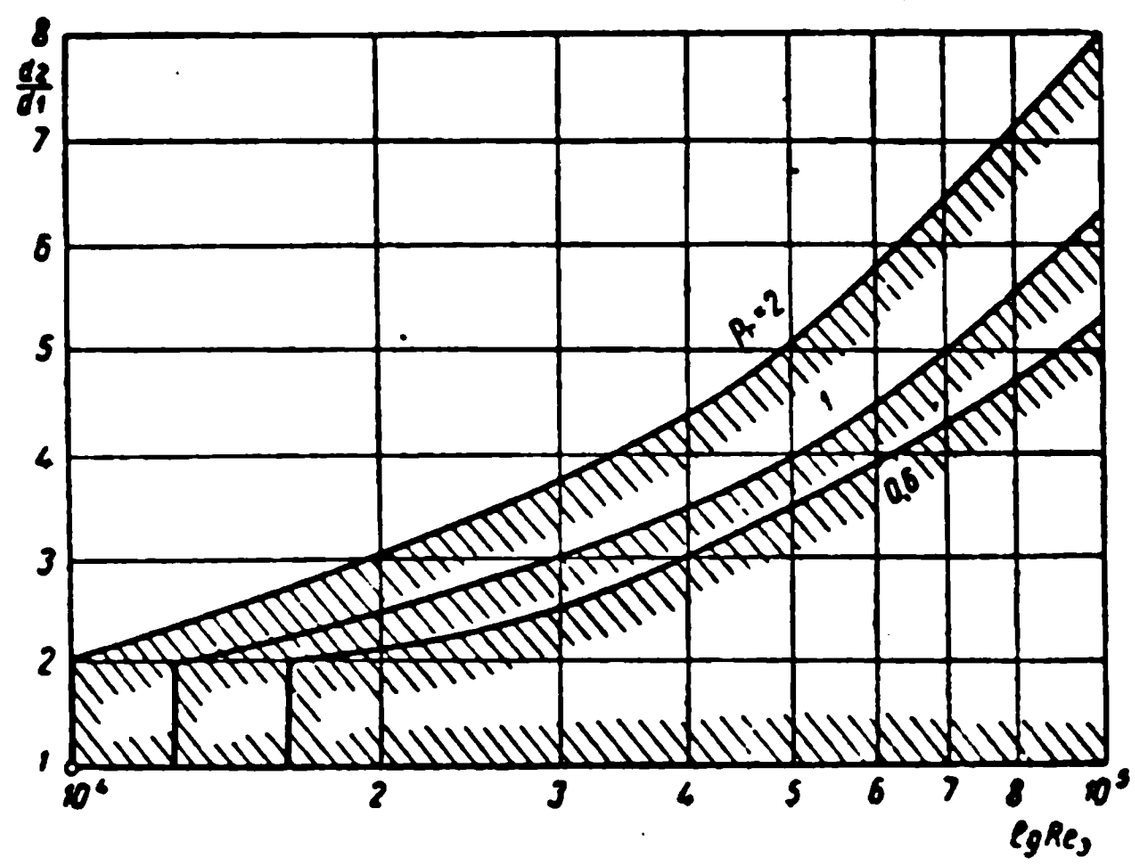

Fig. 76. Region of applicability of equivalent diameter for the calculation of heat transmission in a channel with annular cross section.

For the calculation of the hydraulic resistance

$$\left.\begin{aligned} &Re_э \cdot \xi(Re_э) > 16\varkappa; \\ &1 < \frac{d_2}{d_1} < \frac{Re_э \cdot \xi(Re_э)}{16\varkappa} + 1. \end{aligned}\right\} \qquad (20.12)$$

For the calculation of the heat transmission

$$\left.\begin{aligned} &Nu_э > 2\varkappa; \\ &1 < \frac{d_2}{d_1} < \frac{Nu_э}{2\varkappa} + 1. \end{aligned}\right\} \qquad (20.13)$$

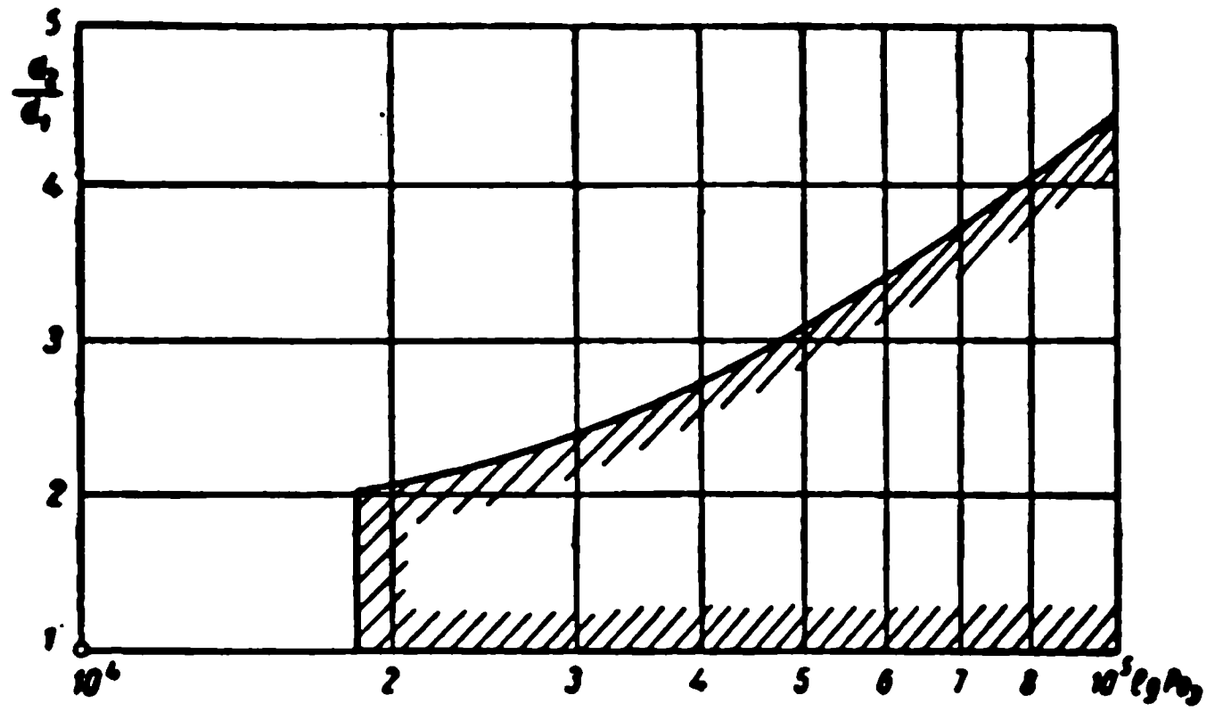

Fig. 77. Region of applicability of equivalent diameter for the calculation of heat transmission of molten metals in a channel with annular cross section.

Let us calculate specific examples of the use of these estimates. If we assume $\kappa = 20 \gg 1$, and take ξ after Blasius to be equal to $0.316/\mathrm{Re}_e^{0.25}$, then calculation yields in place of (20.12)

$$\mathrm{Re}_e \geq 10^4 \text{ and } 1 < \frac{d_2}{d_1} < \frac{Re_e^{0.75}}{1000} + 1. \qquad (20.14)$$

Thus, the limit of the applicability of d_e for the calculation of the hydraulic resistance of annular gaps depends on the Reynolds number and increases with increase in the latter. Since the minimum permissible Reynolds number is equal to 10^4, the tightest limits for the use of d_e have the following form:

$$1 < \frac{d_2}{d_1} < 2.$$

With increasing Re_e the permissible limit of the region of applicability of d_e broadens (Fig. 75).

Let us carry out an analogous estimate of the limits of the region of applicability of d_e for the calculation of heat transmission, if a nonmetallic liquid with $\mathrm{Pr} \geq 0.6$ flows through an annular gap.

Assuming $\kappa = 20$ and $\mathrm{Nu}_e = 0.021\ \mathrm{Re}_e^{0.8} \cdot \mathrm{Pr}_e^{0.43}$, we obtain after calculations in place of (20.13)

$$Re_s^{0.8} \cdot Pr^{0.43} > 1900; \quad 1 < \frac{d_2}{d_1} < \frac{Re_s^{0.8} \cdot Pr^{0.43}}{1900} + 1. \qquad (20.15)$$

Consequently, the limits of the region of applicability of d_e for the calculation of heat transmission in annular gaps, for the flow of non-metallic liquids, depend both on the Reynolds number and on the Prandtl number (Fig. 76). Thus, with increasing Prandtl number the region of applicability of equivalent diameter increases.

To calculate heat transmission of molten metals flowing in channels with annular cross section, we assume $\kappa = 20$ and $Nu_e = 3.3 + 0.014\ Pe_e^{0.8}$. Substituting these quantities into Relations (20.12) we obtain after transformations (Fig. 77)

$$Pe_s > 18600; \quad 1 < \frac{d_2}{d_1} < \frac{3{,}3 + 0{,}014 Pe_s^{0.8}}{40} + 1. \qquad (20.16)$$

Approximate Procedure for the Calculation of the Wall Temperature of a Reactor Channel Cooled by a Nonboiling Liquid

The following are specified: the geometrical shape and dimensions of the cross section f of the cylindrical channel, having a length l (Fig. 78).

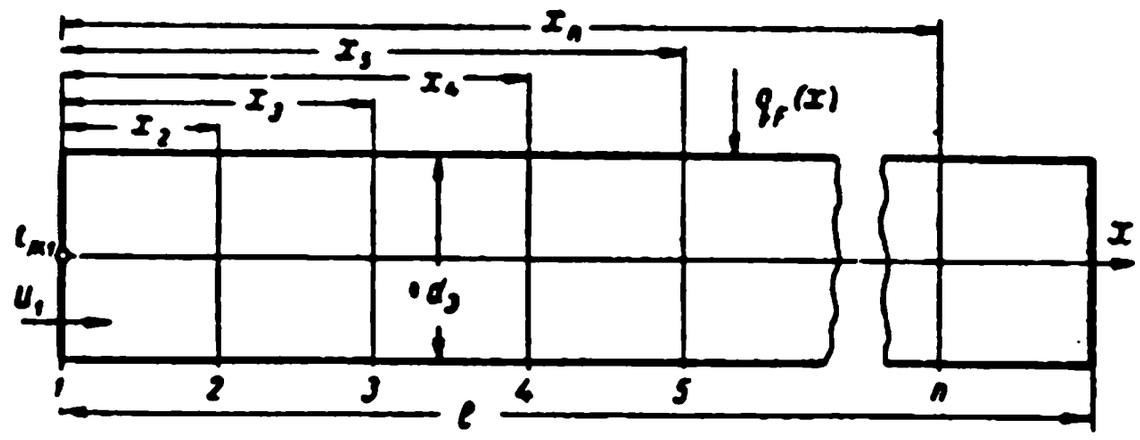

Fig. 78. For the calculation of the wall temperature of a reactor channel.

The temperature t_{zh1} and the pressure p_1 of the cooling liquid ahead of the channel, as well as its thermodynamic parameters.

The gravimetric flow of cooling liquid, G, in kg/sec.

The distribution of the heat rate $q_F(x)$ on the channel walls along the axis.

The calculation of the temperature distribution of the channel

wall along its axis is best carried out in the following sequence:

1) determine the enthalpy i_{zhl}, the number Pr_l, the density ρ_{zhl}, and the viscosity ν_{zhl} of the liquid ahead of the channel;

2) determine the velocity of the liquid u_l, the equivalent diameter $d_e = 4f/P$, and the Reynolds number $Re_l \equiv u_l d_e/\nu_l$ ahead of the channel;

3) determine $\xi = \xi(Re_l)$ and Nu;

4) calculate the thickness of the thermal layer $\underline{k}$ and of the hydrodynamic layer δ by Formulas (19.71) and (19.87);

5) check the possibility of using d_e for the calculation of the heat transmission and the hydraulic resistance. If the form of the cross section is such that for a noticeable part of the perimeter P the inequalities $\delta \ll \Delta n$; R and $k \ll \Delta n$; R are not satisfied, it is necessary to change the shape and dimensions of the cross section $\underline{f}$ in such a way that the foregoing inequalities are satisfied at all points or almost all points of the perimeter P. If this cannot be done, it is necessary to forego the use of the generally accepted formulas for the calculation of heat transmission in round tubes and to carry out an experimental investigation of the local heat transmission in the specified channel;

6) subdivide the channel into $\underline{n}$ equal parts and determine the amount of heat Q_n received by the liquid in each of the parts $0 \leq x \leq x_n$, the start of which is measured from the entrance to the tube (see Fig. 78),

$$Q_n = \Pi \cdot \int_0^{x_n} q_F(x) \cdot dx \quad \text{kcal/hr};$$

7) determine the enthalpy i_{zhn} of the liquid and its temperature in each of the computation sections of the stream:

$$i_{xn} = i_{xl} + \frac{Q_n}{3600G}.$$

The value of t_{zhn} must be determined from the value of i_{zhn}, using for this purpose the corresponding tables or charts;

8) determine the average computed temperature of the liquid in each of the $\underline{n}$ sections $0 \leq x \leq x_n$

$$t_{ж\,n} = 0{,}5\,(t_{ж1} + t_{ж\,n});$$

9) determine for each value of t_{zhn} the thermodynamic parameters of the liquid γ_{zhf}, ν_f, λ_{zhf}, Pr_f;

10) determine for each computation section $0 \leq x \leq x_n$ the average liquid velocity $u_n = G/f \cdot \gamma_{zhf}$ and the Reynolds number $Re_f = u_n d_e/\nu_f$;

11) determine the average value of the heat rate for each of the computation sections

$$\overline{q_F}^{x_n} = \frac{1}{x_n} \cdot \int_0^{x_n} q_F(x) \cdot dx;$$

12) determine the average temperature of the channel wall $\overline{t_F}^{x_n}$ for each computation section from Eq. (20.1)

$$\overline{t_F}^{x_n} = t_{ж\,n} + \frac{\overline{q_F}^{x_n} \cdot d_э}{\lambda_{ж\,f} \cdot 0{,}021 \cdot Re_f^{0.8} \cdot Pr_f^{0.43} \cdot \left(\frac{Pr_f}{Pr_w}\right)^{0.25} \cdot \varepsilon_l\left(\frac{x_n}{d_э}, Re_f\right)}.$$

In the first approximation one must assume $Pr_f/Pr_W = 1$ and determine $\overline{t_F}^{x_n}$, after which determine Pr_W and calculate the factor (Pr_f/Pr_W). If the value of this factor differs noticeably from unity, the calculation must be repeated. After determining the distribution $\overline{t_F}^{x_n}$, it is necessary to plot this function (on a sufficiently large scale);

13) determine the true temperature of the channel wall t_F in each computation section

$$t_F(x) = \overline{t_F}^{x_n} + x_n \left.\frac{d\,\overline{t_F}^{x_n}}{dx}\right|_{x=x_n}.$$

The derivative must be determined here by graphical differentiation of the function $\overline{t_F}^{x_n}(x)$. For control purposes and to make the results ob-

tained more precise, the calculation must be repeated, assuming that Eq. (20.1) describes the local heat transmission in the channel. If the channel is cooled with nonboiling water or with gas, one can use for the determination of $t_F(x)$ Formulas (20.5) or (20.6);

14) for this purpose, knowing the temperature of the liquid t_{zhn} in each of the computation sections x_n, it is necessary to determine the thermodynamic parameters of the liquid γ_{zh}, ν_{zh}, λ_{zh}, and Pr_{zh} in these sections;

15) for each computation section $\underline{x}$ determine the velocity of the liquid $u_{zhn} = G/f \cdot \gamma_{zh}$ and the Reynolds number $Re_{zhn} = u_{zhn} \cdot d_e/\nu_{zh}$;

16) determine the true wall temperature in each of the computation sections, using Eq. (20.1)

$$t_F(x_n) = t_{\text{ж}}(x_n) + \frac{q_F(x_n) \cdot d_e}{\lambda_{\text{ж}} \cdot 0{,}021 Re_{\text{ж}}^{0.8} \cdot Pr_{\text{ж}}^{0.43} \cdot \left(\frac{Pr_{\text{ж}}}{Pr_{\text{ст}}}\right)^{0.25} \cdot \varepsilon_l}$$

or Eqs. (20.5) and (20.6);

17) compare the results of the first and second variants of the calculation $t_F(x)$ and choose the one for which the wall temperature of the channel is the higher.

The calculation of the cooling of a channel by liquid metal is carried out in an analogous sequence. As a rule, it is impossible to use d_e in this case, for the thickness of the thermal layer is usually of the same order of magnitude as the dimensions of the transverse cross section $\underline{f}$ of the channel.

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[Footnotes]

515 To simplify the notation, the sign of averaging with respect to the time is left out from now on.

520 This condition is exactly satisfied when q_F = const.

542 If the liquid stream is twisted about its axis, then the "screw" perturbations of the stream will propagate by more than $20d_e$.

545 If the wall temperature varies along the perimeter of the channel, then t_F is the value averaged over the perimeter in the section x.

545 $t_{zh}(x)$ is the liquid temperature averaged over the section f at a distance x from the entrance.

547 For a derivation of these criteria see §19.

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[List of Transliterated Symbols]

493 ж = zh = zhidkost' = liquid

508 гидр = gidr = gidrodinamicheskiy = hydrodynamic

508 т = t = turbulentnyy = turbulent

514 тепл = tepl = teplovyy = thermal

516 ста[illegible] = stab = stabilizirovannyy = stabilized

520 конв = konv = konvektivnyy = convective

525 э = ekvivalentnyy = equivalent

527 кр = kr = kriticheskiy = critical

527 П = P = perimeter = perimeter

528 макс = maks = maksimum = maximum

540 н = n = nasyshcheniye = saturation

Chapter 10

HEAT TRANSMISSION OF BOILING NONMETALLIC LIQUIDS

§21. MECHANISM OF HEAT TRANSMISSION IN BOILING OF NONMETALLIC LIQUIDS ON THE HEATING SURFACE

Features of Heat-Transmission Mechanism in Boiling

Boiling is defined as such an evaporation of liquids in which the appearance of the vapor is accompanied by formation of new separation surfaces between the phases in the volume of the liquid.

In power reactors one can have volume, surface, or mixed boiling.

In the case of volume boiling:

1) the vapor bubbles can originate at any point of the volume of the liquid;

2) the process can occur without heat being supplied from the outside;

3) the boiling liquid can be isothermal.

Volume boiling occurs in a liquid which is heated above the saturation temperature $t > t_n$ (for example, when its pressure p is rapidly reduced to $p < p_n$).

In the case of surface boiling:

1) the steam bubbles can originate only on the heating surface;

2) the process can occur only when heat is supplied to the boiling liquid through the heating surface;

3) the temperature of the wetting liquid sharply increases in a layer adjacent to the heating surface.

The mixed boiling occurs both in the volume and on the heating

surface. The boiling is possible only in the presence of nuclei of the vapor phase. Let us consider the process of heat transmission in the case of surface boiling of a nonmetallic wetting liquid.

A distinction is made between two modes of surface boiling: bubble and film.

The bubble mode is characterized by the formation on the heating surface of a large number of individual steam bubbles, which are detached from this surface after they reach a certain size. These bubbles break up the viscous sublayer of the liquid adjacent directly to the surface and create the principal thermal resistance for convective heat transmission of nonmetallic liquids. Therefore heat transmission in the case of the bubble mode of boiling is quite intense, for example for water at 1 atmosphere absolute $\alpha \leq 5 \cdot 10^4$ and $q_F \leq 1.25 \cdot 10^6$.

The film mode of boiling is characterized by the presence on the surface of a film of vapor, which envelops this surface and separates it from the liquid.

The intensity of heat transmission in the case of the film mode of boiling is one order of magnitude smaller than that in bubble boiling.

To explain the mechanism of heat transmission in boiling, it is convenient to introduce the gravimetric rate of formation of the vapor phase G_F (kg/m^2-hr) on the heating surface and the maximum possible weight velocity G_* of the removal of the vapor from the heating surface under given conditions of bubble boiling. The value of G_F is determined by the heating rate q_F, by the number of operating vapor-formation centers, by the pressure, and by the physical properties of the liquid in the viscous sublayer. The value of G_* is determined by conditions that are external with respect to the viscous sublayer: the lifting force of the vapor bubbles, the speed of flow of the liquid

outside the sublayer, and in some cases the shape and dimensions of the volume of the boiling liquid and the location of the heating surface in the gravitational field.

Developed bubble boiling is defined as a process of intense formation of vapor bubbles on the heating surface, for which the velocity of vapor generation in the viscous layer is small compared with the maximum possible rate of vapor removal, $G_F < G_*$. In this case the intensity of destruction of the viscous sublayer by the bubbles does not depend on the value of the maximum possible rate G_* of vapor removal. Therefore the heat transmission likewise does not depend on the velocity and character of motion of the boiling liquid in the volume, nor on its shape and dimensions. With increasing heat rate q_F the number of bubbles of vapor which destroy the viscous sublayer and decrease the heat transmission increases. It is clear that the rate G_F of formation of the vapor phase also increases here.

With further increase in q_F, $G_F \rightarrow G_*$. When these quantities become equalized and the former begins to exceed the latter, $G_F \geq G_*$, a layer with increased vapor content is formed near the heating surface. This layer makes it difficult for the liquid to reach the heating surface, i.e., makes cooling of the latter difficult. Therefore, for a fixed heat rate q_F, the surface temperature increases, and the remainder of the liquid in the two-phase layer evaporates and the heating surface becomes covered by a solid film of vapor. The film mode of boiling sets in. The described phenomenon is called the first heat-transmission crisis in surface boiling. The corresponding value of the heat rate is designated q_{Fkr1}.

The second heat-transmission crisis occurs on going over from the film mode to the bubble mode. The corresponding value of the heat rate is designated $q_{F_{kr2}}$.

Thus, the value of q_{Fkrl} depends on the rate of removal of vapor from the heating surface, and consequently on the rate of flow of the boiling liquid, on the form and dimensions of the volume of the boiling liquid, and on other factors which are extraneous with respect to the viscous sublayer.

Nucleation, Development, and Detachment of Vapor Bubbles from the Heating Surface

Bubbles can develop only on nuclei of the vapor phase – centers of vapor production. When the number of these nuclei is small, boiling

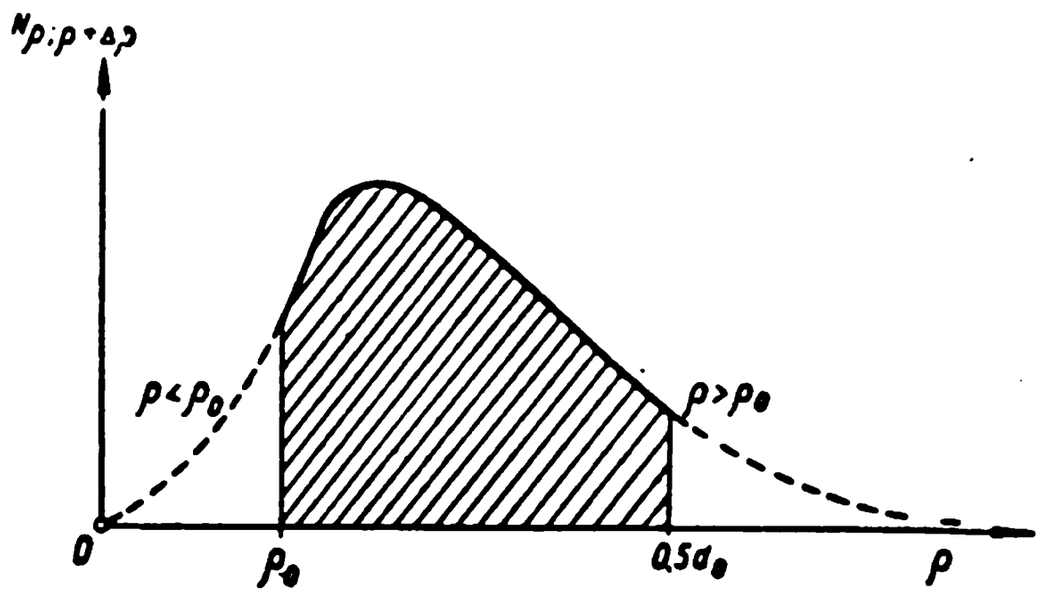

Fig. 79. Probable distribution of the number of nuclei of the vapor phase by dimensions.

is hindered. Thus, for example, water at normal pressure can be superheated almost to 200°C if the number of nuclei is reduced to the possible minimum.

The mechanism of formation and the nature of the nuclei have not yet been investigated in sufficient detail. It is known, however, that the nuclei may be the following:

1) bubbles of gas released from the heating surface;

2) bubbles of vapor occurring as a result of accidental accumulation of fast molecules in the superheated liquid;

3) unevenness in the heating surface.

The experiment with superheating of water to 200°C was carried out with a drop of purified water, which was suspended in heated vase-

line oil. In such a system, the number of nuclei is minimal. On the heating surface, nuclei of various sizes are produced. It is probable that for specified boiling conditions there exists a spectrum of the distribution of nuclei by dimensions which is stable in the mean. A possible form of such a spectrum is shown in Fig. 79. Here $N_{\rho;\ \rho+\Delta\rho}$ is the time-averaged number of nuclei whose radius is a value in the interval between ρ and $\rho + \Delta\rho$ and which occur per unit time on a unit heating surface. A spectrum of similar form was obtained as a result of measuring the number and size of steam bubbles located on the heating surface [49].

It is necessary to distinguish between nuclei of two types: viable and nonviable.

Viable nuclei, once they occur, develop into vapor bubbles, while the nonviable ones rapidly perish after occurrence and produce no bubbles. The radius ρ of the viable nuclei exceeds a certain characteristic value ρ_0, which is called the neutral nucleus radius. Thus, for viable radii $\rho > \rho_0$. The dimensions of the nonviable nuclei satisfy the other condition $\rho < \rho_0$.

The existence of two types of nuclei is attributed to the following causes.

On the surface of the nucleus, i.e., on the separation boundary between the liquid and the vapor, two opposing processes take place: evaporation of the liquid and condensation of the vapor.

In viable nuclei the rate of evaporation exceeds the rate of condensation, while the opposite holds for the nonviable ones.

In turn, the relation between the rates of evaporation and condensation depends on the radius of the nucleus. The reason for it is that the radius of the nucleus determines the vapor pressure in the nucleus.

In accordance with the Laplace law, in the presence of equilibrium

between the phases we have

$$p_\rho = p_{zh} + \frac{2\sigma}{\rho},$$

where p_ρ is the vapor pressure in the nucleus; p_{zh} is the pressure of the liquid around the nucleus; and σ is the coefficient of surface tension in kgm/m^2. The quantity σ characterizes the work that must be consumed to form a unit of new separation surface between phases. According to Bachinskiy, $\sigma \sim (\gamma_1 - \gamma_2)^4$. For water

$$\sigma = 7{,}14 \cdot 10^{-8} \left(\frac{\gamma_1 - \gamma_2}{1000}\right)^4 \text{ kg-m/m}^2.$$

Here γ_1 and γ_2 are the specific gravities of the liquid and water in kg/m^2.

The pressure of the liquid p_{zh} near the nucleus is approximately equal to the pressure of the saturated vapor p_n above a flat free surface of the liquid poured in a vessel. The difference between these quantities is due to the weight of the liquid column, which in most cases can be neglected. Therefore

$$p_{zh} \approx p_n$$

and

$$p_\rho = p_n + \frac{2\sigma}{\rho}. \qquad (21.1)$$

Thus, it follows from the Laplace law that for a specified liquid pressure and for a specified tension coefficient the vapor pressure p_ρ in the nucleus depends essentially on its radius ρ.

Let the radius of the nucleus be such that the vapor pressure in it is equal to the saturation pressure p_{nF} of the given liquid at the temperature t_F on the heating surface.

In such a nucleus, the processes of evaporation and condensation are in equilibrium. Consequently, the nucleus is neutral and its radius is equal to ρ_0. If ρ is so small ($\rho < \rho_0$) that $p_\rho > p_{nF}$, then the rate

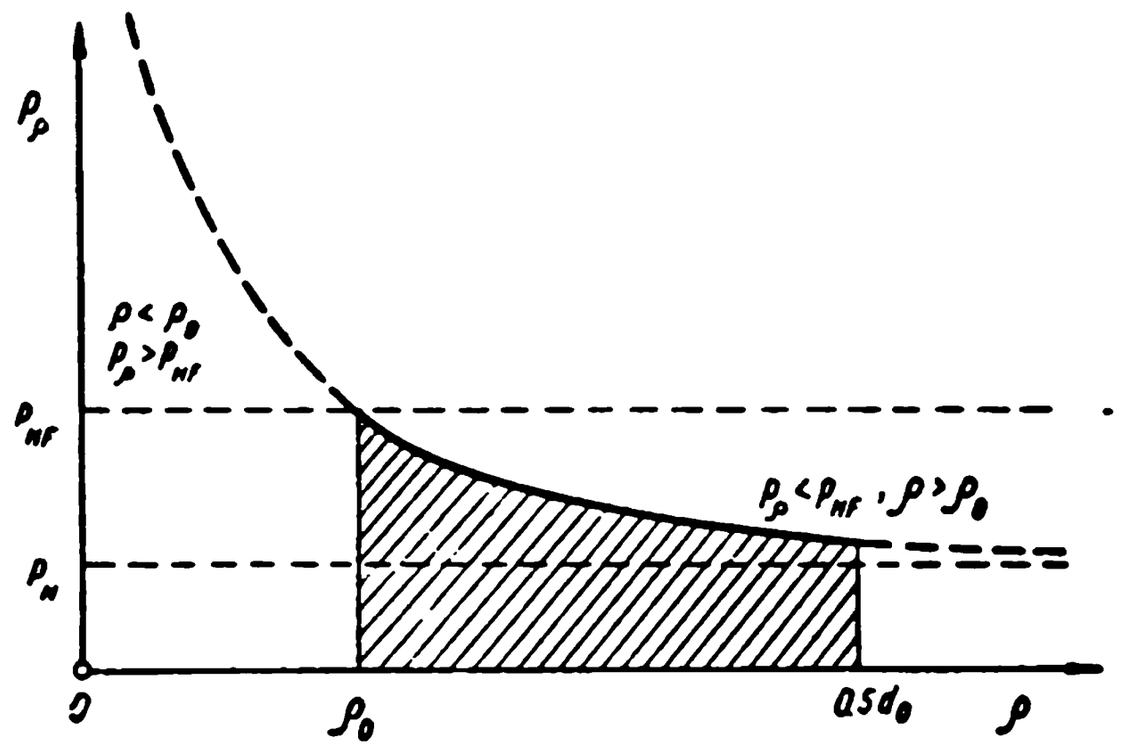

Fig. 80. Region of existence of viable nuclei of the vapor phase.

of condensation in this bubble exceeds the rate of evaporation and the bubble perishes. If ρ_0 is so large ($\rho > \rho_0$) that $p_\rho < p_{nF}$, then the rate of condensation is smaller than the rate of evaporation and the nucleus develops and turns into a vapor bubble (Fig. 80).

It must be noted that the probability that a nucleus having exactly a specified dimension, including ρ_0, will appear during boiling on the heating surface is equal to zero.

The radius ρ_0 of the neutral nucleus can be estimated from the condition of thermodynamic equilibrium between the vapor and the liquid, the free surface of which has a radius of curvature ρ. In the presence of thermodynamic equilibrium between the vapor and liquid phases there is present in the neutral nucleus saturated vapor. The saturation temperature t_{0n} of this vapor is equal to the temperature of the liquid surrounding it. Since the radius of the neutral nucleus is considerably smaller than the thickness of the viscous sublayer, we can assume that the temperature of the liquid surrounding the nucleus is approximately equal to the heating-surface temperature t_F. The saturated vapor pressure in this nucleus is a single-valued function of the saturation temperature

$$p_{0\rho} = p_{0\rho}(t_F),$$

if the radius of curvature ρ_0 is fixed. The pressure of the superheated liquid around the heating surface is approximately equal to the pressure of the supersaturated vapor p_n above a flat free surface, where saturated vapor with temperature t_n is present. This approximation is valid in the case when the weight of the liquid column above the heating surface is small. In turn, $p_n = p_n(t_n)$. We determine the value of ρ_0 with the aid of the Laplace law:

$$p_{0\rho}(t_F) = p_n(t_n) + \frac{2\sigma}{\rho_0}. \qquad (21.2)$$

The saturated vapor pressure $p_{0\rho}(t_F)$ above a curved separation surface between phases is smaller than the saturated vapor pressure $p_n(t_F)$ above a plane surface, for the same saturation temperature t_F. This influence of the curvature of the separation surface is taken into account by the Thomson-Kelvin formula

$$p_{0\rho}(t_F) = p_n(t_F) - \frac{\gamma_2}{\gamma_1 - \gamma_2} \cdot \frac{2\sigma}{\rho_0}. \qquad (21.3)$$

Substituting Eq. (21.3) in Eq. (21.2) and noting that $t_F = t_n + \Delta t$, we obtain

$$p_n(t_n + \Delta t) - p_n(t_n) - \frac{\gamma_2}{\gamma_1 - \gamma_2} \cdot \frac{2\sigma}{\rho_0} = \frac{2\sigma}{\rho_0} \qquad (21.4)$$

or

$$\frac{p_n(t_n + \Delta t) - p_n(t_n)}{\Delta t} \Delta t = \frac{2\sigma}{\rho_0} \cdot \frac{\gamma_1}{\gamma_1 - \gamma_2}. \qquad (21.5)$$

In turn

$$\frac{p_n(t_n + \Delta t) - p_n(t_n)}{\Delta t} \approx \left(\frac{\partial p}{\partial T}\right)_n = \frac{r_n \cdot \gamma_1 \cdot \gamma_2 \cdot A}{T_n \cdot (\gamma_1 - \gamma_2)}. \qquad (21.6)$$

Substituting (21.6) in (21.5), we get

$$\frac{r_n \cdot \gamma_1 \cdot \gamma_2 \cdot A \cdot \Delta t}{T_n \cdot (\gamma_1 - \gamma_2)} \approx \frac{2\sigma}{\rho_0} \cdot \frac{\gamma_1}{\gamma_1 - \gamma_2}.$$

Consequently,

$$\rho_0 = \frac{2\sigma \cdot T_n}{A \cdot r_n \cdot \gamma_2 \cdot \Delta t} = \frac{4{,}76\sigma \cdot T_n}{r_n \cdot \gamma_2 \cdot \Delta t} \text{ мм}. \qquad (21.7)$$

For water at p_n = 1 atm abs, $t_n = 100^\circ C$, and $t_F = 125^\circ C$ we have $\rho \approx$ $\approx 1.3 \cdot 10^{-3}$ mm, and for p_n = 100 atm abs we have $\rho_0 \approx 0.12 \cdot 10^{-3}$ mm.

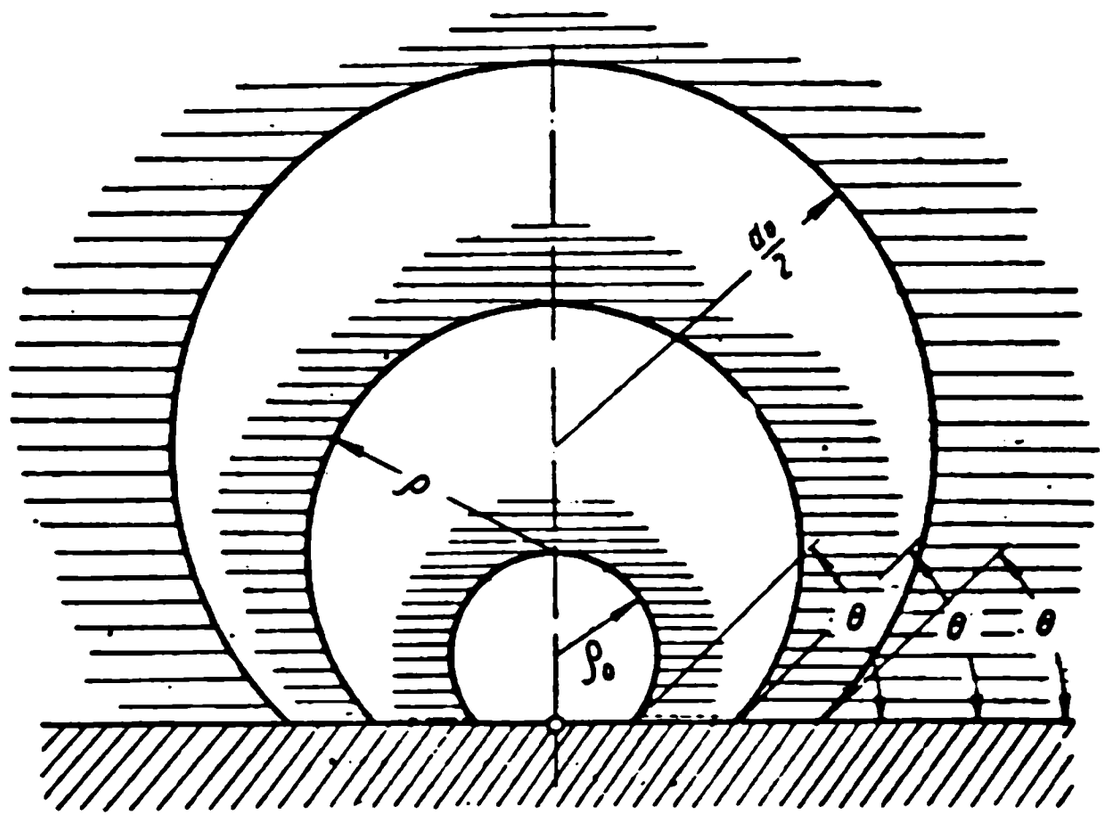

Fig. 81. Simplified scheme of formation of vapor bubble from a viable nucleus.

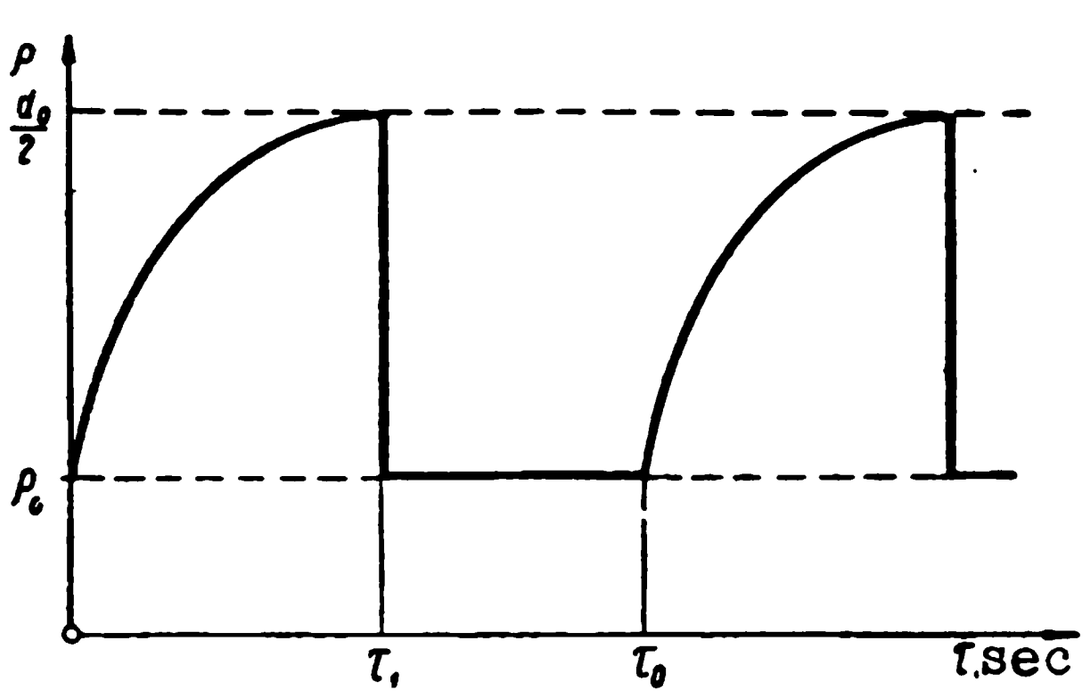

Fig. 82. Diagram of the periodic (in the mean) process of development of a vapor bubble and its detachment from the heating surface.

If a nucleus with $\rho > \rho_0$ has been created somewhere on the heating surface, a vapor bubble begins to form in that place (Fig. 81). But since the heat conductivity λ_{zh} of the liquid is much larger than the heat conductivity of the vapor ($\lambda_p \ll \lambda_{zh}$), almost all the heat from the heating surface will be transferred to the liquid, whose near-wall layer will start to be superheated. The superheated liquid evaporates in a bubble, the diameter of which increases rapidly; then, un-

der the influence of various factors, the bubble breaks away from the heating surface and is replaced by the less heated liquid. Upon detachment, the bubble has a diameter d_0. As a result of the growth and detachment of the vapor bubbles, the viscous sublayer of the liquid is intensely disturbed, i.e., it becomes turbulent. This leads to very intense heat transmission. After detachment of the bubble, a small amount of vapor apparently remains in its place and serves as a nucleus for the development of the next bubble. This repeating process has a random (statistical) character. In the mean, however, it can be regarded as periodic, and the values of the period τ_0 of bubble formation, breakaway diameter d_0 of the bubble, and nucleus diameter 2ρ can be regarded as statistical averages.

Under this condition, the time dependence of the diameter $\underline{d}$ of a bubble located on the heating surface can be qualitatively represented by the scheme shown in Fig. 82. For water boiling under normal thermodynamic conditions $\tau_0 \sim 2 \cdot 10^{-2}$ sec. Approximately half of this time is consumed in formation of the steam bubble with breakaway diameter d_0 from a nucleus $2\rho_0$. The other half is consumed in superheating new batches of liquid flowing into the place of the detached bubble. The breakaway of the bubble of vapor from the heating surface occurs under the influence of various factors: one can observe both the washing away of the bubble by the forced stream of the liquid, as well as its breakaway under the action of the lifting force. Sometimes vapor bubbles separate abruptly from a vertical heating surface and move for some time in a horizontal direction transversely to the vertical stream of the liquid. The mechanism of detachment of bubbles from the heating surface depends essentially under the circumstances under which the boiling process takes place. The simplest mechanism is the detachment of the vapor bubble under the action of the lifting forces

from a horizontal heating surface facing upward.

In this case it is natural to assume that d_0 depends on Θ, σ, and $\gamma_1 - \gamma_2$. Here Θ is the wetting angle. Consequently, $d_0 = f(\Theta, \sigma, \gamma_1 - \gamma_2)$. Reducing this system of parameters to dimensionless form, we obtain

$$d_0 = \varphi(\Theta) \cdot \sqrt{\frac{\sigma}{\gamma_1 - \gamma_2}}. \qquad (21.8)$$

It has been found by experiment that for wetting liquids that have $0 < \Theta < 90^\circ$, the function $\varphi(\Theta)$ is approximately equal to

$$\varphi(\Theta) \approx 0.02\Theta^\circ.$$

For water $\Theta \approx 50^\circ$ and

$$d_0 = 2{,}67 \sqrt{\frac{\gamma_1 - \gamma_2}{1000}} \text{ мм}. \qquad (21.9)$$

For water at $p = 1$ kg/cm^2 we have $d_0 = 2.5$ mm and $\rho_0 = 1.3 \cdot 10^{-3}$ mm. The ratio $d_0/2\rho_0$ is equal to $2.5/(2 \cdot 1.3 \cdot 10^{-3}) \approx 10^3$.

Thus, the diameter of a vapor bubble increases approximately three orders of magnitude within the very short time that it stays on the heating surface. Such an explosive process of the growth of the steam bubble exerts a strong influence on the viscous sublayer of liquid, and leads to an increase in the intensity of the heat exchange.

After detachment from the heating surface, in the case of boiling in a large volume, the bubble rises under the influence of the lifting forces. The process of the lifting of the bubble is slow and occupies much more time than the process of its development from 2ρ to d_0 on the heating surface. Therefore, in spite of the fact that the steam bubble enters into the region of weakly superheated liquid, during the time of its rising it increases in size because of evaporation of the liquid, by at least one order of magnitude compared with the value of its breakaway diameter d_0.

It must be noted that the bulk of the steam (approximately 95%

for water) is produced during the time that the liquid is evaporated in bubbles as they rise, and only an insignificant part (≈5% for water) is produced during the time of development of the bubbles on the heating surface.

Influence of Some Factors on the Intensity of Heat Transmission and its Crisis in the Case of Surface Bubble Boiling in a Large Volume

Influence of temperature difference. For a fixed type of liquid and saturation pressure, to increase the temperature difference $\Delta t = t_F - t_n$ it is necessary to increase the heating-surface temperature t_F. The radius ρ_0 of the neutral nuclei then decreases, since

$$\rho_0 \sim \frac{\sigma T_n}{r_n \gamma_2 (t_F - t_n)}.$$

With decreasing ρ_0, the number of viable nuclei per unit heating surface increases (see Figs. 79 and 80). Consequently, the number of vapor bubbles that destroy the viscous sublayer increases. As a result, an increase in the temperature difference leads to intensification of the heat transmission in the case of surface boiling.

Figure 83 shows in coordinates log α and log Δt the character of the dependence of the heat-transmission coefficient on the temperature difference for bubble boiling of water in a large volume at a pressure of 1 atm abs. It is seen from the figure that at small values of the temperature difference, when $\Delta t < 5^\circ C$, an increase in Δt exerts a small influence on α.

This boiling mode is called convective, since the number of steam bubbles produced is not sufficient for intense destruction of the viscous sublayer. Consequently, the heat transmission in the convective boiling mode obeys the laws governing the natural convection of liquids.

With further increase in the temperature difference ($\Delta t > 5^\circ C$), a developed bubble boiling mode sets in. Here the heat-transmission co-

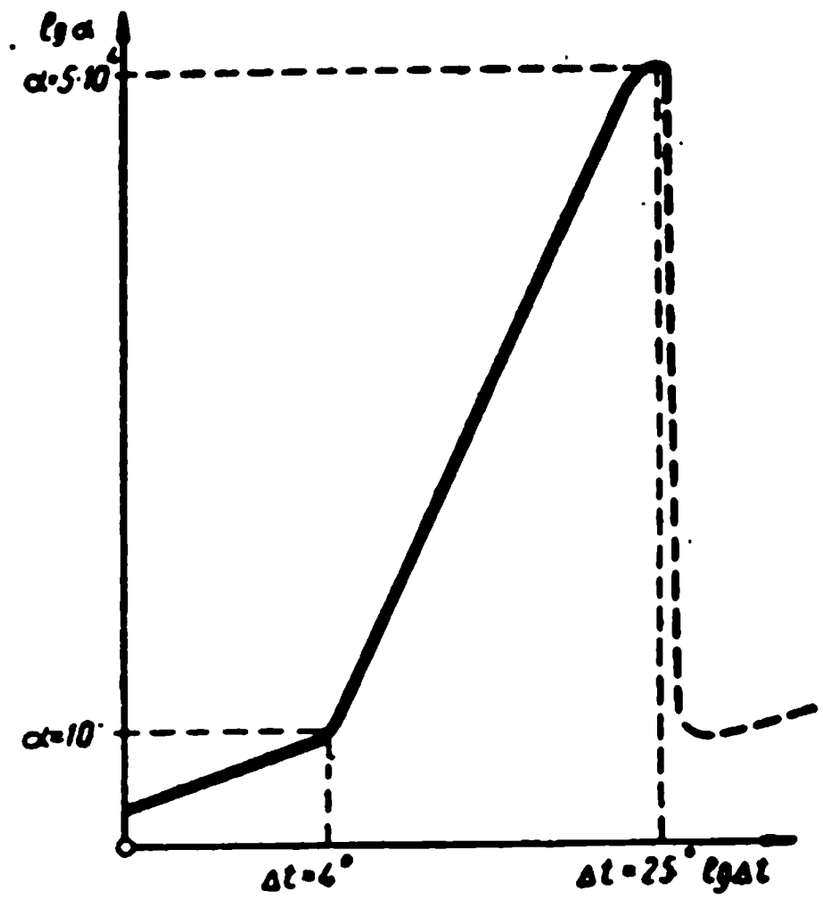

Fig. 83. Character of dependence of the heat-transmission coefficient on the temperature difference for bubble boiling of water in a large volume (p_n = 1 atm abs).

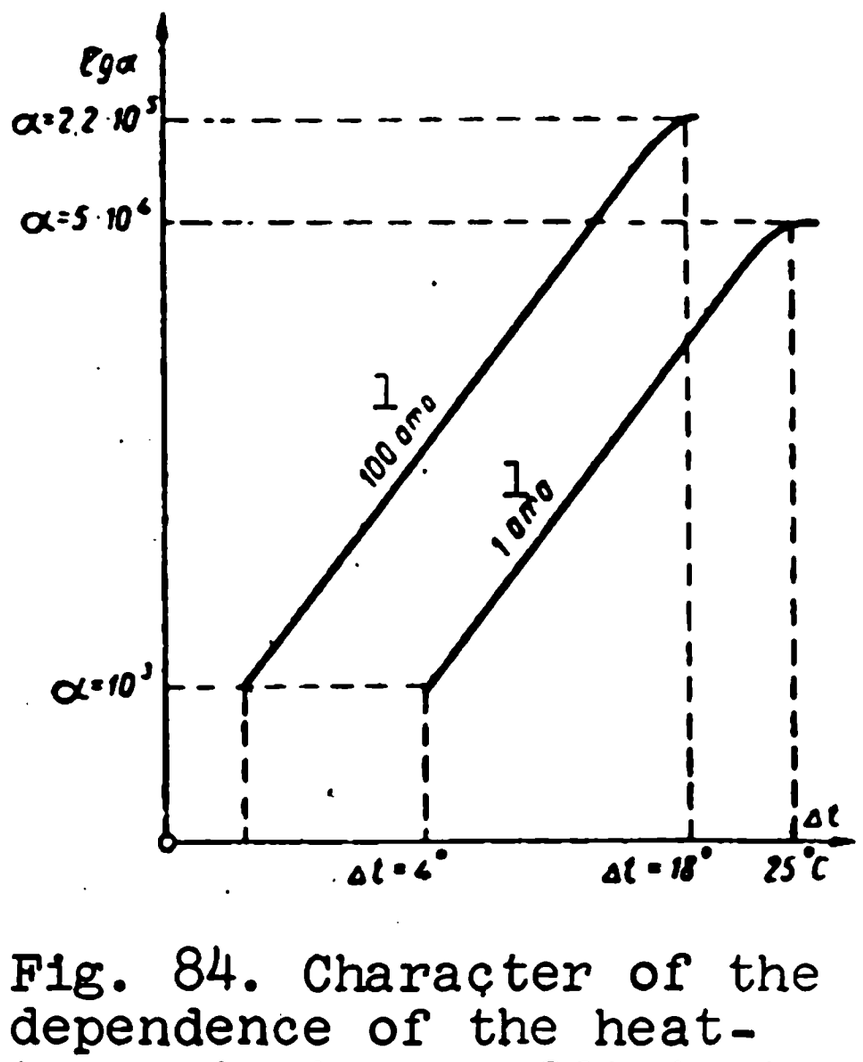

Fig. 84. Character of the dependence of the heat-transmission coefficient on the saturation pressure in bubble boiling in a large volume. 1) atm abs.

efficient becomes essentially dependent on the temperature difference. Whereas for $\Delta t = 5^{\circ}C$ the heat-transmission coefficient in the considered case of boiling of water is equal to $\alpha \approx 10^3$ kcal/m^2-hr-deg, as Δt is increased to $25^{\circ}C$ the heat-transmission coefficient rises to $\alpha \approx 5 \cdot 10^4$ kcal/m^2-hr-deg, i.e., by 50 times. For comparison we note that a similar increase in heat transmission in the case of turbulent flow of water in a tube can be obtained by increasing the velocity of the forced stream by approximately $50^{1/0.8} = 132$ times, which in turn leads to an increase in the power consumed in pumping the liquid by $132^3 = 2.3 \cdot 10^6$ times.

For $\Delta t = 25^{\circ}C$, the rate of steam generation on the heating surface exceeds somewhat the maximum possible rate of removal of this steam in bubble boiling, which is governed by hydrodynamic conditions in the liquid stream. Therefore a film mode of surface boiling sets in: the heating surface becomes separated from the liquid by a film of steam.

Owing to the relatively low heat conductivity of the steam, the heat transmission in the film boiling mode is sharply reduced. For boiling water the coefficient of heat transmission in film boiling is

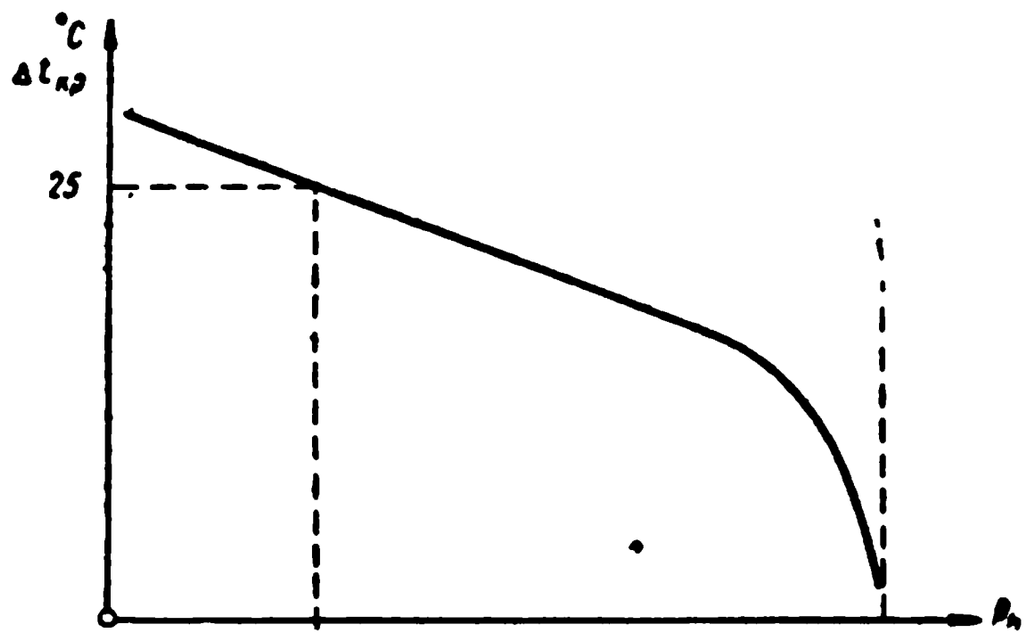

Fig. 85. Character of dependence of critical temperature difference on the saturation pressure in boiling of water in a large volume. 1) atm abs.

approximately 50 times lower than for bubble boiling (at normal pressure). If the heat rate q_F is maintained constant (for example, on the surface of the fuel elements in nuclear power reactors), then the changeover to the film boiling mode leads to a sharp increase in the temperature t_F of the cooled surface, and as a result this surface may go out of order. If the temperature of the heating surface is maintained constant, then the occurrence of the film boiling mode leads to a sharp decrease in the amount of heat transferred.

Thus, the appearance of the heat-transmission crisis on the heating surface during the boiling of a liquid leads to emergency conditions.

Influence of surface-tension coefficient. Since the size of the radius ρ_0 of the neutral nucleus is directly proportional to the surface-tension coefficient, $\rho_0 \sim \sigma$, an increase in the latter, other conditions being fixed, leads to a reduction in the number of viable nuclei of the vapor phase. Consequently, with increasing σ the heat-transmission coefficient decreases.

Influence of saturation pressure. For a fixed value of the tem-

to a decrease in the factor $\sigma T_n/r_n\gamma_2$. Consequently, ρ_0 decreases and the intensity of the heat transmission increases (Fig. 84); the increase in the number of steam bubbles leads here to an earlier heat-transmission crisis during boiling. The character of the dependence of the critical temperature difference Δt_{krl} and the critical heat rate q_{Fkrl} on the saturation pressure for water are shown in Figs. 85 and 86 in logarithmic coordinates.

Influence of the contact angle Θ. The influence of the value of the contact angle Θ on the intensity of the heat transmission during boiling on a heating surface has not been investigated in sufficient detail as yet. It is known [2] that an increase in Θ (Fig. 87) leads to an increase in the heat-transmission coefficient in the case of bubble boiling of water (at normal pressure). It is also known that in the boiling of nonwetting liquids, which have $\Theta > 90^\circ$, only the film mode of boiling is observed.

Influence of the height of the boiling-liquid layer. The height H of the liquid layer in the case of surface boiling in a large volume does not influence the intensity of the heat transmission if $H \gg d_0$ (Fig. 88). In boiling on an external surface of a horizontal tube, the intensity of heat transmission remains constant until the upper point of the tube becomes bare. Consequently, the character of motion of the liquid in a large volume ($l \gg d_0$) for developed bubble boiling does not exert a noticeable influence on the heat transmission.

Influence of roughness of the heating surface. An increase in the roughness of the heating surface leads to the appearance of new viable nuclei and therefore intensifies the heat transmission. However, in the case when the heat rate q_F exceeds $0.2 \cdot 10^6$, this influence practically disappears.

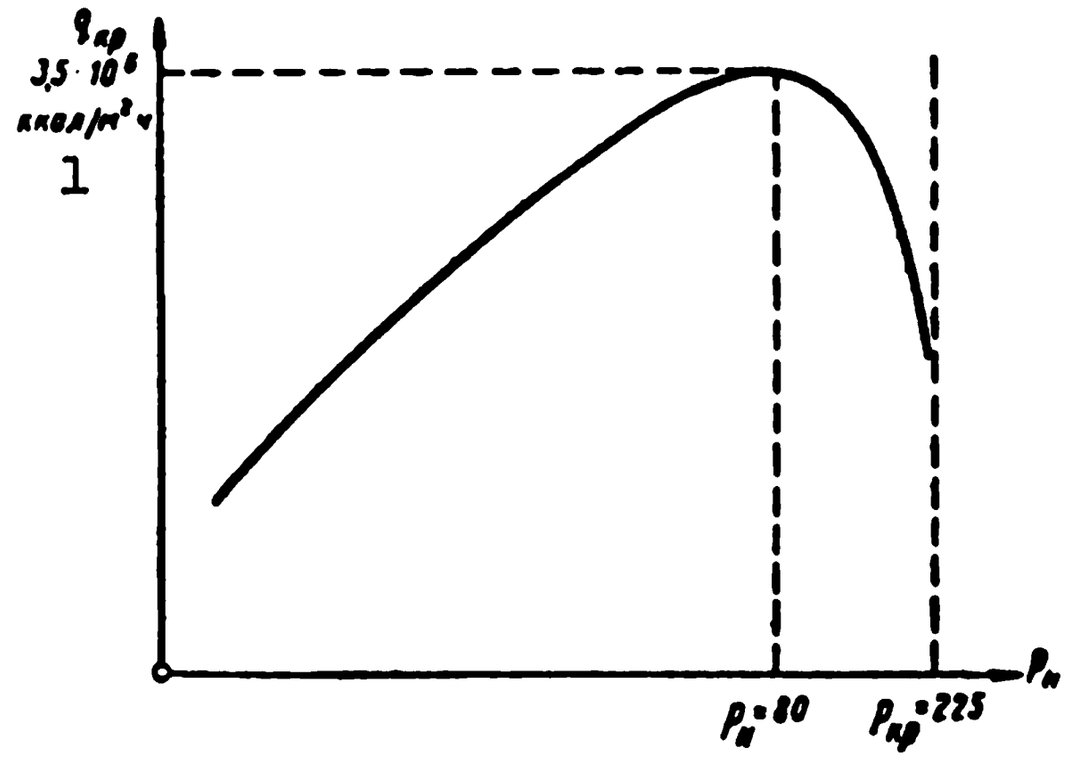

Fig. 86. Dependence of the critical heat rate on the pressure for boiling water in a large volume. 1) kcal/m^2-hr.

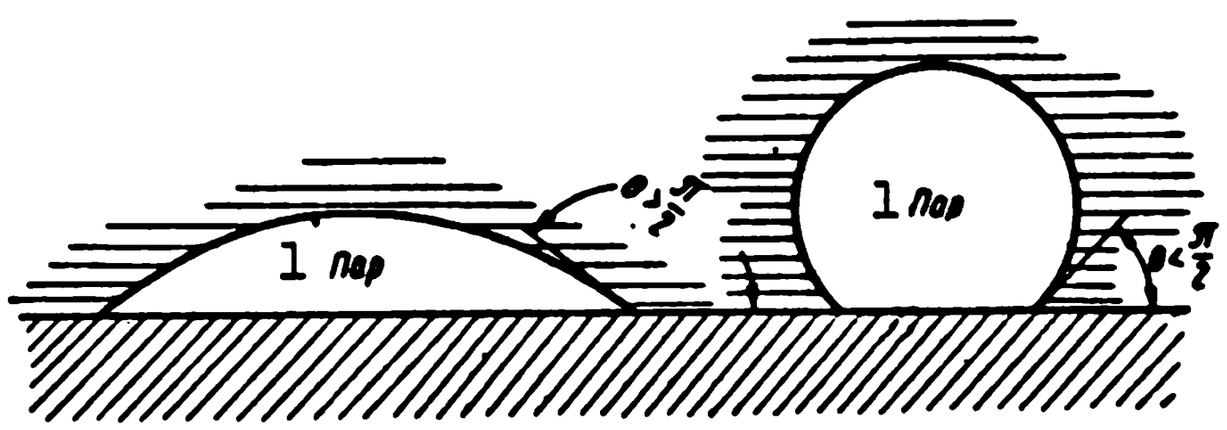

Fig. 87. Form of the vapor bubble on the heating surface for wetting and nonwetting liquids. 1) Vapor.

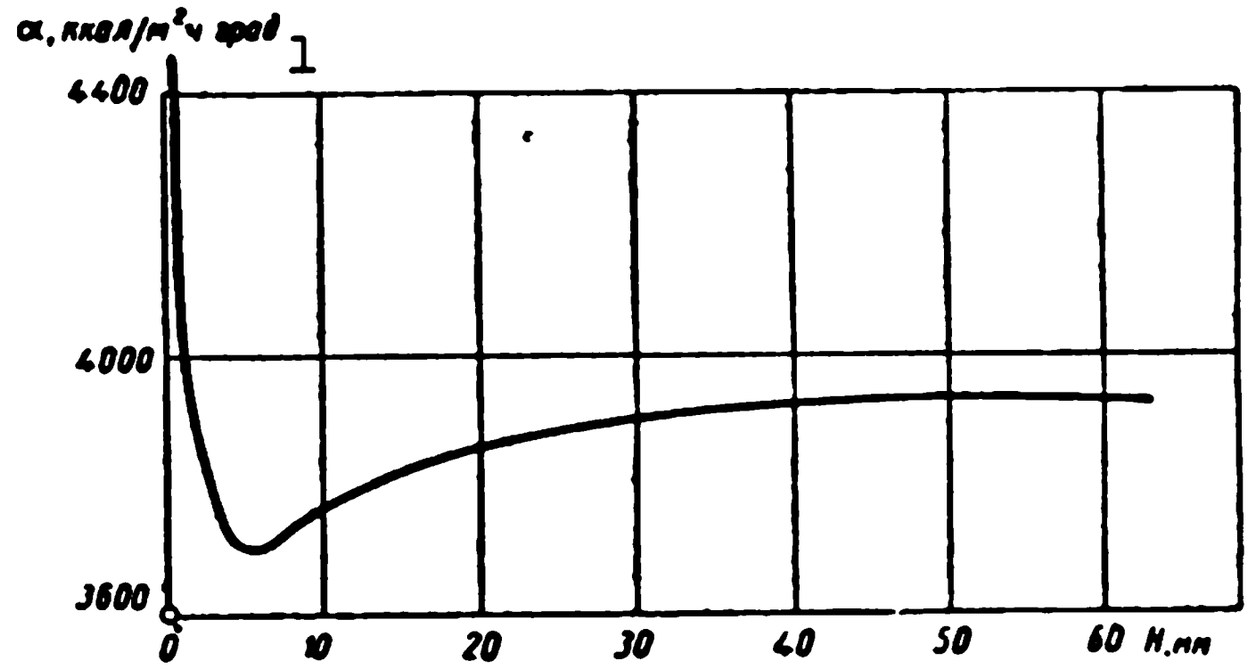

Fig. 88. Character of influence of the height of the boiling-water layer on the heat-transmission coefficient. 1) kcal/m^2-hr-deg.

oxide film on the heating surface increases its roughness, but creates

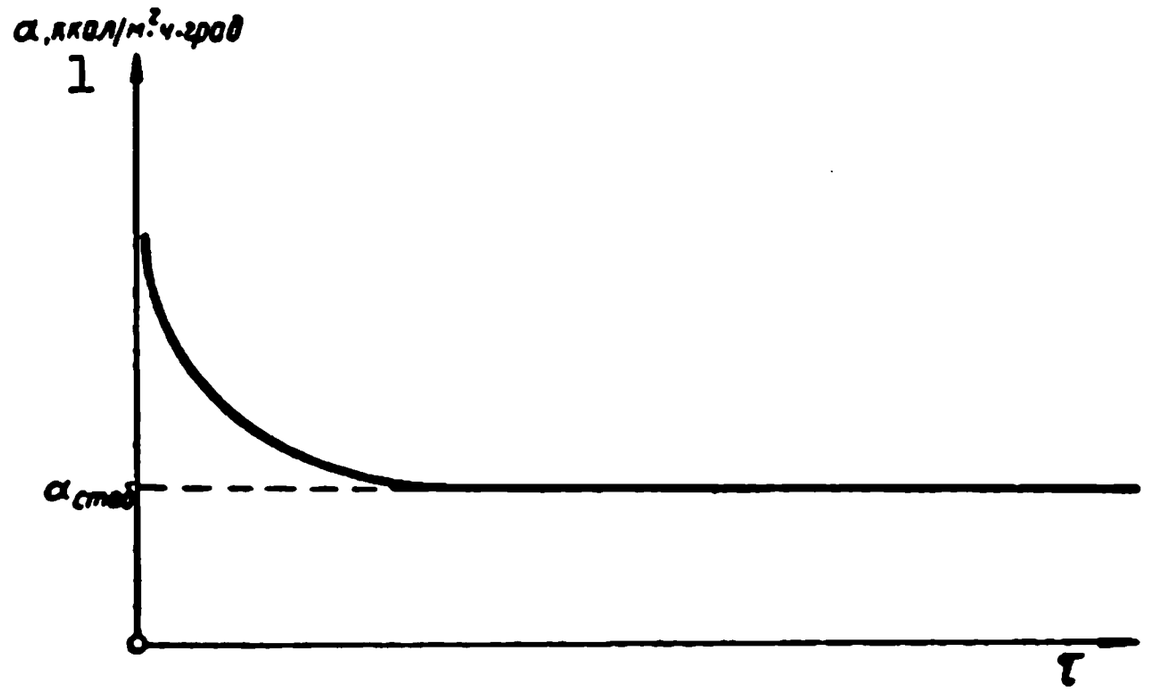

Fig. 89. Character of influence of the duration of boiling on the heat-transmission coefficient. 1) kcal/m^2-hr-deg.

mission in bubble boiling on oxidized surfaces at low heat rates ($q_F \leq \leq 2 \cdot 10^5$) is higher than on clean smooth surfaces. When $q_F > 2 \cdot 10^6$ the influence of the roughness ceases to come into play, but the thermal resistance of the oxides remains. Therefore, at higher heat rates under these conditions the heat transmission is lower than in boiling on clean smooth tubes [31].

Influence of thermodynamic properties of the heating-surface material. At the present time this influence has not yet been sufficiently investigated. There are data to indicate that when $q_F \geq 2 \cdot 10^5$ the thermodynamic properties of the heating surface (thermal conductivity and thermal diffusivity) have no influence upon the intensity of heat transmission in bubble boiling. However, these properties of the heating surface may exert a certain influence on the condition under which the heat-transmission crisis occurs during boiling, since this phenomenon usually has an essentially nonstationary character.

Influence of the duration of the boiling. The character of the influence of the duration of the boiling on the intensity of heat transmission in bubble boiling of water on a heating surface made of polished copper is shown in Fig. 89. It is seen from this figure that

with the increase in duration of boiling the intensity of the heat transmission decreases monotonically, approaching asymptotically a stabilized value. This phenomenon can be attributed to the presence of a large number of random nuclei of the vapor phase, which are removed in the course of time.

Influence of the rate of forced motion of the boiling liquid. The character of this influence (Fig. 90) is determined by the ratio of the intensities of the destruction of the viscous sublayer both due to the forced flow of the liquid and as the consequence of the formation and detachment of the vapor bubbles. At a small speed of forced flow u_0, it does not exert any influence on the intensity of heat transmission, which is determined here principally by the destruction of the viscous sublayer by the vapor bubbles. This is indeed the developed bubble-boiling mode.

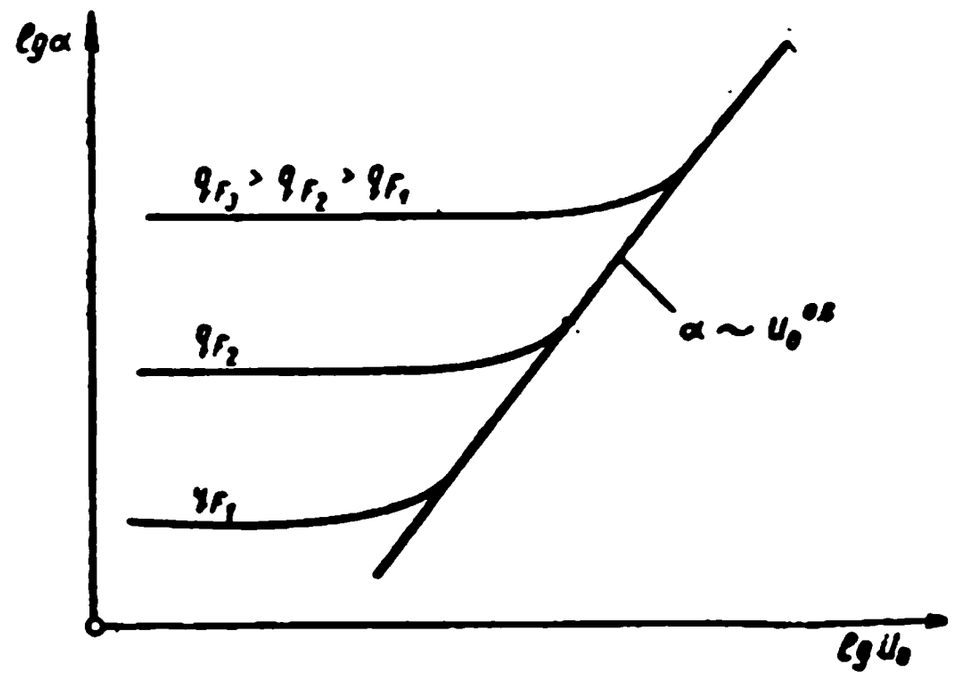

Fig. 90. Character of the influence of the speed of forced flow of the liquid on the heat transmission in bubble boiling.

However, at large values of u_0 the vapor bubbles do not have time to develop and disturb the sublayer, since the forced stream of liquid washes them away and the heat transmission obeys here the laws of convective heat exchange without a change in the aggregate state.

The foregoing far from complete survey of the influence of different factors on heat transmission in bubble boiling of liquids on heat-

ing surfaces indicates the relative complexity of this phenomenon. This circumstance makes it difficult to investigate the process and is the reason why at present there are no generally accepted dimensionless computation equations which take reliably and completely into account all the various factors that influence heat exchange during boiling.

§22. HEAT TRANSMISSION AND ITS CRISIS IN THE CASE OF BUBBLE BOILING IN A LARGE VOLUME

Formulation of the Problem

We consider a model of heat transmission in bubble boiling of a wetting nonmetallic liquid in a large volume, based on the analysis of the operation of one vapor-production center [26]. The heat from the heating surface is transferred to the liquid, which becomes superheated and evaporates in a vapor bubble. The latter increases in size, then breaks away, floats up, and continues to grow as it rises (Fig. 91).

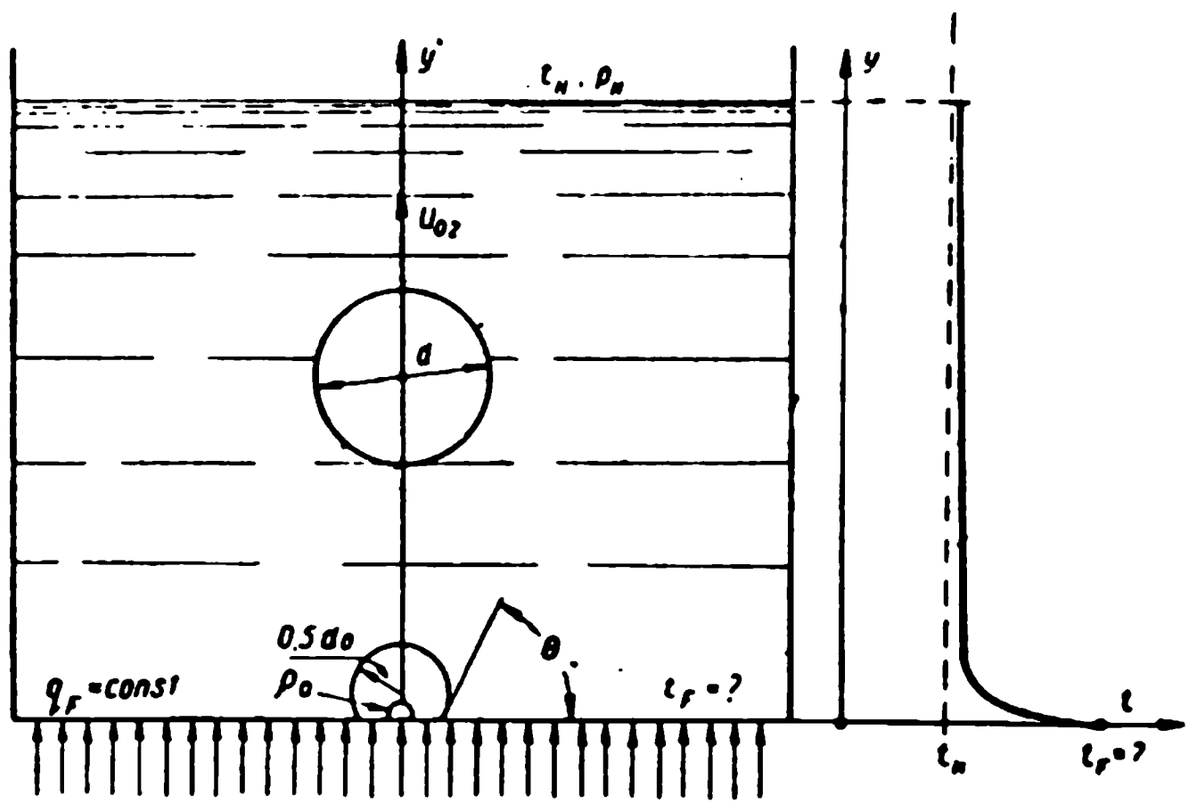

Fig. 91. Formation and detachment of a vapor bubble in the operation of one vapor-production center.

The geometrical uniqueness conditions indicate that we are considering the process of boiling on a horizontal heating surface in a

uid and the dimensions of the heating surface are considerably larger than the breakaway diameter d_0 of the bubble, so that these dimensions do not influence the heat-transmission process.

The physical conditions specify the thermodynamic parameters of the homogeneous liquid, which wets the heating surface, the contact angle $\Theta < 90^\circ$, and also the saturation temperature and pressure (t_n, p_n). In addition, the distribution of the size of the nuclei of the vapor phase on the heating surface $N_{\rho,\ \rho+\Delta\rho} = f(\rho)$ is specified.

The time conditions indicate that the process of formation and detachment of the vapor bubbles is in the mean periodic and does not depend on the initial conditions.

The boundary conditions for the temperature and velocity fields in the liquid are:

on the free surface

$$t|_{y=H} \approx t_H \quad \text{and} \quad u|_{y=H} \approx 0;$$

on the heating surface a constant heat rate q_F is specified:

$$-\lambda_m \frac{\partial t}{\partial y}\Big|_{y=+0} = q_F.$$

In addition, we have here

$$u|_{y=0} = 0;$$

on the surface of the floating vapor bubble we have

$$\frac{\partial}{\partial \tau}\left(\gamma_2 \cdot r_n \cdot \frac{\pi d^3}{6}\right) = -\lambda_1 \oint_{S=4\pi d^2} \frac{\partial t}{\partial r}\Big|_{r=\frac{d}{2}+0} \cdot dS.$$

This equation describes the process wherein the bubble increases as it rises after detachment from the heating surface. At the instant of detachment

$$d|_{\tau=0} = d_0.$$

If we assume that the liquid adheres to the uniformly rising bubble, which has spherical form, then

$$u_{x^2+z^2+[y-u_0\cdot\tau]^2=\frac{d^2}{4}} = u_0.$$

Here $\vec{u}_0$ is the velocity of the uniform rise of the bubble, determined from the condition of equilibrium between its hydraulic resistance force and the lifting force

$$\frac{\pi d^3}{6}\cdot(\gamma_1-\gamma_2)=\xi\left(\frac{u_0 d}{\nu_1}\right)\cdot\frac{\pi d^2}{4}\cdot\frac{\gamma_1\cdot u_0^2}{2g}.$$

Under these conditions it is necessary to determine the average with respect to time and with respect to the surface F of the temperature $\bar{t}_F$ of the surface during the operation of one vapor-production center

$$t_F=\frac{1}{F\tau_0}\cdot\int_0^F df\int_0^{\tau_0} t_F\cdot d\tau.$$

If we assume that there is no thermal resistance whatever on the separation boundary between the liquid and the heating surface, then the temperature t_F at any point of the surface F and at any instant of time is equal to the temperature $t_1|_{y=0}$ of the liquid which is adjacent to the surface at the same point and at the same instant of time

$$t_F=t_1|_{y=0}.$$

To determine t_1 it is necessary to use the differential equations of convective heat exchange

$$\frac{\partial t_1}{\partial\tau}+(\mathbf{u}_1,\ \operatorname{grad} t_1)-a_1\cdot\nabla^2 t_1=0;$$

$$\operatorname{rot}\left\{\frac{\partial\mathbf{u}}{\partial\tau}+(\mathbf{u}_1,\ \operatorname{grad})\,\mathbf{u}_1-\nu_1\cdot\nabla^2\mathbf{u}_1\right\}=0.$$

If we introduce these equations the excess temperature $\vartheta_1\equiv t_1-t_n$, then the mathematical formulation of the problem assumes the following form:

$$\left.\begin{aligned}
&\bar{\vartheta}_F=\frac{1}{F\tau_0}\int_0^F df\int_0^\tau \vartheta_1|_{y=+0}\,d\tau\equiv\overline{\Delta t}\ ;\\
&\frac{\partial\vartheta_1}{\partial\tau}+(\mathbf{u}_1,\ \operatorname{grad}\vartheta_1)-a_1\cdot\nabla^2\vartheta_1=0;\\
&\operatorname{rot}\left\{\frac{\partial\mathbf{u}_1}{\partial\tau}+(\mathbf{u}_1,\ \operatorname{grad})\,\mathbf{u}_1-\nu_1\cdot\nabla^2\mathbf{u}_1\right\}=0;\\
&\vartheta|_{y=H}\approx 0;\quad \mathbf{u}|_{y=H}\approx 0;\quad \mathbf{u}|_{y=0}=0;\\
&-\lambda_1\cdot\frac{\partial\vartheta_1}{\partial y}\Big|_{y=+0}=q_F;
\end{aligned}\right\}\qquad(22.1)$$

$$\left.\begin{aligned}
&\frac{\partial}{\partial\tau}(d^3)=\frac{6\cdot\lambda_1}{\gamma_2\cdot r_n}\int\limits_{S=4\pi d^2}\frac{\partial\vartheta}{\partial r}\Big|_{r=\frac{d}{2}+0}\cdot ds;\quad d|_{\tau=0}=d_0;\\
&\qquad\qquad u_{x^2+z^2+(y-u_0\tau)^2=\frac{d^2}{4}}=u_0;\\
&\frac{3}{4}g\cdot\frac{\gamma_1-\gamma_2}{\gamma_1}=\xi\left(\frac{u_0\cdot d}{\nu_1}\right)\cdot u_0^2.
\end{aligned}\right\}\qquad(22.1)$$

In Eqs. (22.1) the quantities q_F, a_1, ν_1, γ_1, γ_2, g, r_n, $d_0 = f(\theta) \times \sqrt{\sigma/(\gamma_1 - \gamma_2)}$ are specified by the uniqueness conditions, while $\overline{\Delta t}$, ϑ_1, $\vec{u}_1$, τ_0, u_0 are unknown.

Similarity Conditions

From Eqs. (22.1) we obtain a system of parameters determining the average temperature difference $\overline{\Delta t}$ during the operation of one viable nucleus of the vapor phase:

$$\overline{\Delta t}=f_1\left(a_1,\ \nu_1,\ \lambda_1,\ \frac{\gamma_1-\gamma_2}{\gamma_1}\cdot g,\ d_0,\ q_F\right).\qquad(22.2)$$

In bubble boiling under real conditions, there act on a unit surface Z viable nuclei

$$Z=\int\limits_{Q_0}^{0.5d_0}N_{Q_0,\ Q_0+\Delta Q}(Q)\cdot dQ=f_2(Q_0,\ d_0).$$

In this case the average temperature difference $\overline{\Delta t}$ will be smaller than during the operation of one vapor-production center, other conditions the same, since Z nuclei turbulize the viscous sublayer more strongly than a single nucleus. This leads to a better cooling of the heating surface, i.e., to a reduction in its temperature t_F, which decreases the temperature difference, if the saturation temperature remains the same. Therefore the quantity $\overline{\Delta t}$ in real conditions depends on Z. Then we obtain in place of (22.2)

$$\Delta t=f_3\left(a_1,\ \nu_1,\ \lambda_1,\ \gamma_2 r_n,\ \frac{\gamma_1-\gamma_2}{\gamma_1}g,\ d_0,\ q_F,\ Z\right)$$

or

$$\Delta t=f_4\left(a_1,\ \nu_1,\ \lambda_1,\ \gamma_2\cdot r_n,\ \frac{\gamma_1-\gamma_2}{\gamma_1}g,\ d_0,\ q_F,\ Q_0\cdot\overline{\Delta t}\right).\qquad(22.3)$$

where

$$Q_0 \cdot \overline{\Delta t} = \frac{2\sigma A T_н}{r_н \gamma_2}.$$

The quantities

$$a_1,\ \nu_1,\ \lambda_1,\ \gamma_2,\ r_н,\ \frac{\gamma_1 - \gamma_2}{\gamma_1} g,\ d_0 = f(\theta) \times \sqrt{\frac{\sigma}{\gamma_1 - \gamma_2}}$$

and $\rho_0 \cdot \overline{\Delta t}$, for a specified distribution of dimensions of the nuclei, depend only on the type of boiling liquid, the saturation pressure p_n, and the wetting angle Θ. Therefore the equation (22.3) can be written in simpler form:

$$\overline{\Delta t} = f_5 \text{ (kind of liquid, } \theta,\ q_F,\ p_n). \qquad (22.4)$$

If we introduce into this equation the heat-transmission coefficient $\overline{\alpha} = q_F \sqrt{\Delta t}$, then for a specified value of q_F we obtain

$$\overline{\alpha} = f_6 \text{ (kind of liquid, } \Theta,\ q_F,\ p_n), \qquad (22.5)$$

and for a specified heat rate

$$\overline{\alpha} = f_7 \text{ (kind of liquid, } \Theta,\ \overline{\Delta t},\ p_n). \qquad (22.6)$$

Relations of the type (22.4)-(22.6) are convenient for use in investigation of heat transmission of any single liquid.

To study the heat transmission of various liquids it is advantageous to use the general relation (22.3), which must be reduced to dimensionless form. These equations contain $n = 9$ independent dimensional quantities and $k = 4$ groups of homogeneous quantities with measurement units that are independent of one another. Therefore after transformation to dimensionless form we obtain $n - k = 9 - 4 = 5$ similarity criteria as follows:

$$\frac{q_F \cdot f(\Theta)}{\lambda_1 \cdot \overline{\Delta t}} \left(\frac{\sigma}{\gamma_1 - \gamma_2}\right)^{0.5} = f_8 \left[\frac{\nu_1}{a_1};\ \frac{g}{\nu_1^2} \cdot \left(\frac{\gamma_1 - \gamma_2}{\gamma_1}\right) \cdot \left(\frac{\sigma}{\gamma_1 - \gamma_2}\right)^{\frac{3}{2}} \times \right.$$

$$\left. \times f^3(\Theta);\ \frac{q_F r_н \gamma_2 f^2(\Theta)}{A T_н \lambda_1 (\gamma_1 - \gamma_2)};\ \frac{A \cdot T_н \sigma \lambda_1}{f(\Theta)\, a_1 (r_н \cdot \gamma_2)^2} \times \left(\frac{\gamma_1 - \gamma_2}{\sigma}\right)^{0.5}\right]. \qquad (22.7)$$

From this equation it follows that in the case of bubble boiling in a

large volume, the direct proportionality between the heat rate q_F and the temperature difference is violated, i.e., under these conditions the Newton-Rikhman law loses its force, in spite of the assumption that the thermodynamic parameters are independent of the temperature.

The similarity equation (22.7) determines also the conditions of the heat-transmission crisis during boiling. Here the criterion $\frac{q_{F\,кр1}\cdot r_н\cdot\gamma_2\cdot f^2(\Theta)}{A\cdot T_н\cdot\lambda_1\cdot(\gamma_1-\gamma_2)}$ becomes nondefining, since it contains a sought quantity – the critical heat rate q_{Fkr1}, which depends on the conditions under which the process occurs and cannot be chosen arbitrarily if these conditions are specified.

Therefore the indicated nondefining criterion under the conditions of the boiling crisis depends on three defining criteria which enter into Eq. (22.7). Consequently, to determine the critical heat rate q_{Fkr1} and the critical temperature difference we obtain two similarity equations

$$\frac{q_{F\,кр1}\cdot r_н\cdot\gamma_2\cdot f^2(\Theta)}{A\cdot T_н\cdot\lambda_1\cdot(\gamma_1-\gamma_2)}=f_9\left[\frac{\nu_1}{a_1};\ \frac{g}{\nu_1^2}\cdot\left(\frac{\sigma}{\gamma_1-\gamma_2}\right)^{\frac{3}{2}}\cdot f^3(\Theta)\times\right.$$

$$\left.\times\left(\frac{\gamma_1-\gamma_2}{\gamma_1}\right),\ \frac{A\cdot T_н\cdot\sigma\cdot\lambda_1}{f(\Theta)\cdot a_1\cdot(r_н\gamma_2)^2}\left(\frac{\gamma_1-\gamma_2}{\sigma}\right)^{0.5}\right];\qquad(22.8)$$

$$\frac{q_{F\,кр1}}{\lambda_1\cdot\Delta t_{кр1}}\left(\frac{\sigma}{\gamma_1-\gamma_2}\right)^{0.5}\cdot f(\Theta)=f_{10}\left[\frac{\nu_1}{a_1};\frac{g}{\nu_1^2}\left(\frac{\sigma}{\gamma_1-\gamma_2}\right)^{\frac{3}{2}}\cdot f^2(\Theta)\times\right.$$

$$\left.\times\left(\frac{\gamma_1-\gamma_2}{\gamma_1}\right);\ \frac{A\cdot T_н\cdot\sigma\cdot\lambda_1}{f(\Theta)\cdot a_1\cdot(r_н\gamma_2)^2}\cdot\left(\frac{\gamma_1-\gamma_2}{\sigma}\right)^{0.5}\right].\qquad(22.9)$$

If we use the notation

$$\frac{q_F}{\lambda_1\cdot\overline{\Delta t}}\cdot\left(\frac{\sigma}{\gamma_1-\gamma_2}\right)^{0.5}\equiv\overline{Nu}$$

(Nusselt number, or the reciprocal of the dimensionless temperature difference);

$$\nu_1/a_1\equiv Pr_1\ \text{(the Prandtl number)};$$

$$\frac{g}{\nu_1^2}\left(\frac{\sigma}{\gamma_1-\gamma_2}\right)^{\frac{3}{2}}\cdot\left(\frac{\gamma_1-\gamma_2}{\gamma_1}\right)\equiv Ar\quad\text{(the Archimedes number)};$$

$$\frac{q_F \cdot r_M \cdot \gamma_2}{A \cdot T_M \cdot \lambda_1 \cdot (\gamma_1 - \gamma_2)} \equiv K_q \quad \text{and} \quad \frac{A \cdot T_M \cdot \sigma \cdot \lambda_1}{a_1 \cdot (r_M \cdot \gamma_2)^2} \left(\frac{\gamma_1 - \gamma_2}{\sigma}\right)^{0.5} \equiv K_0$$

then Eqs. (22.7)-(22.9) can be rewritten in the form:

$$\overline{Nu} \cdot f(\Theta) = f_8 \left[Pr_1,\ Ar \cdot f^3(\Theta),\ K_q \cdot f^2(\Theta),\ \frac{K}{f(\Theta)}\right]; \tag{22.10}$$

$$K_{q\,кр} \cdot f^2(\Theta) = f_9 \left[Pr_1,\ Ar \cdot f^3(\Theta),\ \frac{K}{f(\Theta)}\right]; \tag{22.11}$$

$$\overline{Nu}_{кр} \cdot f(\Theta) = f_{10} \left[Pr_1,\ Ar \cdot f^3(\Theta),\ \frac{K}{f(\Theta)}\right]. \tag{22.12}$$

Some Results of Experimental Investigations of Heat Transmission and its Crisis in Bubble Boiling in a Large Volume

The functional connections f_8, f_9, and f_{10} between the similarity criteria in Eqs. (22.10)-(22.12) must be determined by experiment. This raises a difficulty connected with accounting for the influence of the wetting angle Θ on the heat transmission in bubble boiling. Experimental investigations of this problem are quite scanty. There is only one known work [2], in which it is shown that the value of Θ essentially influences the heat-transmission coefficient in the case of water boiling at normal pressure. However, at high pressure, and also for other liquids, this question has not been investigated. Therefore, in processing the results of measurements of heat transmission, the dependence on the contact angle in Eqs. (22.10)-(22.12) was not taken into account. This is equivalent to using as the characteristic linear dimension not the breakaway bubble diameter d_0 but the quantity $\delta \equiv \frac{d_0}{f(\Theta)} = \sqrt{\frac{\sigma}{\gamma_1 - \gamma_2}}$.

It is further necessary to note that many experiments offer evidence that mixing processes occurring away from the heating surface and caused by the lifting of the vapor bubbles* exert a weak influence on heat transmission in the case of bubble boiling of nonmetallic liquids in large volumes. Consequently, the Archimedes number, which is a measure of the ratio of the buoyant forces to the viscosity forces, can be eliminated from Eq. (22.10). However, this number must be re-

tained in Eqs. (22.11) and (22.12), since the transition to film boiling is connected with the action of lifting forces, characterizing the "pressing away" of the liquid from the cooled surface.

If the foregoing is taken into account, then Eqs. (22.10)-(22.12) can be rewritten in the following form:

$$\overline{Nu} = f_8(Pr_1, K_q, K); \tag{22.13}$$

$$\overline{K}_{q\,кр} = f_9(Pr_1, K, Ar); \tag{22.14}$$

$$\overline{Nu}_{кр} = f_{10}(Pr_1, K, Ar). \tag{22.15}$$

The results of the measurements of the heat transmission in bubble boiling of wetting nonmetallic surfaces in a large volume can be represented in the form of the following equation:

$$\overline{Nu} = 0{,}0825 Pr_1^{-0.50} \cdot K_q^{0.7} \cdot K^{\frac{1}{3}}. \tag{22.16}$$

If we substitute here the values of the similarity criteria, then we obtain after calculations an expression for the temperature difference

$$\overline{\Delta t} = \frac{10^6}{6{,}9} \frac{\mu_1^{0.45} \cdot C_{p1}^{0.12} \cdot T_н^{0.37} \cdot q_F^{0.3}}{\lambda_1^{0.75} \cdot \left(\frac{\gamma_2 r_н}{\gamma_1 - \gamma_2}\right)^{0.033} \cdot \left(\frac{\gamma_1}{\sigma}\right)^{\frac{1}{3}}}. \tag{22.17}$$

For the heat-transmission coefficient we have

$$\bar{\alpha} = 6{,}9 \cdot 10^{-3} \frac{\lambda_1^{0.75} \cdot q_F^{0.7} \cdot \left(\frac{\gamma_2 \cdot r}{\gamma_1 - \gamma_2}\right)^{0.033} \cdot \left(\frac{\gamma_1}{\sigma}\right)^{\frac{1}{3}}}{\mu_1^{0.45} \cdot C_{p1}^{0.12} \cdot T_н^{0.37}} \tag{22.18}$$

It is interesting to note that $\bar{\alpha} \sim \lambda_1^{0.75}$. Therefore, in the boiling of organic liquids, which have a low heat-conduction coefficient compared with water ($\sim \lambda_{H_2O}/6$), the heat transmission is appreciably lower than in the boiling of water.

Equation (22.16) makes it possible to estimate approximately the influence of the contact angle Θ on the heat transmission in bubble boiling. Comparing Expressions (22.16) and (22.10) we obtain

$$\overline{Nu}\cdot f(\Theta) = 0{,}0825 Pr_l^{-0.50}\cdot [K_q\cdot f^2(\Theta)]^{0.7}\cdot \left[\frac{K}{f(\Theta)}\right]^{\frac{1}{3}}$$

or

$$\overline{Nu} = 0{,}082 Pr_l^{-0.50}\cdot K_q^{0.7}\cdot K^{\frac{1}{3}}\cdot [f(\Theta)]^{0.07}.$$

If we assume $f(\Theta) \approx 0.02\Theta$, we get

$$\overline{Nu} \sim \Theta^{0.06}.$$

In accordance with Eq. (22.5), Relation (22.18) without account of the influence of the angle Θ can be rewritten in simpler form

$$\bar{\alpha} = f_{11}\,(\text{type of liquid},\ p_n)\cdot q_F^{0.7} \qquad (22.19)$$

When water boils on a wettable surface and $p \leq 100$ atm abs, the following approximate relation is valid [35b]

$$f_{11}(H_2O,\ p_н) \approx 3 p_н^{0.18}.$$

We therefore have for water

$$\bar{\alpha} \approx 3 p_н^{0.18}\cdot q_F^{0.7}. \qquad (22.20)$$

From Eqs. (22.4) and (22.17) it follows that

$$\overline{\Delta t} = f_{12}\,(\text{type of liquid},\ p_n)\cdot q_F^{0.3}.$$

For water

$$f_{12}(H_2O,\ p_н) \approx \frac{1}{3} p_н^{0.18};$$

$$\overline{\Delta t} \approx \frac{1}{3} p_н^{0.18}\cdot q_F^{0.3}. \qquad (22.21)$$

Analogous simple computation formulas can be obtained also for heat transmission of other boiling liquids.

The results of the measurements of the critical heat rate, corresponding to the transition from the bubble boiling mode to the film mode (first crisis) can be represented in the form of the following equation [27]:

$$K_{q\,кр\,1} = 142 Pr_l^{0.6}\cdot K^{-0.68}\cdot Ar^{0.37} \qquad (22.22)$$

Substituting into this equation the values of the similarity criteria and carrying out transformations, we obtain

$$q_{F\,kp\,1} = 4700\,\frac{T_н^{0.32}\cdot\sigma^{0.21}\cdot(\gamma_1-\gamma_2)^{0.48}\cdot(\gamma_2 r)^{0.36}\cdot\lambda_1^{0.4}}{\gamma_1^{0.31}\cdot C_{p1}^{0.08}\cdot\mu_1^{0.14}}. \qquad (22.23)$$

An approximate estimate of the influence of the contact angle Θ on the critical heat rate q_{Fkr1} is obtained by comparing Eqs. (22.11) and (22.22). As a result of the calculations we obtain

$$q_{F\,kp\,1} \approx \Theta^{-1.57}.$$

Hydrodynamic Theory of the Crisis of Bubble Boiling in a Large Volume [30a]

The hydrodynamic theory of the crisis of bubble boiling in a large volume of wetting nonmetallic liquid is based on the following simplifications:

1) the boiling crisis is determined essentially by the conditions of motion of the liquid and vapor in the two-phase boundary layer;

2) turbulization in this layer is so intense, that the molecular viscosities of the liquid and vapor, μ_1 and μ_2, do not exert a noticeable influence on the occurrence of the crisis;

3) the inertia forces in the vapor stream greatly exceed the inertia forces in the liquid jets, which practically do not arrive at the heating surface after the occurrence of the crisis;

4) the dimensions of the heating surface are sufficiently large compared with the dimensions of the bubbles and films of the vapor, and therefore do not exert any influence on the conditions for the occurrence of the crisis.

It follows from the first simplification that the quantity q_{Fkr1} is determined by the purely hydrodynamic characteristics of the two-phase layer, and in particular by the definite value of the average vapor velocity u_{2kr}. The latter depends on the action of the lifting force, and also the viscosity forces, the pressure, the surface tension, and inertia. In accordance with the second simplification, the viscosity coefficients μ_1 and μ_2 drop out of consideration. The value

of the pressure gradient is the consequence of the hydrodynamic conditions under which the process takes place and cannot be specified beforehand. The action of the lifting forces is reflected by the quantity $\gamma_1 - \gamma_2$, while the action of the surface-tension forces is reflected by σ.

In accordance with the third limitation, it is essential to take into account the inertia forces in the vapor stream, i.e., the quantity γ_2/g. Thus, the "critical" vapor velocity is determined by three quantities:

$$u_{2кр} = \varphi\left(\gamma_1 - \gamma_2,\ \sigma,\ \frac{\gamma_2}{g}\right).$$

Reducing this equation to dimensionless form, we obtain

$$\frac{u_{2кр} \cdot \sqrt{\frac{\gamma^2}{g}}}{\sqrt[4]{\sigma \cdot (\gamma_1 - \gamma_2)}} = \text{const}.$$

The average velocity of the vapor phase u_{2kr} has the same order of magnitude as the vapor-production rate [30c], i.e.,

$$u_{2кр} \sim \frac{q_{F\,кр}}{\gamma_2 \cdot r_м}.$$

Substituting the last relation in the preceding one, we obtain

$$\frac{q_{F\,кр}}{r_м \cdot \sqrt{g\gamma_2} \cdot \sqrt[4]{\sigma \cdot (\gamma_1 - \gamma_2)}} = \text{const} \equiv k. \qquad (22.24)$$

A comparison of this formula with the results of the measurements of q_{Fkr1} for large volumes of various boiling liquids makes it possible to conclude that the presented approximate analysis discloses the most essential features of the heat-transmission crisis during boiling, although it does not take into account all the possible factors that influence it. At the same time, the value of k_1 for the given liquid is practically independent of the pressure.

If we take $k_1 = 0.16$ as an average value for the first crisis and

$k_2 = k_1/6$ for the second, we obtain the following formulas for estimating the critical heat rates for the first and second boiling crises [31]:

$$q_{F\,кр1} = 1800 r_н \cdot \sqrt{\gamma_2} \cdot \sqrt[4]{\sigma \cdot (\gamma_1 - \gamma_2)} \text{ kcal/m}^2\text{-hr}; \qquad (22.25)$$

$$q_{F\,кр2} = 300 r_н \cdot \sqrt{\gamma_2} \cdot \sqrt{\sigma \cdot (\gamma_1 - \gamma_2)} \text{ kcal/m}^2\text{-hr}. \qquad (22.26)$$

Formulas (22.22) and (22.24) give approximately equal values of the critical heat rate, in spite of the fact that they have been obtained on the basis of different premises. In this connection, it is interesting to note that Eq. (22.24) has been obtained in [75] from an analysis of the conditions of the stability of a vapor film on the heating surface.

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[Footnote]

584 In bubble boiling of molten metals in a large volume, the mixing processes away from the heating surface should apparently exert an influence on the heat transmission.

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[List of Transliterated Symbols]

560 н = n = nasyshcheniye = saturation

562 кр = kr = kriticheskiy = critical

565 ж = zh = zhidkost' = liquid

565 н = n = nasyshchennyy = saturated

576 стаб = stab = stabilizirovannyy = stabilized

Chapter 11

HEAT TRANSMISSION UPON CONDENSATION OF VAPORS OF NONMETALLIC LIQUIDS

§23. PRINCIPLES OF THE THEORY OF HEAT TRANSMISSION IN FILM CONDENSATION ON A VERTICAL PLATE

Features of the Mechanism of Heat Transmission in Film Condensation of Vapors of Nonmetallic Liquids

Let us consider the features of the mechanism of heat transmission upon condensation of vapors of nonmetallic liquids on solid surfaces.

The process of condensation of a vapor on a surface can proceed if the following two conditions are satisfied:

1) heat is carried away through the condensation surface;

2) there exist on this surface condensation centers, i.e., nuclei of the liquid phase (small drops, dust particles, charges).

A distinction is made between two types of condensation on a surface.

Film condensation is observed on wettable surfaces when the condensate formed on the solid surface flows along the latter under the influence of some forces (for example, gravity) in the form of a continuous film. This form of condensation is encountered most frequently in heat-exchange apparatus.

Drop condensation occurs on nonwettable surfaces, and then the condensate is formed on the solid surface in the form of individual drops, which are removed from the surface under the influence of some forces. Drop condensation is usually observed in mercury-vapor con-

For some cases mixed condensation is typical, when the film mode is established on some parts of the surface, and the drop mode on others.

It has been established as a result of measurements that in film condensation of nonmetallic vapors the intensity of heat transmission is approximately one order of magnitude lower than in drop condensation. The reason for this phenomenon is that the film of condensate has a relatively large thermal resistance, the value of which determines essentially the intensity of the heat transmission in film condensation of nonmetallic vapors.*

Heat transmission in film condensation has been studied so far in much more detail than for drop condensation. Figure 92 shows the distribution of the temperature in film condensation. Saturated vapor with temperature t_n condenses on the external surface of the condensate film, which flows downward under the action of the forces of gravity. The temperature t_{pl} of this film surface differs somewhat from the saturation temperature t_n. The temperature difference $t_n - t_{pl}$ is due to the finite rate of displacement of the vapor molecules from the volume to the surface of the film.

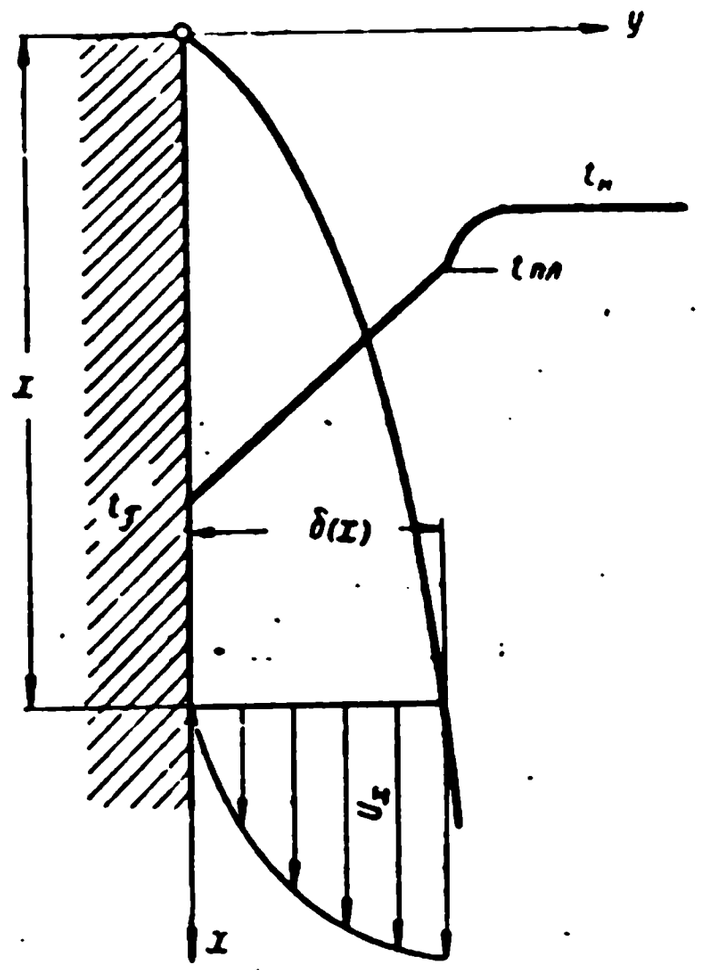

Fig. 92. Diagram showing the distribution of the temperature in the film of condensate.

The heat released upon formation of the liquid phase is transmitted through the condensate film by heat conduction and convection to the solid surface, which has a temperature $t_F < t_{pl} < t_n$. Therefore a temperature drop $t_{pl} - t_F$ is observed in the film. Part of this heat is "washed down" by the film.

Since $t_F < t_n$, the condensate in the film is somewhat supercooled.

Influence of Different Factors on the Intensity of Heat Transmission in Film Condensation

The intensity of heat transmission in film condensation is mostly influenced by the following factors:

1) the velocity and direction of flow of the vapor;

2) admixture of air to the condensing vapor;

3) form, dimension, and location of the solid surface of condensation;

4) pressure of the saturated vapor;

5) state of the solid surface.

Velocity and direction of vapor flow. The velocity and direction of the vapor flow exert an appreciable influence by means of friction between the vapor and the film. At pressures $p \geq 1$ atm abs and at water-vapor velocities $u_0 \geq 10$ m/sec this influence is insignificant, since the condensate film is destroyed by the stream of vapor.

Admixtures of air. Admixtures of air or of some other gas in the condensing vapor greatly increase the thermal resistance on the outer surface of the film. Gas molecules are gathered together and the vapor molecules diffuse through the gas layer toward the film. The intensity of heat transmission is reduced to less than half if the stationary vapor contains an admixture of merely 1% (by weight) of noncondensing gas.

Form, dimensions, and location of the solid condensation surface in the field of the forces due to gravity exert an appreciable influence on the intensity of heat transmission in condensation, since all these factors determine the conditions under which the film flows down, and consequently its thickness and thermal resistance.

With increasing saturated-vapor pressure the intensity of heat transmission in film condensation increases (for the fixed geometry of

the condensation surface and for a specified temperature difference).

State of solid surface. On oxidized and rough surfaces, the film turns out to be thicker than on smooth surfaces. This leads to a considerable decrease in heat-transmission intensity, reaching 20-30%.

Superheating of the vapor. This phenomenon does not exert a noticeable influence on heat transmission in the case of film condensation, since the vapor has the saturation temperature near the film.

With increasing viscosity of the condensate the thickness of the film increases and the heat transmission becomes worse.

An increase in the density and heat conductivity of the condensate leads to an increase in the heat-transmission intensity.

More detailed information on the influence of various factors on the heat transmission in the case of film condensation can be found in [30c, 31, 35b].

Setup and Mathematical Formulation of the Problem of Film Condensation on a Vertical Plate

Let us consider the calculation of the intensity of heat transmission in the case of film condensation on a vertical plate. A solution of this relatively simple problem makes it possible to demonstrate in clear fashion the influence of various factors (including the influence of the dependence of the thermodynamic parameters of the condensate on the temperature) on heat transmission in the case of film condensation.

It is required to determine the local heat rate q_F on a vertical condensation surface for the following uniqueness conditions.

The geometrical conditions specify a vertical smooth plate, on which film condensation takes place (see Fig. 92). It is assumed that the thickness of the condensate film $\delta(x)$ is small compared with the distance x to the upper edge of the plate $\delta(x) \ll x$. In other words,

for the condensate film there is satisfied the fundamental approximation of the theory of hydrodynamic boundary layer.

The physical conditions indicate that a pure, dry, saturated, and stationary vapor is being condensed, with a saturation temperature t_n. The flow of the condensate film is laminar and creeping, i.e., the inertial forces in the film are small compared with the viscosity and weight forces. Convective heat transfer in the film is small compared with the heat conductivity, while the friction of the film against the vapor is small compared with the friction of the condensate against the plate.

The temperature on the outer surface of the film is equal to the saturated-vapor temperature t_n. The viscosity, thermal conductivity, and the specific weight of the condensate are specified functions of the temperature:

$$\mu_1 = \mu_1(t),\ \lambda_1 = \lambda_1(t),\ \gamma_1 = \gamma_1(t).$$

The action of the surface-tension forces and the supercooling of the condensate are neglected.

The time conditions indicate that the condensation process is stationary.

The boundary conditions are:

on the surface of the plate

$$u_x|_{y=0} = 0\ ;\ t|_{y=0} = t_F;$$

on the surface of the film

$$t|_{y=\delta} = t_n > t_F;\ \frac{\partial u_x}{\partial y}\Big|_{y=\delta-0} \approx 0. \qquad (23.1)$$

Since the condensate adheres to the condensation surface, and there is no radiant heat exchange, to determine the local value of the heat rate q_F it is necessary to use the Biot-Fourier law

$$-q_F = -\left[\lambda_1(t)\cdot\frac{\partial t}{\partial y}\right]_{y=+0}. \qquad (23.2)$$

The temperature field in the condensate film is described by the

Fourier-Ostrogradskiy equation, which for the assumed uniqueness conditions has the following form:

$$\frac{\partial}{\partial y}\left[\lambda_1(t)\cdot\frac{\partial t}{\partial y}\right]=0. \qquad (23.3)$$

The boundary conditions (25.1) contain the thickness of the condensate film $\delta(x)$. Let us determine it from the heat balance, which we set up for a film element dx of 1 meter width:

$$q_F=r_n\cdot\frac{d}{dx}\int_0^{\delta(x)}u_x\cdot\gamma_1(t)\cdot dy. \qquad (23.4)$$

The velocity field u_x in the condensate film is described by the Navier-Stokes equation, which with allowance for the assumed uniqueness conditions has the following form:

$$\gamma_1(t)-\gamma_2+\frac{\partial}{\partial y}\left[\mu_1(t)\cdot\frac{\partial u_x}{\partial y}\right]=0. \qquad (23.5)$$

An analogous setup of the heat-transmission problem for film condensation, but at constant values of the thermodynamic parameters of the condensate (γ_1, λ_1, μ_1 = const) was proposed by Nusselt [69].

Solution of the Problem of Film Condensation on a Vertical Plate with Account of the Dependence of the Physical Parameters of the Condensate on the Temperature

Integrating (23.3) we get

$$\frac{\partial t}{\partial y}=\frac{C(x)}{\lambda_1(t)}, \qquad (23.6)$$

where C(x) is an arbitrary function of x. To determine C(x) let us integrate Eq. (23.6) between the limits y = 0 and y = $\delta(x)$. After transformation we obtain

$$C(x)=\frac{\lambda_{1cp}\cdot\Delta t}{\delta(x)},$$

where

$$\lambda_{cp}\equiv\frac{1}{\Delta t}\int_{t_F}^{t_n}\lambda_1(t)\cdot dt$$

is the mean integrated value of the heat conduction coefficient of the

condensate in the temperature interval $t_F \leq t \leq t_n$; $\Delta t \equiv t_n - t_F$ is the temperature difference in the film. Substituting the obtained expression for C(x) into Eq. (23.6) we get

$$\frac{\partial t}{\partial y} = \frac{\lambda_{1cp} \cdot \Delta t}{\lambda_1 (t) \cdot \delta (x)}. \qquad (23.7)$$

Substituting Expression (23.7) into (23.2), we obtain the following expression for the heat rate:

$$q_F = \lambda_{1cp} \cdot \frac{\Delta t}{\delta (x)}. \qquad (23.8)$$

To determine the thickness of the film $\delta(x)$ we integrate Eq. (23.5)

$$\int_y^{\delta(x)} [\gamma_1 (t) - \gamma_2] \cdot dy + \int_{\mu_1 (t) \cdot \frac{\partial u_x}{\partial y}}^{0} d \left[\mu_1 (t) \cdot \frac{\partial u_x}{\partial y} \right] = 0,$$

hence

$$\mu_1 (t) \cdot \frac{\partial u_x}{\partial y} = \int_y^{\delta(x)} [\gamma_1 (t) - \gamma_2] \cdot dy.$$

Integrating again, we obtain

$$u_x = \int_0^y \frac{dy}{\mu_1 (t)} \int_y^{\delta(x)} [\gamma_1 (t) - \gamma_2] \cdot dy.$$

Substituting the last equation, and also (23.8), into (23.4), we get

$$\lambda_{1cp} \frac{\Delta t}{\delta (x)} = r_н \cdot \frac{d}{dx} \left\{ \int_0^{\delta(x)} \gamma_1 (t) \cdot dy \int_0^y \frac{dy}{\mu_1 (t)} \int_0^{\delta(x)} [\gamma_1 (t) - \gamma_2] \cdot dy \right\}.$$

In the last equation we replace y by a new variable. From (23.7) we get

$$\partial y = \frac{\lambda_1 (t) \cdot \delta (x)}{\lambda_{1cp} \cdot \Delta t} \partial t.$$

Substituting the last relation in the preceding one and replacing the integration limits, we obtain

$$\frac{\lambda_{1cp} \cdot \Delta t}{\delta (x)} = r_н \cdot \frac{d}{dx} \left[\frac{\delta^3 (x)}{\lambda_{1cp}^3 \cdot \Delta t^3} \int_{t_F}^{t_н} \lambda_1 (t) \cdot \gamma_1 (t) \cdot dt \int_{t_F}^{t} \frac{\lambda_1 (t) \cdot dt}{\mu_1 (t)} \times \right.$$

$$\left. \times \int_t^{t_н} \lambda_1 (t) [\gamma_1 (t) - \gamma_2 \cdot dt \right].$$

If we put

$$\int_{t_F}^{t_н} \lambda_1(t)\cdot\gamma_1(t)\cdot dt \int_{t_F}^{t} \frac{\lambda_1(t)\cdot dt}{\mu_1(t)} \int_{t}^{t_н} \lambda_1(t)\cdot[\gamma_1(t)-\gamma_2]\cdot dt \equiv$$

$$\equiv F(t_н,\ t_F), \qquad (23.9)$$

then the preceding equation can be rewritten in simpler form:

$$\frac{\lambda_{ср}^4\cdot\Delta t^4}{r_н\cdot F(t_н,\ t_F)} = \delta(x)\cdot\frac{d\delta^3(x)}{dx} = 3\delta^3(x)\cdot\frac{d\delta(x)}{dx}.$$

Integrating this expression and taking into account the boundary condition $\delta(0) = 0$, we obtain the following expression for the thickness of the film:

$$\delta(x) = \left[\frac{4}{3}\frac{x}{r_н\cdot F(t_н,\ t_F)}\right]^{0,25}\cdot\Delta t\cdot\lambda_{1ср}.$$

Substituting $\delta(x)$ in (23.8) we obtain the solution of the problem [9b]:

$$q_F = \left[\frac{3}{4}\frac{r_н\cdot F(t_н,\ t_F)}{x}\right]^{0.25}. \qquad (23.10)$$

From Eq. (23.10) it follows that the Newton-Rikhman law is not satisfied here.

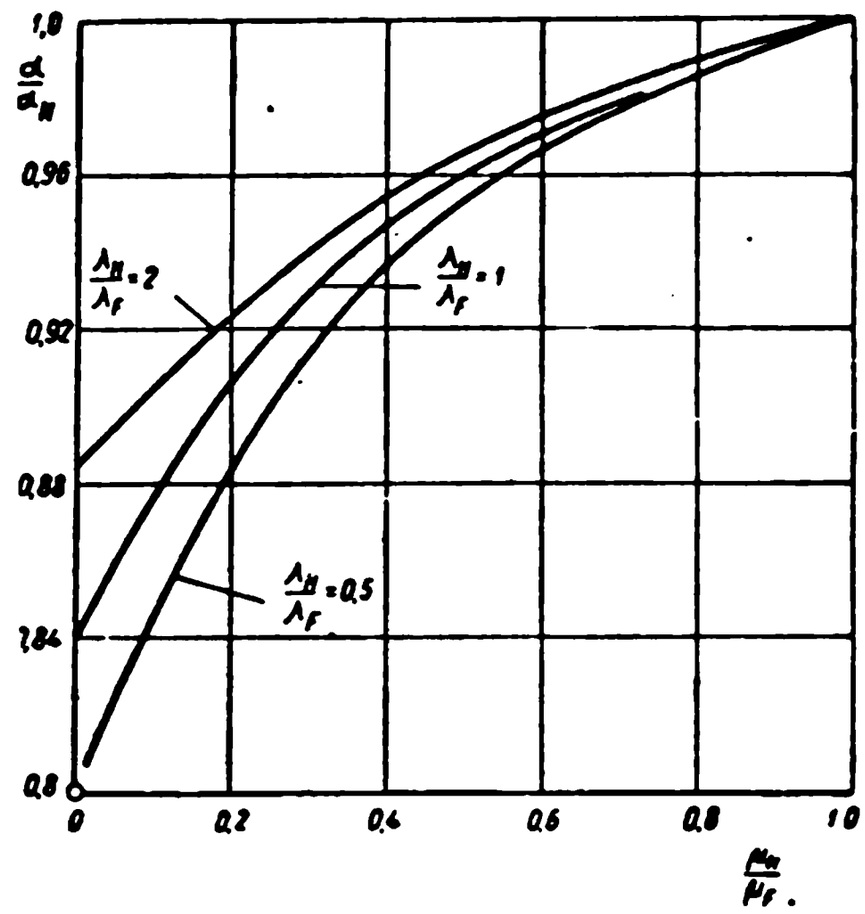

Fig. 93. The effect of the temperature dependence of the thermodynamic parameters of the condensate on heat transmission in film condensation.

ness of the result obtained for film condensation of glycerine vapor [18].

If the thermodynamic parameters of the condensate vary little within the specified temperature interval, then the quantities λ_1, γ_1, and μ_1 can be regarded as constants chosen for the mean temperature of the film $t_{sr} = 0.5(t_F + t_n)$.

In this case Relation (23.9) simplifies greatly:

$$F(t_F, t_H) \equiv \frac{\lambda_{1cp}^3 \cdot \gamma_{1cp}^2}{\mu_{1cp}} \cdot \int_{t_F}^{t_H} dt \int_{t}^{t_H} \left(1 - \frac{\gamma_2}{\gamma_{1cp}}\right) \cdot dt \int_{t_F}^{t} dt =$$

$$= \frac{\lambda_{1cp}^3 \cdot \gamma_{1cp}^2}{3\mu_{1cp}} \left(1 - \frac{\gamma_2}{\gamma_{1cp}}\right) \cdot \Delta t^3.$$

Substituting this expression for $F(t_n, t_F)$ into (23.10) we obtain the following expression for q_F, originally derived by Nusselt [69]*:

$$q_{FN} = \left\{\frac{1}{4} \frac{r_H \cdot \lambda_{1cp}^3 \cdot \gamma_{1cp}^2}{\mu_{1cp} \cdot \Delta t \cdot x} \left(1 - \frac{\gamma_2}{\gamma_{1cp}}\right)\right\}^{0.25}. \qquad (23.11)$$

Thus, the Newton-Rikhman law for heat transmission does not hold in film condensation even for constant thermodynamic parameters of the condensate.

In order to clarify the character of the influence of the temperature dependence of the thermodynamic condensate parameters on the heat transmission, calculations were made of the ratio $q_F/q_{FN} = \alpha/\alpha_N$ for different values of μ_{1n}/μ_{1F}, $\lambda_{1n}/\lambda_{1F}$ with γ = const, $\gamma_2 \ll \gamma_1$ and under the condition that the functions $\lambda_1(t) \cdot \gamma_1(t)$ and $\lambda_1(t)/\mu_1(t)$ in (23.9) can be represented by straight lines. The results of the calculations [32a,b] are shown in Fig. 93 in the form of a plot of the function

$$\frac{\alpha}{\alpha_N} = f\left(\frac{\mu_{1H}}{\mu_{1F}}, \frac{\lambda_{1H}}{\lambda_{1F}}\right).$$

Thus, the change in viscosity and heat conductivity in the condensate film can exert a noticeable influence on the heat transmission

in the case of film condensation.

The results of experimental investigations of heat transmission in film condensation [30b] of a slowly moving vapor, when $\rho_2 \cdot u_2^2 \leq 30$, lead to the conclusion that Formula (23.11) is valid if the Reynolds number for the film $\overline{\overline{Re}}_1 \equiv \overline{\overline{u}}_1 \cdot \delta(x)/\nu_{1sr}$ does not exceed 100.

§24. HEAT TRANSMISSION IN CONDENSATION OF A MOVING VAPOR OF A NONMETALLIC LIQUID ON A HORIZONTAL CYLINDER

Formulation of the Problem

Let us consider the process of heat transmission in film condensation of a vapor of nonmetallic liquid on a horizontal cylinder under the following uniqueness conditions.

The geometrical conditions specify a long horizontal cylinder of diameter d with smooth surface. It is situated in a stream of vapor directed downward (at a right angle to the cylinder axis) (Fig. 94).

The physical conditions indicate that the vapor of the nonmetallic liquid is clean, dry, and saturated, and its condensate wets the surface of the cylinder. This results in film condensation. The specified quantities are the constant physical parameters of the vapor (γ_2, ν_2) and of the condensate (ν_1, a_1, λ_1) as well as the latent heat of vapor production r_n. It is assumed that the surface-tension forces can be neglected.

The time conditions specify a stationary heat-transmission process.

The boundary conditions are:

on the cylinder surface

$$u|_{r=r_0} = 0 \quad \text{and} \quad t|_{r=r_0} = t_F,$$

on the outer free surface of the condensate film the following two conditions are satisfied:

$$\mu_1 \cdot \frac{\partial u_{1\varphi}}{\partial r}\Big|_{r_0+\delta(\varphi)-0} = \xi(Re_2, \varphi) \cdot \frac{\gamma_2 \cdot u_{02}^2}{2g};$$

$$-\lambda_1 \cdot \frac{\partial t_1}{\partial r}\Big|_{r_0+\delta(\varphi)-0} = \frac{\gamma_1 \cdot r_n}{r_0} \cdot \frac{d}{d\varphi} \int_{r_0}^{r_0+\delta(\varphi)} u_{1\varphi} \cdot dr,$$

i.e., the conditions of friction between the condensate film and the vapor as well as the condition of heat exchange on the outer surface of the film are specified. Here $\xi(\mathrm{Re}_2, \varphi)$ is the local coefficient of friction between the film and the vapor, and

$$Re_2 = \frac{u_0 \cdot \left(d + \varphi_{2\bar{\delta}}\right)}{\nu_2} \approx \frac{u_0 \cdot d}{\nu_2}$$

is the Reynolds number of the vapor stream.

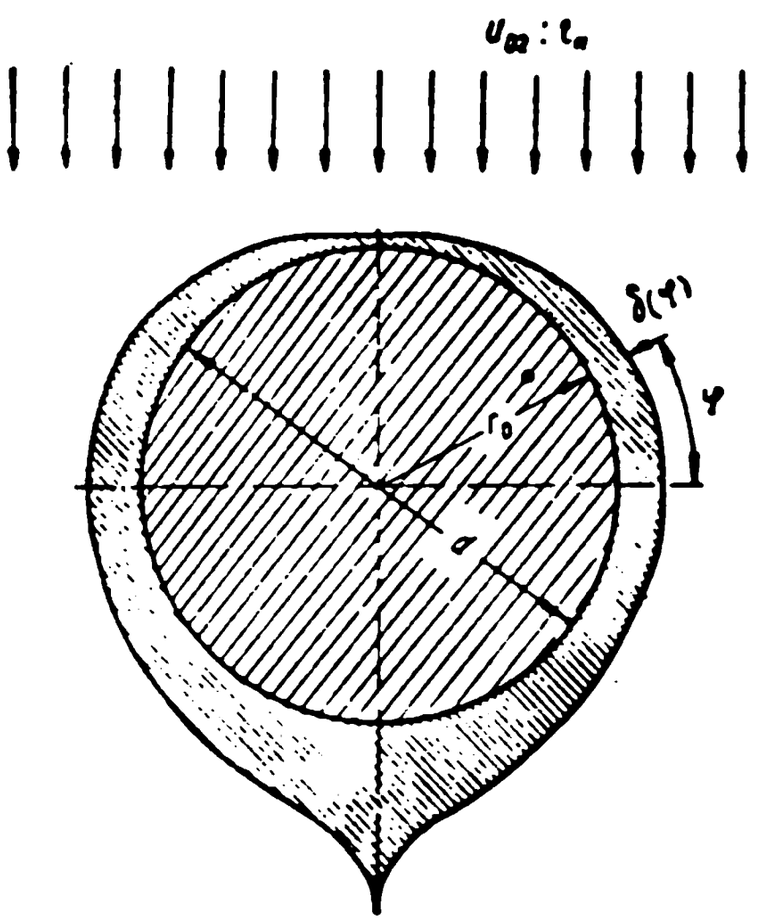

Fig. 94. Diagram of film condensation of moving vapor on a horizontal cylinder.

In addition, the temperature of the free surface of the film is specified:

$$t|_{r_0+\delta(\varphi)} = t_n > t_F.$$

Under these conditions it is necessary to determine the heat rate on the surface of the cylinder, averaged over the perimeter,

$$\bar{q}_F = -\frac{\lambda_1}{2\pi}\int_0^{2\pi} \frac{\partial t_1}{\partial r}\Big|_{r_0+0} \cdot d\varphi.$$

The distribution of the temperature t_1 and of the velocity $\vec{u}_1$ in the film obey the following equations:

$$(\mathbf{u}_1, \operatorname{grad} t_1) - a_1 \cdot \nabla^2 t_1 = 0;$$

$$(\mathbf{u}_1, \operatorname{grad})\mathbf{u}_1 = \mathbf{g}\left(1 - \frac{\gamma_2}{\gamma_1}\right) - \frac{1}{\varrho_1}\cdot\operatorname{grad} p_1 + \nu_1 \nabla^2 \mathbf{u}_1;\ \operatorname{div}\mathbf{u}_1 = 0.$$

If we introduce into these equations and the boundary conditions the excess temperature $\vartheta_1 \equiv t_n - t_1$, we obtain

$$\left.\begin{gathered} q_F = \frac{\lambda_1}{2\pi}\int_0^{2\pi} \frac{\partial \vartheta}{\partial r}\Big|_{r_0+0} \cdot d\varphi; \\ (\mathbf{u}, \operatorname{grad}\vartheta_1) - a_1 \cdot \nabla^2 \vartheta_1 = 0; \\ (\mathbf{u}_1, \operatorname{grad})\mathbf{u}_1 = \mathbf{g}\left(1 - \frac{\gamma_2}{\gamma_1}\right) - \frac{1}{\varrho_1}\cdot\operatorname{grad} p_1 + \nu_1 \cdot \nabla^2 \mathbf{u}_1; \\ \operatorname{div}\mathbf{u}_1 = 0; \\ \mathbf{u}_1|_{r_0} = 0;\ \ \vartheta_1|_{r_0} = t_n - t_F \equiv \Delta t;\ \ \vartheta_1|_{r_0+\delta(\varphi)} = 0; \\ + \lambda_1 \frac{\partial \vartheta_1}{\partial r}\Big|_{r_0+\delta(\varphi)-0} = \frac{\gamma_1 \cdot r_n}{r_0} \cdot \frac{d}{d\varphi}\int_{r_0}^{r_0+\delta(\varphi)} u_{1\varphi} \cdot d\varphi; \end{gathered}\right\} \qquad (24.1)$$

$$\mu_1 \frac{\partial u_{1\varphi}}{\partial r}\Big|_{r_0+\delta(\varphi)-0} = \xi\left(\frac{u_{02}d}{\nu_2},\ \varphi\right)\cdot\frac{\gamma_2\cdot u_{02}^2}{2g}. \qquad (24.1)$$

Similarity Conditions

From Eqs. (24.1) we obtain a system of parameters, which determine the average heat rate on the surface of the cylinder:

$$^{\varphi}\overline{q}_F = f_1\left(\lambda_1,\ a_1,\ \nu_1,\ \gamma_1-\gamma_2,\ \frac{\gamma_1}{g},\ d,\ \Delta t,\ \frac{\gamma_1 r_H}{r_0},\ \frac{\gamma_2\cdot u_{02}^2}{g},\ Re_2\right) \qquad (24.2)$$

Relation (24.2) contains n = 10 dimensional independent quantities of k = 5 groups of quantities with units that are independent of one another. Therefore after reducing it to dimensionless form we obtain n − k = 10 − 5 = 5 criteria:

$$\frac{^{\varphi}\overline{q}_F\cdot d}{\lambda_1\cdot\Delta t} \equiv {}^{\varphi}\overline{Nu},\ \frac{\nu_1}{a_1}\equiv Pr_1,\ \frac{g\cdot d^3}{\nu_1^2}\left(1-\frac{\gamma_2}{\gamma_1}\right)\equiv Ar,\ \frac{r_H}{C_{p1}\cdot\Delta t}\equiv K;$$

$$\pi \equiv \frac{\gamma_2\cdot u_{02}^2}{g\cdot d\cdot\gamma_1\cdot\left(1-\frac{\gamma_2}{\gamma_1}\right)},$$

and Eq. (24.2) assumes the form

$$^{\varphi}\overline{Nu} = f_2(Pr_1,\ Ar,\ K,\ \pi,\ Re_2). \qquad (24.3)$$

If the condensation occurs at low saturated-vapor pressure, then the specific gravity of the vapor is a small quantity compared with the specific gravity of the condensate, i.e., $\gamma_2 \ll \gamma_1$. In this case $\gamma_2/\gamma_1 \ll 1$. Consequently,

$$Ar \equiv \frac{gd^3}{\nu_1^2}\left(1-\frac{\gamma_2}{\gamma_1}\right)\approx\frac{gd^3}{\nu_1^2}\equiv Ga \text{ (the Galileo number)}$$

and

$$\pi \equiv \frac{\gamma_2\cdot u_{02}^2}{g\cdot d\cdot\gamma_1\cdot\left(1-\frac{\gamma_1}{\gamma_2}\right)}\approx\frac{\gamma_2\cdot u_{02}^2}{g\cdot d\cdot\gamma_1}.$$

Then we have in place of (24.3)

$$^{\varphi}\overline{Nu} = f_3(Pr_1,\ Ga,\ K,\ \pi,\ Re_2). \qquad (24.4)$$

Heat Transmission in the Condensation of Stationary Vapor on a Horizontal Cylinder

If the vapor condensing on the cylinder is stationary, then

$$\vec{u}_{02} = 0,\ \pi = 0 \text{ and } Re_2 = 0.$$

In this case Eq. (24.4) assumes the simpler form

$$^{\varphi}\overline{Nu}_0 = f_3(Pr_1,\ Ga,\ K,\ 0,\ 0) = f_4(Pr_1,\ Ga,\ K). \qquad (24.5)$$

In this case the dependence of $^{\varphi}\overline{Nu}$ on Pr_1, Ga, and K has been found analytically [69]. In addition to the foregoing uniqueness conditions it was assumed here that the flow of the film is creeping and that the convective heat transport along the film is small compared with the thermal conductivity transversely to the film.

As a result of integration of Eqs. (24.1), which under these limitations assume a much simpler form, it has been found in [69] that

$$^{\varphi}\overline{Nu}_0 = 0{,}725\sqrt[4]{Pr_1 \cdot Ga \cdot K} \qquad (24.6)$$

This formula can be used to calculate heat transmission at low velocity of the condensing vapor, when

$$\gamma_2 \cdot u_{02}^2 < 30. \qquad (24.7)$$

The thermodynamic parameters of the condensate must be chosen here for the average film temperature

$$t_{1cp} = \frac{1}{2}(t_F + t_w).$$

Heat Transmission in Condensation of a Moving Vapor on a Horizontal Cylinder

If the velocity of the condensing vapor is so large that Condition (24.7) is not satisfied, i.e., $\gamma_2 \cdot u_{02}^2 > 30$, then to calculate the heat transmission it is necessary to take into account also the similarity criteria π and Re_2, which enter into Eq. (24.4). This equation can be greatly simplified by assuming the criteria Pr_1, Ga, and K influence the heat transfer of the moving vapor in the same way as is observed for stationary vapor.* We then obtain in lieu of (24.4)

$$^{\varphi}\overline{Nu} = f_5(Pr_1 \cdot Ga \cdot K,\ \pi,\ Re). \qquad (24.8)$$

Since in the condensation of stationary vapor we have

$$^{\varphi}\overline{Nu} = f_7\left(^{\varphi}\overline{Nu}_0,\ \pi,\ Re_2\right). \qquad (24.10)$$

A more detailed analysis of the similarity conditions for the considered heat-transmission process has shown [3] that Eq. (24.10) is more conveniently represented in a different, equivalent form

$$\frac{^{\varphi}\overline{Nu}}{^{\varphi}\overline{Nu}_0} = f_8\left(^{\varphi}\overline{Nu}_0,\ ^{\varphi}\overline{Nu}_0\cdot\pi,\ Re_2\right). \qquad (24.11)$$

where

$$\frac{^{\varphi}\overline{Nu}}{^{\varphi}\overline{Nu}_0} = \frac{^{\varphi}\overline{\alpha}\cdot d}{\lambda_1}\cdot\frac{\lambda_1}{^{\varphi}\overline{\alpha}_0\cdot d} = \frac{^{\varphi}\overline{\alpha}}{^{\varphi}\overline{\alpha}_0};$$

$$^{\varphi}\overline{Nu}_0\cdot\pi = \frac{^{\varphi}\overline{\alpha}_0\cdot d_1}{\lambda_1}\cdot\frac{\gamma_2\cdot u_{02}^2}{g\cdot d\cdot\gamma_1} = \frac{\gamma_2\cdot u_{02}^2\alpha_0}{g\cdot\lambda_1\cdot\gamma_1} \equiv \pi_1.$$

Therefore Relation (24.11) can be written in the form

$$\frac{^{\varphi}\overline{\alpha}}{^{\varphi}\overline{\alpha}_0} = f_8\left(^{\varphi}\overline{Nu}_0,\ \pi_1,\ Re_2\right). \qquad (24.12)$$

To determine the form of the function f_8 in Eq. (24.12), the heat transmission was measured [53] in film condensation of saturated steam at different downward velocities and at different saturation pressures ($p_n \approx 0.09$-0.5 atm abs). The results of these measurements can be represented, with an error of $\pm 7.5\%$, in the form of the following approximate equation:

$$\frac{^{\varphi}\overline{\alpha}}{^{\varphi}\overline{\alpha}_0} \approx 173\,\frac{\pi_1^{0.167}}{Re_2^{0.175}\cdot\overline{Nu}_0^{0.7}} \qquad (24.13)$$

If we substitute in Relation (24.13) the values of the criteria π_1, Re_2, and Nu_0, then after transformation the formula for the calculation of the heat-tranmission coefficient assumes the following form [31]:

$$\frac{^{\varphi}\overline{\alpha}}{^{\varphi}\overline{\alpha}_0} \approx 33{,}2\left(\frac{u_{02}^2\cdot\gamma_2}{g\cdot\gamma_1\cdot d}\right)^{0.08}\left(\frac{3600\gamma_1\cdot r_H\cdot d^3}{\lambda_1\cdot\mu_1\cdot\Delta t}\right)^{-0.125} \qquad (24.14)$$

This formula is valid for

$$\gamma_2 \cdot u_2^2 = 400\text{-}6000 \text{ kg/sec}^2\text{-m}.$$

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[Footnotes]

591 In film condensation of metal vapors, the thermal resistance of the condensate film is relatively small. The fundamental factor here is the thermal resistance "from the vapor to the film" and "from the film to the heating surface," i.e., the "thermal resistance of the contact."

598 The Nusselt formula is obtained if one assumes $\gamma_2/\gamma_{1sr} \ll 1$ in Eq. (23.11).

602 Such a simplification can be justified by assuming the system of limitations used in [69].

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[List of Transliterated Symbols]

591 н = ˙n = nasyshcheniye = saturation

591 пл = pl = plenochnyy = film

595 ср = sr = srednyy = average

Chapter 12

RADIANT HEAT EXCHANGE

§25. RADIANT HEAT EXCHANGE IN A SYSTEM OF TWO BODIES

Radiant heat exchange in existing stationary power reactors is of secondary significance compared with convective heat exchange and heat conductivity. In many cases it becomes possible to confine oneself to a calculation of radiant heat exchange between two bodies separated by a transparent medium. We therefore develop in the present chapter only the basic concepts and laws of thermal radiation and have presented the simplest calculation for radiant heat exchange in a system consisting of two bodies.

Phenomenological Method of Investigating Radiant Heat Exchange

In the phenomenological method of investigating the simplest radiant heat exchange processes we do the following:

1) we disregard the discrete structure of the radiant energy, and also the discrete character of the processes of its radiation and absorption;

2) we use the following fundamental laws of physics:

a) the law of energy conservation, in order to obtain computation relations between the densities of the different radiant fluxes on the surface of the body;

b) the first and second laws of thermodynamics in the form of the Stefan-Boltzmann and Kirchhoff laws for the intrinsic equilibrium radiation;

3) we use by way of a hypothesis the assumption that local thermo-

dynamic equilibrium obtains, and also the Lambert hypothesis that the brightness of the intrinsic and reflected radiation is independent of the direction;

4) the coefficient of radiation of a black body, C_0, and also the coefficients of absorption A and the degree of blackness ε of real bodies participating in radiant heat exchange are determined experimentally.

Simplest Concepts and Basic Limitations

Following the first feature of the phenomenological method, we introduce into consideration the basic concepts of radiant heat exchange without using discrete microscopic characteristics of this process.

Radiant energy is the energy of oscillation of the continuous electromagnetic field. The properties of radiant energy depend essentially on the wavelength λ of these oscillations. In the interval $0.8\ \mu \leq \lambda \leq 40.0\ \mu$, the radiant energy has the properties of thermal radiation. These properties include first of all the ability of thermal radiation of being emitted, propagated, absorbed, reflected, and transmitted.

Radiation refers to a process of conversion of internal energy of a body into radiant energy. This energy leaves the bulk of the body through its surface into the surrounding space. Radiation is a characteristic of all bodies and depends on their temperature and physical properties. In opaque solids the radiant energy going to the outside is generated in very thin layers near the surface of the body. This is why one investigates the radiation from the surface of the body, without considering the process of formation of radiant energy in its bulk.

Radiation can be either continuous or selective. Continuous radiation covers the entire spectrum of radiation (from $\lambda = 0$ to $\lambda \to \infty$).

while selective radiation covers only individual bands (for certain gases and vapors). Radiation can be either diffuse or directional. In diffuse radiation the radiant energy is radiated uniformly in all directions of the hemisphere.

The transport of radiant energy refers to the process of its propagation. The character of this process is determined by the physical properties of the medium in which the radiation is transported, and also the spectral composition of the radiation.

Absorption refers to the process of conversion of part of the radiant energy, incident on a body, into internal energy. Absorption, like emission, is a characteristic of any real medium. The radiant energy striking the body is absorbed in part, reflected in part, and part passes through the body without being converted into other types of energy.

Reflection can be either diffuse or specular. In the former case the reflected radiant energy is uniformly distributed in all directions within the limits of a hemisphere. In specular reflection the well-known law of geometrical optics is satisfied: the angle of reflection of the radiant energy from the body is equal to the angle of incidence.

The totality of the processes of emission, transport, absorption, reflection, and transmission of thermal energy is called radiant heat exchange. Radiant heat exchange can also occur between bodies having the same temperature. In this case, however, the resultant heat transport from one body to the other will be zero (by virtue of the second law of thermodynamics). Such a process of radiant heat exchange is called equilibrium, and such an equilibrium is dynamic.

Thermal radiation is the only form of radiation which can be in equilibrium. Radiant heat exchange can occur also in the absence of a material medium between the bodies.

Let us introduce the concepts of hypothetical bodies with limiting properties with respect to thermal radiation.

A body is called white if it reflects diffusely all the thermal radiation incident on it, absorbing none and transmitting none.

A body is called transparent (diathermal) if it transmits all the thermal radiation incident on it, absorbing none and reflecting none.

A body is called black if it absorbs all the thermal radiation incident on it, transmitting nothing and reflecting nothing.

In calculations of radiant heat exchange the following simplest concepts are employed:

the integral radiant flux Q in kcal/hr is the quantity of radiant energy radiated in the entire spectrum ($0 \leq \lambda < \infty$) and passing through a surface F;

the integral density of hemispherical radiation E in kcal/m^2-hr is the integral flux of radiant energy passing through a unit surface F in all directions of the hemispherical solid angle

$$E \equiv dQ/dF.$$

Consequently, the radiant flux is

$$Q = \int_F E \cdot dF.$$

If the quantity E is constant on the surface F, then

$$Q = E \cdot F. \qquad (25.1)$$

The spectral or monochromatic density of hemispherical radiation E_λ in kcal/m^3-hr is the elementary radiation density dE in the narrow interval of wavelength variation $d\lambda$, referred to this interval

$$E_\lambda \equiv dE/d\lambda. \qquad (25.2)$$

The radiation brightness B (kcal/m^2-hr-steradian) is the radiant flux d^2Q passing in a given direction ϑ, referred to a unit solid angle $d\omega$ and a unit surface dF_ϑ, orthogonal to the chosen direction ϑ

(Fig. 95):

$$B \equiv \frac{d^2Q}{dF_\phi \cdot d\omega}. \qquad (25.3)$$

If an observer is situated on the surface of the body, then the observer registers the following types of hemispherical-radiation density (Fig. 96):

E_{sob} is the density of intrinsic radiation, which arises within the bulk of the body, passes through its surface, and is determined by the physical nature and temperature of the body;

E_{pad} is the radiation density incident on the body from the outside;

E_{otr} is the density of reflected radiation. E_{otr} is that part of E_{pad} which is reflected from the body.

E_{pogl} is the density of absorbed radiation, which is the part of E_{pad} absorbed by the body and converted into its intrinsic energy.

E_{eff} is the density of effective radiation from the body into space. By definition E_{eff} is equal to the sum of E_{sob} and E_{otr}:

$$E_{эфф} \equiv E_{соб} + E_{отр}.$$

For a black body $E_{eff0} = E_{sob0}$, so that in this case $E_{otr0} = 0$;

E_{rez} is the density of resultant radiation. By definition it is equal to the difference between the influx and outflow of radiant energy either on the outward or on the inward side of the surface F of the body. For a nontransparent body on the inward side (n = −0) of its surface F we have

$$E_{рез}\Big|_{n=-0} \equiv E_{погл} - E_{соб},$$

and on the outward side (n = +0) we have

$$E_{рез}\Big|_{n=+0} \equiv E_{пад} - E_{эфф}.$$

All these forms of hemispherical-radiation density can be defined

both for the entire radiation spectrum $0 \leq \lambda \leq \infty$, or for the elementary interval of variation of the wavelength $d\lambda$. In the former case the quantities E are integral densities, and in the latter they are spectral ones (monochromatic). The density of the resultant radiation E_{rez} can be positive, negative, or zero.

Processes of propagation of radiant energy are considered in the approximation of geometrical (i.e., ray) optics. This means that the dimensions of the bodies and the radii of curvature of their surfaces are large compared with the greatest wavelength of thermal radiation.

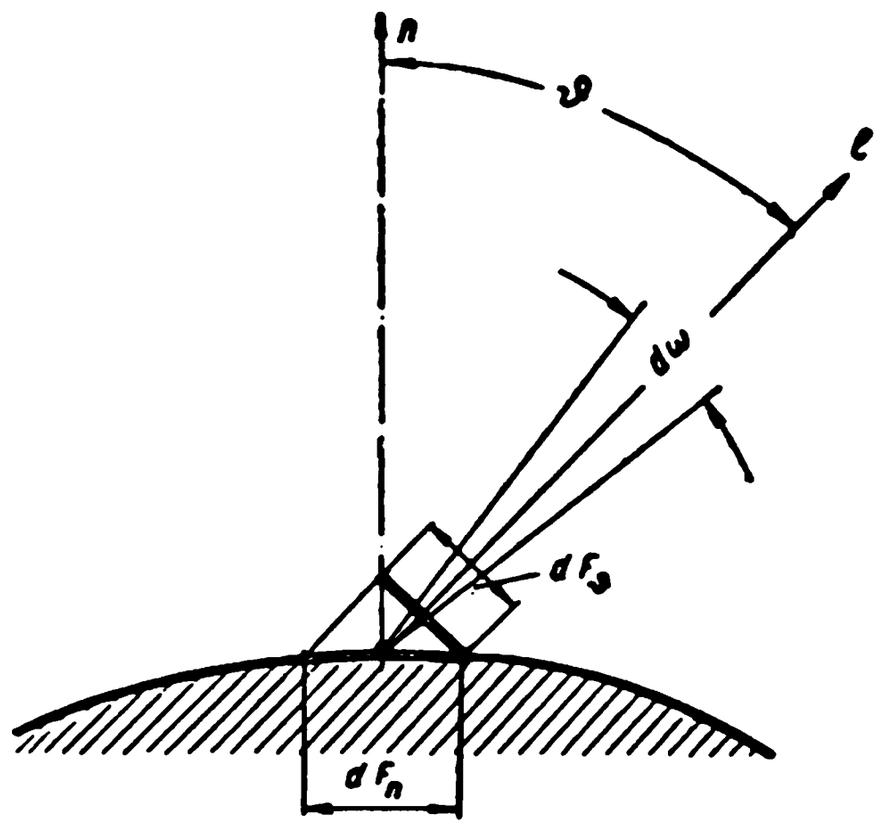

Fig. 95. Illustrating the definition of radiation brightness.

We are considering radiant heat exchange processes that are stationary in the mean. This means that the time averaged characteristics of the process do not depend on the time, and the averaging is carried out over a sufficiently large time interval compared with the largest period of oscillation of the thermal radiation.

Law of Energy Conservation for Radiant Flux Densities

Following the second feature of the phenomenological method, let us set up an expression for the law of energy conservation for the densities E_{pad}, E_{pogl}, E_{otr}, and E_{prokh}:

$$E_{пад} + E_{отр} + E_{прох} = E_{изл}.$$

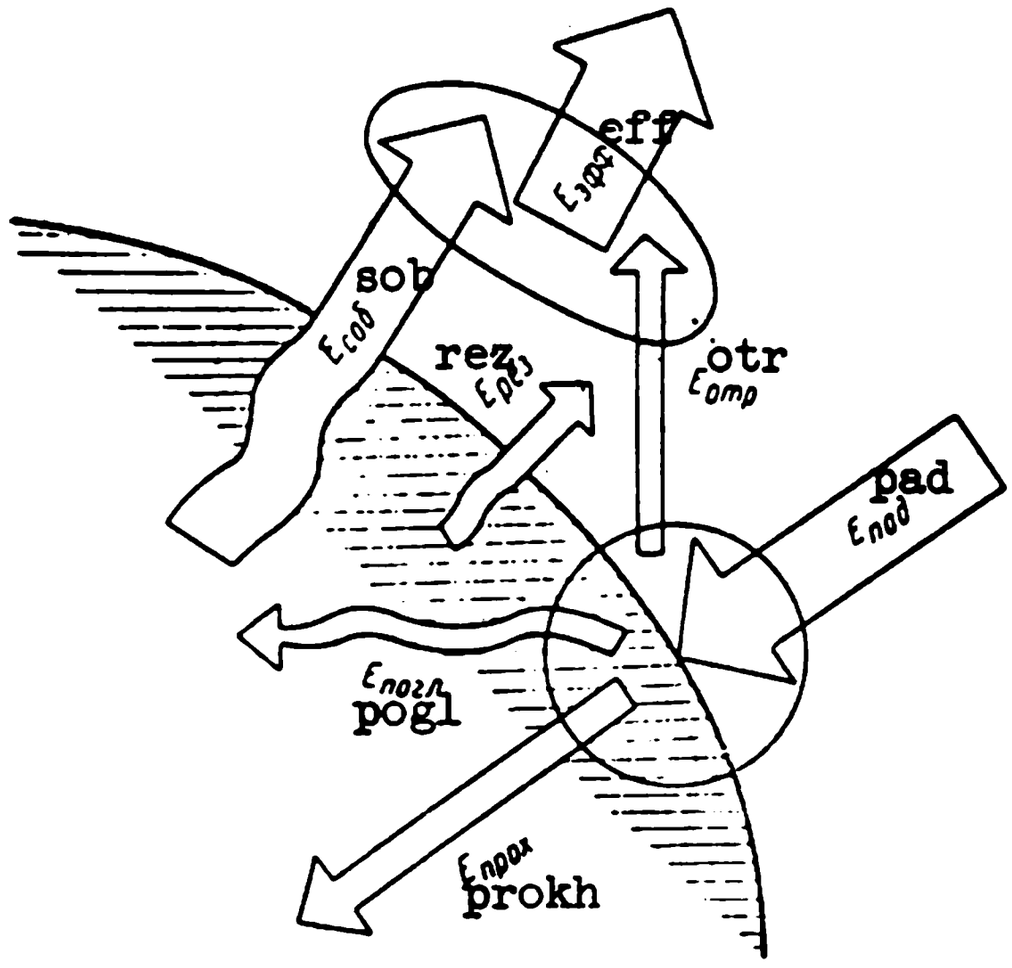

Fig. 96. Classifications of form of integral hemispherical radiation density on the surface of a semitransparent body.

Dividing the last relation by E_{pad}, we obtain

$$A + R + D = 1.$$

Here

$$A \equiv \frac{E_{погл}}{E_{пад}}; \quad R \equiv \frac{E_{отр}}{E_{пад}}; \quad D \equiv \frac{E_{прох}}{E_{пад}}$$

are the coefficients of absorption, reflection, and transmission, respectively. These coefficients are positive, finite, and dimensionless quantities, which characterize the ability of the body to absorb, reflect, or transmit thermal radiation. If the densities E_{pogl}, E_{otr}, E_{prokh}, and E_{pad} are integral, then the coefficients A, R, and D are called average over the spectrum. If these densities are monochromatic, then the quantities A_λ, R_λ, and D_λ are likewise designated as monochromatic. For real bodies the coefficients A, R, and D lie in the ranges $0 < A \leq 1$; $0 < R < 1$, and $0 \leq D < 1$. For nontransparent bodies, $D = 0$ and $A + R = 1$. For diathermal, white, and black bodies we have, respectively,

$$R_{\text{д}} = 0; \quad A_{\text{д}} = 0; \quad D_{\text{д}} = 1;$$
$$R_{\text{б}} = 1; \quad A_{\text{б}} = 0; \quad D_{\text{б}} = 0;$$
$$R_{\text{о}} = 0; \quad A_{\text{о}} = 1; \quad D_{\text{о}} = 0.$$

The average emissivity ε is the ratio of the density of the intrinsic radiation E_{sob} of a real body to the density of the intrinsic radiation E_{sob0} of a black body at the same temperature:

$$\varepsilon \equiv \frac{E_{\text{соб}}}{E_{\text{соб}\,\text{о}}}. \tag{25.4}$$

For the spectral emissivity, we have the following analogous expression:

$$\varepsilon_{\lambda} \equiv \frac{E_{\text{соб}\lambda}}{E_{\text{соб}\lambda_{\text{о}}}}.$$

A gray body is defined as a nontransparent body, the absorption (or reflection) coefficient of which $0 < A < 1$ does not depend on either the direction of the incident radiation or its spectral composition.

Let us use once more the law of energy conservation to obtain the computation relation between the densities of the effective, intrinsic, and resultant radiations [44, 48]. Inasmuch as

$$E_{\text{отр}} = (1 - A) \cdot E_{\text{пад}}$$

and

$$E_{\text{пад}} = E_{\text{рез}} + E_{\text{эфф}},$$

we have

$$E_{\text{отр}} = (1 - A) \cdot (E_{\text{рез}} + E_{\text{эфф}}).$$

Therefore

$$E_{\text{эфф}} \equiv E_{\text{соб}} + E_{\text{отр}} = E_{\text{соб}} + (1 - A) \cdot (E_{\text{рез}} + E_{\text{эфф}}).$$

From the last equation we get

$$E_{\text{эфф}} = \frac{E_{\text{соб}}}{A} + \left(\frac{1}{A} - 1\right) \cdot E_{\text{рез}}. \tag{25.5}$$

This relation is quite convenient for calculations of the resultant radiant fluxes. It follows from (25.5) that

$$\text{if } E_{\text{рез}} = 0, \quad E_{\text{эфф}} = \frac{E_{\text{соб}}}{A};$$

$$\text{if } E_{рез} > 0, \quad E_{эфф} > \frac{E_{соб}}{A};$$

$$\text{if } E_{рез} < 0, \quad E_{эфф} < \frac{E_{соб}}{A}.$$

In accordance with the Kirchhoff law we have

$$\frac{E_{соб}}{A} = E_{соб0}.$$

Therefore the density of effective radiation of a real body can, in accordance with the sign of E_{rez}, be either larger or smaller than the density of the intrinsic radiation of a black body of the same temperature.

If the densities of effective, intrinsic, and resultant radiations are constant on the surface F of the body, then we get from (25.1) and (25.5) the following computation relation for the radiant fluxes

$$Q_{эфф} = \frac{Q_{соб}}{A} + \left(\frac{1}{A} - 1\right) \cdot Q_{рез}. \tag{25.6}$$

By way of a very simple example of the use of this relation, let us consider the calculation of the resultant radiant flux: in a closed system consisting of two gray concentric spheres, separated by a transparent medium (Fig. 97). We are given the surfaces F_1 and F_2 of the spheres, their temperatures T_1 and $T_2 < T_1$, and the average absorption coefficients A_1 and A_2.

For each of these spheres Relation (25.6) holds true:

$$\left.\begin{aligned} Q_{эфф1} &= \frac{Q_{соб1}}{A_1} + \left(\frac{1}{A_1} - 1\right) \cdot Q_{рез1}; \\ Q_{эфф2} &= \frac{Q_{соб2}}{A_2} + \left(\frac{1}{A_2} - 1\right) \cdot Q_{рез2}. \end{aligned}\right\} \tag{25.7}$$

We obtain an additional equation by using the law of energy conservation for the resultant radiant fluxes in a closed system, consisting of two bodies:

$$Q_{рез1} + Q_{рез2} = 0. \tag{25.8}$$

The second additional equation is obtained by using the law of

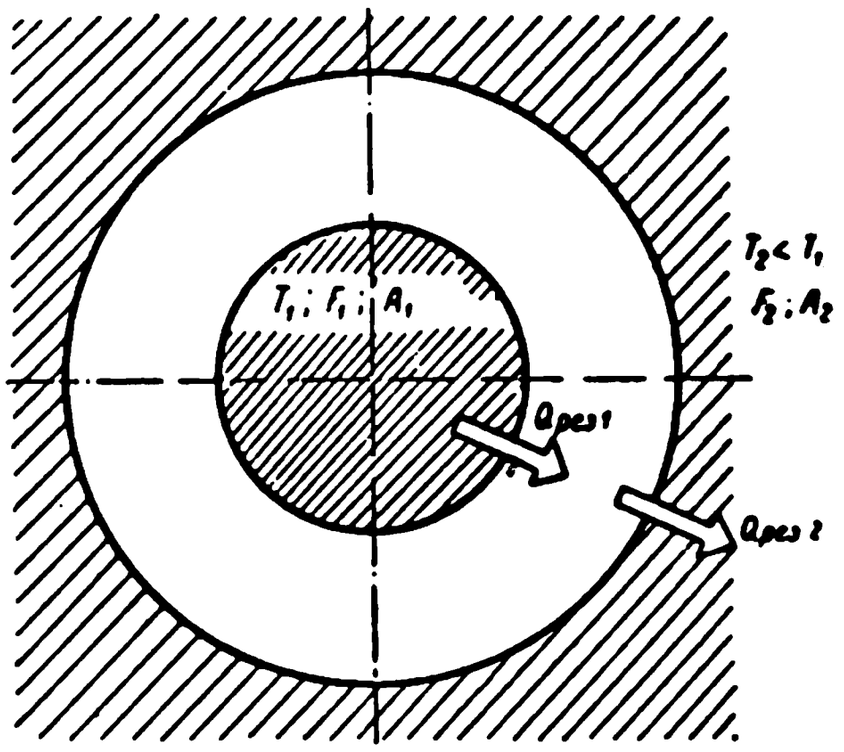

Fig. 97. Illustrating the calculation of radiant heat exchange between two spheres.

energy conservation for radiant fluxes, obtained and transmitted by some of the bodies (for example, by the second sphere). The entire effective radiant flux $Q_{eff\ 1}$ of the first body strikes the second body. However, only a certain part $\varphi_{2\to1} \cdot Q_{eff2}$ of the effective radiation of the second sphere arrives at the first sphere from the second one. Therefore

$$Q_{рез2} = Q_{эфф1} - \varphi_{2\to1} \cdot Q_{эфф2}. \quad (25.9)$$

The quantity $\varphi_{2\to1}$ is called the angle coefficient. This dimensionless quantity shows what fraction of the effective radiant flux of the second sphere strikes the first.

Solving the system of equations (25.7)-(25.9) with respect to Q_{rez2}, we obtain

$$Q_{рез2} = \frac{\frac{Q_{соб1}}{A_1} - \varphi_{2\to1}\frac{Q_{соб2}}{A_2}}{\frac{1}{A_1} + \varphi_{2\to1}\left(\frac{1}{A_2} - 1\right)}. \quad (25.10)$$

Consequently, to determine the resultant heat flux Q_{rez2} in the very simple case considered here of heat exchange between two spheres, it is necessary to know the values of the intrinsic radiant fluxes $Q_{sob\ 1}$ and Q_{sob2}, the absorption coefficients A_1 and A_2, and also the value of the angle coefficient $\varphi_{2\to1}$.

First and Second Laws of Thermodynamics for Equilibrium Thermal Radiation (the Stefan-Boltzmann and the Kirchhoff Laws)

Following the second feature of the phenomenological method, let us use the fundamental laws of thermodynamics for the determination of the connection between the hemispherical density of intrinsic integral

each of the bodies participating in the radiant heat exchange.

For the equilibrium radiation of a black body, Boltzmann obtained (1884) the following relation:

$$E_{\text{соб}0} = C_0 \cdot \left(\frac{T}{100}\right)^4 \text{ kcal/m}^2\text{-hr}. \qquad (25.11)$$

Somewhat earlier, Stefan (1879) obtained an analogous relation as a result of measurement of the intrinsic radiation of a model of a black body. The Stefan-Boltzmann law reads as follows:

the hemispherical density of the intrinsic integral radiant flux of a black body in thermodynamic equilibrium with surrounding bodies is directly proportional to the fourth power of its absolute temperature.

The proportionality coefficient C_0, in kcal/m^2-hr-deg^4, in Eq. (25.11) is called the black-body radiation coefficient. It must be noted that the intrinsic radiation of real bodies differs from that of a black body. It is therefore impossible to make direct use of the Stefan-Boltzmann law for practical calculations of the radiant heat exchange between real bodies in general and between two gray spheres in particular. This difficulty is overcome with the aid of the second law of thermodynamics. Application of this law to the analysis of the properties of equilibrium radiation of real bodies leads to the following Kirchhoff law:

the ratio of the hemispherical density of the intrinsic integral radiant flux E_{sob} of any real body to its mean absorption coefficient A under conditions of thermodynamic equilibrium is the same for all bodies and depends only on the temperatures of these bodies:

$$\frac{E_{\text{соб}}}{A} = \ldots = \frac{E_{\text{соб}0}}{A_0} = E_{\text{соб}0}(T). \qquad (25.12)$$

Kirchhoff's law is satisfied also for monochromatic equilibrium

$$\frac{E_{\cos\lambda}}{A_\lambda} = E_{\lambda\cos 0}(T). \qquad (25.13)$$

Equation (25.12) determines the value of E_{sob} of a nonblack body as a function of $E_{sob\,0}$ of a black body, i.e., as a function of the temperature T, and also as a function of the physical properties, whose influence is reflected by the value of the absorption coefficient A.

It must be emphasized that the Stefan-Boltzmann and Kirchhoff laws are <u>perfectly rigorous only for equilibrium thermal radiation</u>. Therefore under conditions of nonequilibrium radiant heat exchange in a system of bodies having different temperatures, Eqs. (25.11) and (25.12) become approximate. Their use for practical calculations is related with the assumption of the presence of local thermodynamic equilibrium at each point on the surface of each of the bodies participating in the heat exchange.

From Kirchhoff's law it is possible to draw several conclusions:

1) the larger the absorption coefficient of the body, the larger the density of its intrinsic radiant flux for a specified temperature;

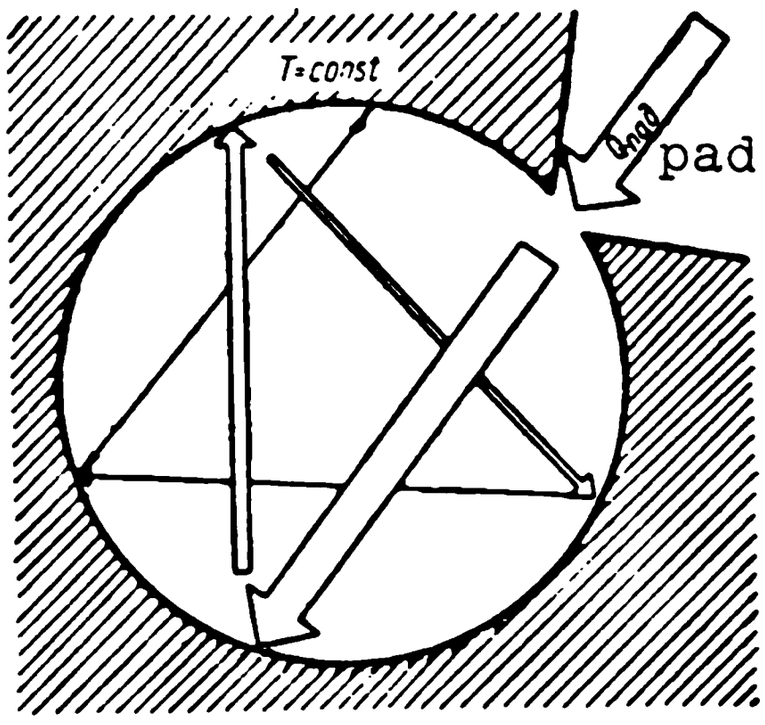

Fig. 98. Diagram of approximate model of a black body.

2) the largest absorption coefficient is possessed by a black body, for which $A = A_0 = 1$. Consequently, for a specified temperature a black body has also the largest value of intrinsic emission as compared with other bodies;

3) one can use for an approximate model of a black body a small opening in an isothermal shell surrounding a large cavity (Fig. 98). In this case the radiant flux incident on the aperture is absorbed almost completely

each of the bodies participating in the radiant heat exchange.

For the equilibrium radiation of a black body, Boltzmann obtained (1884) the following relation:

$$E_{\text{соб}0} = C_0 \cdot \left(\frac{T}{100}\right)^4 \text{ kcal/m}^2\text{-hr}. \qquad (25.11)$$

Somewhat earlier, Stefan (1879) obtained an analogous relation as a result of measurement of the intrinsic radiation of a model of a black body. The Stefan-Boltzmann law reads as follows:

the hemispherical density of the intrinsic integral radiant flux of a black body in thermodynamic equilibrium with surrounding bodies is directly proportional to the fourth power of its absolute temperature.

The proportionality coefficient C_0, in kcal/m^2-hr-deg^4, in Eq. (25.11) is called the black-body radiation coefficient. It must be noted that the intrinsic radiation of real bodies differs from that of a black body. It is therefore impossible to make direct use of the Stefan-Boltzmann law for practical calculations of the radiant heat exchange between real bodies in general and between two gray spheres in particular. This difficulty is overcome with the aid of the second law of thermodynamics. Application of this law to the analysis of the properties of equilibrium radiation of real bodies leads to the following Kirchhoff law:

the ratio of the hemispherical density of the intrinsic integral radiant flux E_{sob} of any real body to its mean absorption coefficient A under conditions of thermodynamic equilibrium is the same for all bodies and depends only on the temperatures of these bodies:

$$\frac{E_{\text{соб}}}{A} = \ldots = \frac{E_{\text{соб}0}}{A_0} = E_{\text{соб}0}(T). \qquad (25.12)$$

Kirchhoff's law is satisfied also for monochromatic equilibrium

$$\frac{E_{\cos\lambda}}{A_\lambda} = E_{\lambda\cos 0}(T). \qquad (25.13)$$

Equation (25.12) determines the value of E_{sob} of a nonblack body as a function of $E_{sob\,0}$ of a black body, i.e., as a function of the temperature T, and also as a function of the physical properties, whose influence is reflected by the value of the absorption coefficient A.

It must be emphasized that the Stefan-Boltzmann and Kirchhoff laws are <u>perfectly rigorous only for equilibrium thermal radiation</u>. Therefore under conditions of nonequilibrium radiant heat exchange in a system of bodies having different temperatures, Eqs. (25.11) and (25.12) become approximate. Their use for practical calculations is related with the assumption of the presence of local thermodynamic equilibrium at each point on the surface of each of the bodies participating in the heat exchange.

From Kirchhoff's law it is possible to draw several conclusions:

1) the larger the absorption coefficient of the body, the larger the density of its intrinsic radiant flux for a specified temperature;

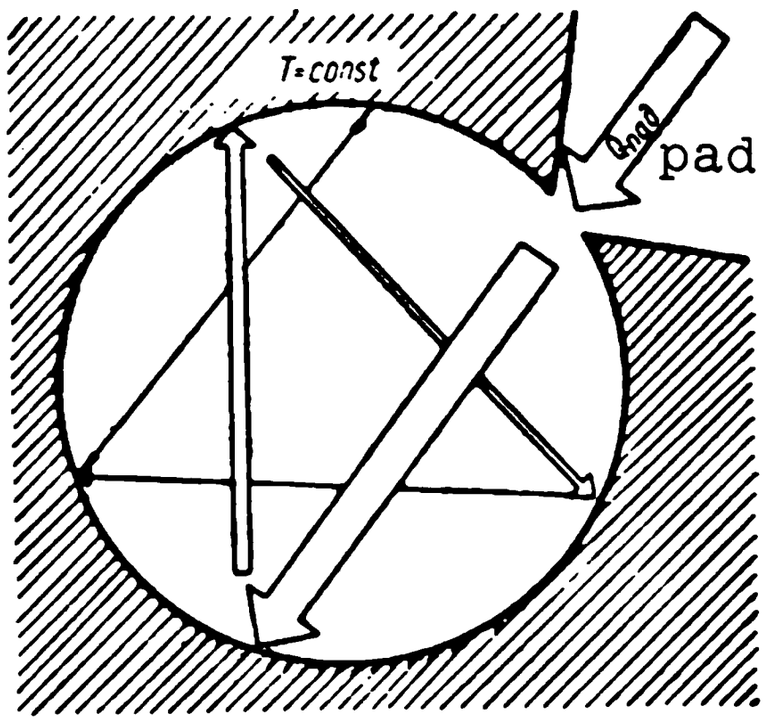

Fig. 98. Diagram of approximate model of a black body.

2) the largest absorption coefficient is possessed by a black body, for which $A = A_0 = 1$. Consequently, for a specified temperature a black body has also the largest value of intrinsic emission as compared with other bodies;

3) one can use for an approximate model of a black body a small opening in an isothermal shell surrounding a large cavity (Fig. 98). In this case the radiant flux incident on the aperture is absorbed almost completely

inside the cavity as a result of multiple reflections from the shell and as a result of absorptions inside the shell. Therefore the coefficient of absorption of such an aperture is close to unity;

4) the degree of blackness of a body under conditions of equilibrium radiant heat exchange is numerically equal to its coefficient of absorption

$$\varepsilon = A. \qquad (25.14)$$

This conclusion follows from a comparison of Eqs. (25.4) and (25.12).

5) For equilibrium monochromatic radiation an analogous relation holds true

$$e_\lambda = A_\lambda.$$

Consequently, a body with a selective spectrum radiates only in those wavelength intervals, in which it can absorb radiant energy.

Comparing (25.11) with (25.12), we obtain the Stefan-Kirchhoff equation for the hemispherical density of the integral intrinsic radiation of a nonblack body as a function of the temperature

$$E_{соб} = A \cdot C_0 \cdot \left(\frac{T}{100}\right)^4. \qquad (25.15)$$

If the surface F of the body is isothermal, gray, and homogeneous, then E_{sob} is constant on this surface.

Under these conditions, the intrinsic integral radiant flux is equal to

$$Q_{соб} = F \cdot A \cdot C_0 \cdot \left(\frac{T}{100}\right)^4. \qquad (25.16)$$

Substituting Relation (25.16) into (25.10), we obtain the resultant radiant flux in a system of two gray spheres

$$Q_{рез\,2} = \frac{F_1 \cdot C_0}{\frac{1}{A_1} + \varphi_{2\to1} \cdot \left(\frac{1}{A_2} - 1\right)} \times$$

$$\times \left[\left(\frac{T_1}{100}\right)^4 - \varphi_{2\to1} \cdot \frac{F_2}{F_1} \cdot \left(\frac{T_2}{100}\right)^4\right]. \qquad (25.17)$$

Thus, a calculation of Q_{rez2} reduces to a determination of the physical parameters C_0, A_1, A_2, and the angle coefficient φ_{2-1}.

Hypothesis of Local Thermodynamic Equilibrium and Lambert's Law

Following the third feature of the phenomenological method, we introduce hypotheses which enable us to use the laws of equilibrium thermal radiation for the calculation of nonequilibrium and radiant heat exchange processes, and which also help determine the value of the angle coefficient φ_{2-1}. The first of these consists in the assumption that at each point of the surface of the body participating in the nonequilibrium radiant heat exchange, local thermodynamic equilibrium obtains. The second hypothesis (Lambert, 1760) reads as follows:

the brightness of the intrinsic and also of the reflected thermal radiation does not depend on the direction from the given point on the surface of the body.

An experimental verification leads to the conclusion that Lambert's hypothesis is satisfied for rough bodies and for directions whose angle ϑ with the normal to the surface of the body does not exceed 60° ($0 \leq \vartheta \leq 60^\circ$). With further increase in this angle, i.e., as $\vartheta \to \pi/2$, the radiation brightness B decreases sharply. Examples of the dependence of the brightness B on the angle ϑ are shown in Fig. 99.

From an examination of these examples it follows that the value of the brightness averaged over different directions differs little from its constant value at $0 < \vartheta < 60^\circ$. Therefore in technical calculations of the radiant heat exchange between rough bodies it is usually assumed that Lambert's law is satisfied for all directions. In this case it becomes possible to prove that the value of the angle coefficient $\varphi_{2\to1}$ has a purely geometrical character, i.e., it depends only on the dimensions and mutual placement of the two spheres. Consequently, the quantity $\varphi_{2\to1}$ does not depend on the temperatures T_1 and

T_2, and can be determined under equilibrium heat-exchange conditions, when both bodies have the same temperature $T_1 = T_2 = T$. Substituting the last condition in (25.17), we obtain

$$\varphi_{2\to 1} = \frac{F_1}{F_2}. \qquad (25.18)$$

Thus, the share of the effective radiation of the second sphere $\varphi_{2\to 1}$ striking the first sphere is equal to the ratio of the surfaces of these spheres.

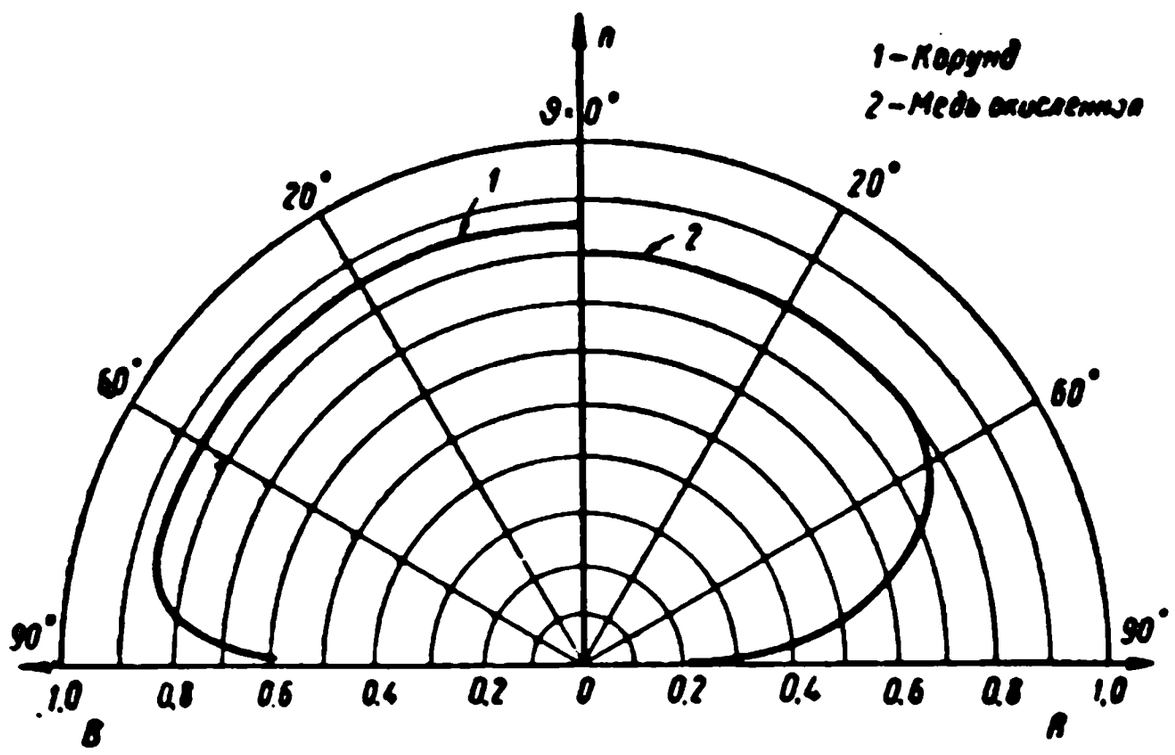

Fig. 99. Dependence of the brightness of thermal radiation on the direction for certain body roughnesses. 1) Corundum; 2) oxidized copper.

With the aid of Relations (25.17) and (25.18) we obtain the resultant flux

$$Q_{рез} = \frac{F_1 C_0}{\frac{1}{A_1} + \frac{F_1}{F_2}\cdot\left(\frac{1}{A_2} - 1\right)}\left[\left(\frac{T_1}{100}\right)^4 - \left(\frac{T_2}{100}\right)^4\right]. \qquad (25.19)$$

The quantity

$$\frac{1}{\frac{1}{A_1} + \frac{F_1}{F_2}\cdot\left(\frac{1}{A_2} - 1\right)} \equiv A_{1\to 2}$$

is called the reduced coefficient of absorption of the system of bodies 1 and 2. Then Relation (25.19) assumes the form

$$Q_{рез2} = A_{1\to 2} C_0\left[\left(\frac{T_1}{100}\right)^4 - \left(\frac{T_2}{100}\right)^4\right] \qquad (25.20)$$

Thus, calculation of Q_{rez2} reduces to a determination of the phys-

ical parameters C_0, A_1, and A_2.

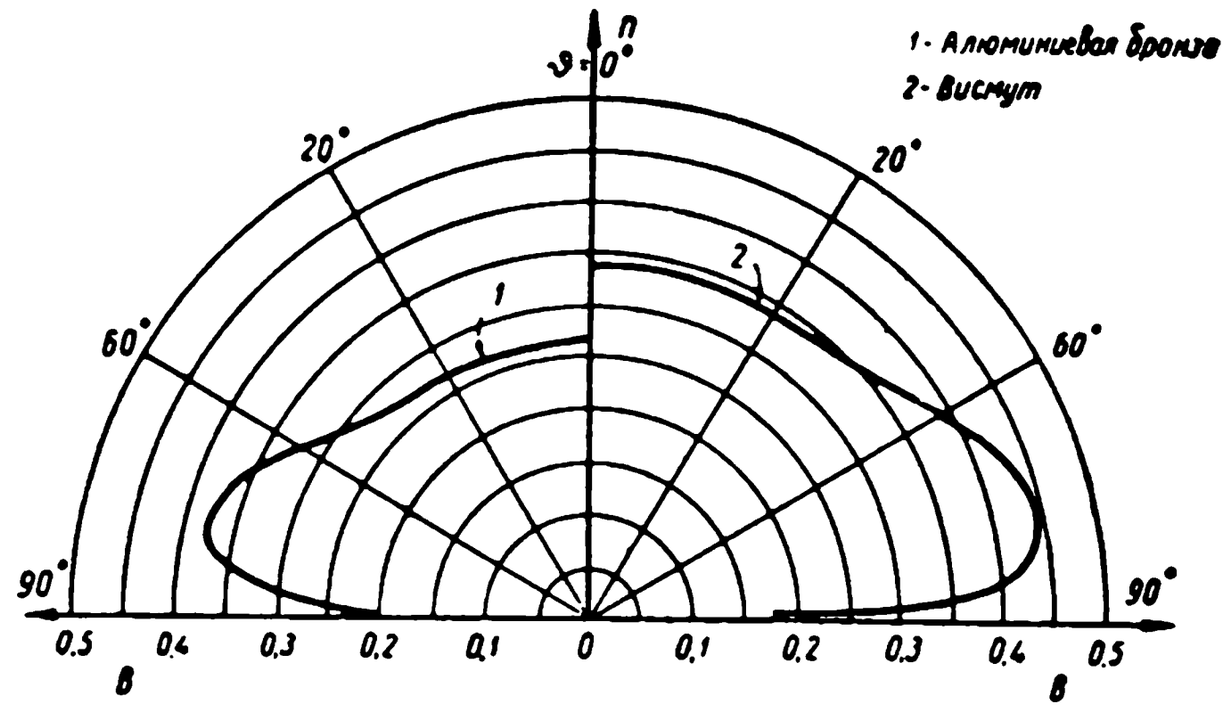

Fig. 100. Dependence of the brightness of thermal radiation on the direction for polished metals. 1) Aluminum bronze; 2) bismuth.

It must be noted that Lambert's law does not hold for the brightness of the intrinsic and reflected radiations of polished metals (Fig. 100).

Emission and Absorption Coefficients and the Emissivity

Following the fourth feature of the phenomenological method, we determine by experiment the emission coefficient of a black body, and also the coefficients of absorption or the degree of blackness for each of the bodies participating in the radiant heat exchange.

As a result of numerous measurements of the intrinsic radiation of a black-body model, the following value was obtained for C_0:

$$C_0 = 4.9 \text{ kcal/m}^2\text{-hr-}^\circ\text{K}^4.$$

The values of the average emissivity ε for certain metals are listed in the table.

In some cases the quantity ε or A changes appreciably with the temperature of the body. An example of such a dependence of the emissivity ε for a cleaned surface of nonmachined and cast steel is shown in Fig. 101. Figure 102 shows the dependence of A for graphite at room temperature on the temperature of the source of the black-body

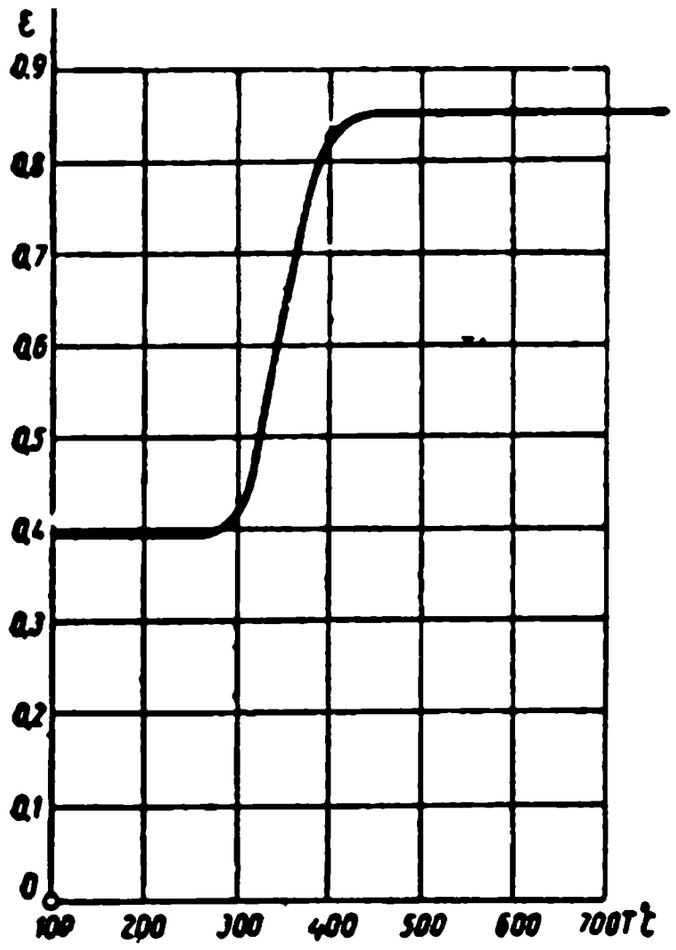

Fig. 101. Dependence of the emissivity of steel on its temperature.

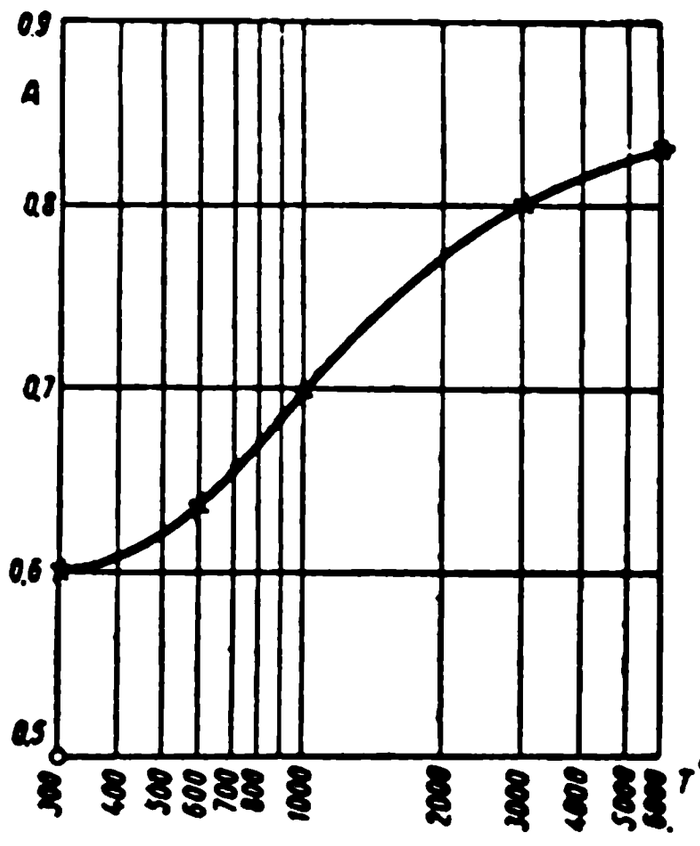

Fig. 102. Dependence of the coefficient of absorption of graphite on the temperature of the source of black-body radiation incident on the graphite.

1 Материал	Температура, °С 2	ε
3 Алюминий полированный	50—500	0,04—0,06
4 • сильно окисленный	50—500	0,2—0,3
5 Вольфрам	200	0,05
•	600—1000	0,1—0,16
•	1500—2200	0,24—0,31
6 Медь полированная	50—100	0,02
7 • окисленная	50	0,6
• •	500	0,88
8 • жидкая	1100—1300	0,13—0,15
9 Молибден	600—1000	0,08—0,13
•	1500—2200	0,19—0,26
10 Молибденовая нить	700—2500	0,1—0,3
11 Платина	1000—1500	0,14—0,18
12 Ртуть чистая	0—100	0,09—0,12
13 Серебро чистое, полированное	200—600	0,02—0,03
14 Хром полированный	500—1000	0,28—0,38

1) Material; 2) temperature; 3) aluminum polished; 4) aluminum strongly oxidized; 5) tungsten; 6) copper polished; 7) copper oxidized; 8) copper liquid; 9) molybdenum; 10) molybdenum filament; 11) platinum; 12) mercury pure; 13) silver pure polished; 14) chromium polished.

radiation incident on the graphite [70]. It is seen from this figure that as the temperature of the incident radiation is increased from 300 to 6000°K

Region of Applicability of the Formula for Radiant Heat Exchange between Two Concentric Spheres

Expression (25.19) for the resultant radiant flux Q_{rez2} has been obtained for a system of two gray concentric spheres, each of which has a constant temperature and a constant absorption coefficient at all points of their surfaces. This equation, however, can be used also for the calculation of Q_{rez2} in different systems, consisting of two isothermal gray bodies, separated by a transparent medium [44]. These include a system of two long coaxial cylinders, and also a system of two plates parallel to each other (Fig. 103), the distance between which is small compared with their height and width.

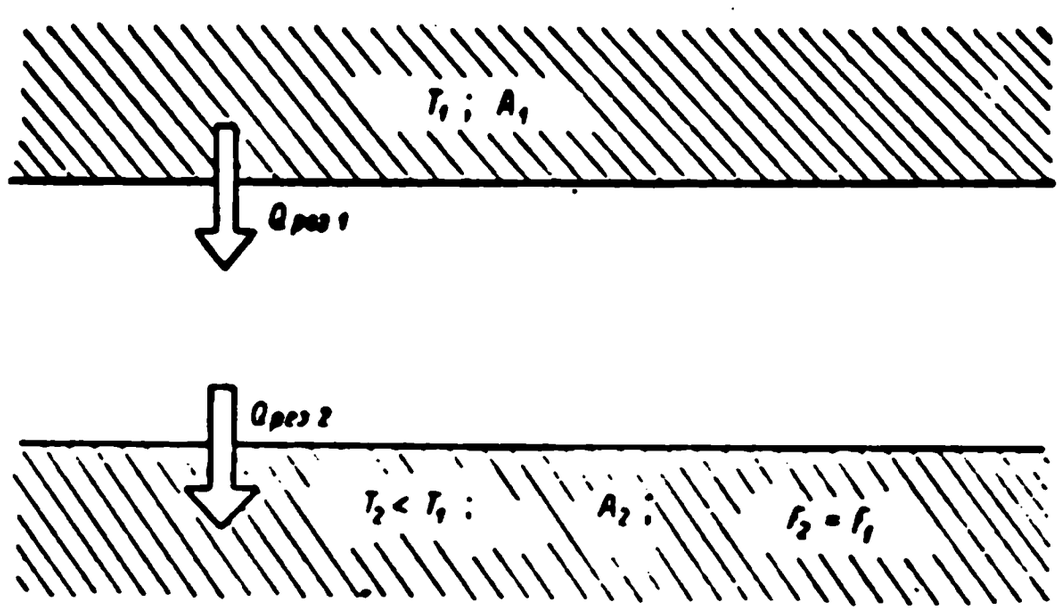

Fig. 103. Illustrating the calculation of radiant heat exchange between two plates that are parallel to each other.

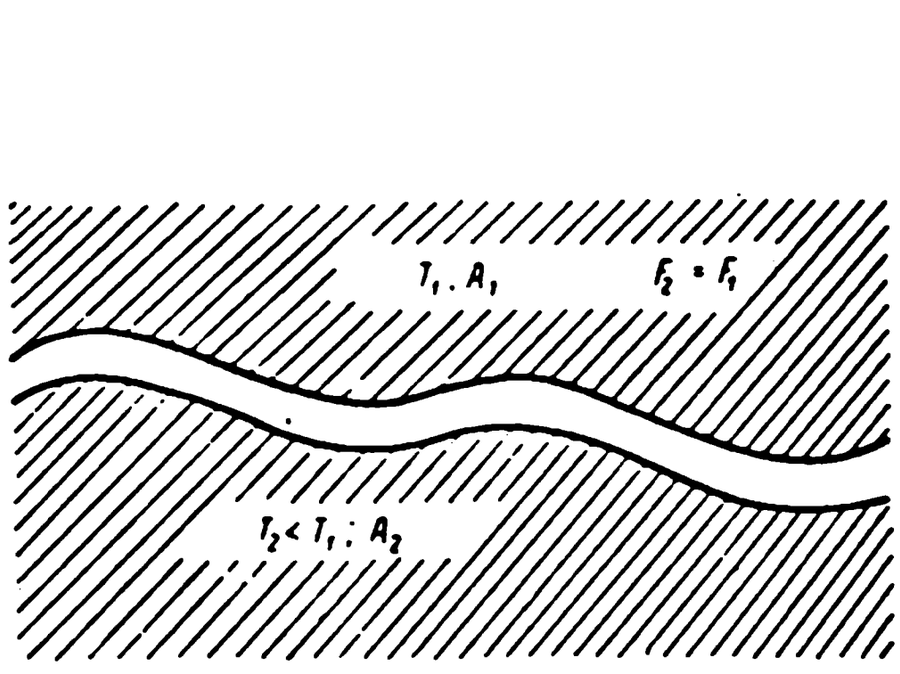

Fig. 104. Illustrating the calculation of radiant heat exchange between two equidistant curvilinear surfaces that are located close to each other.

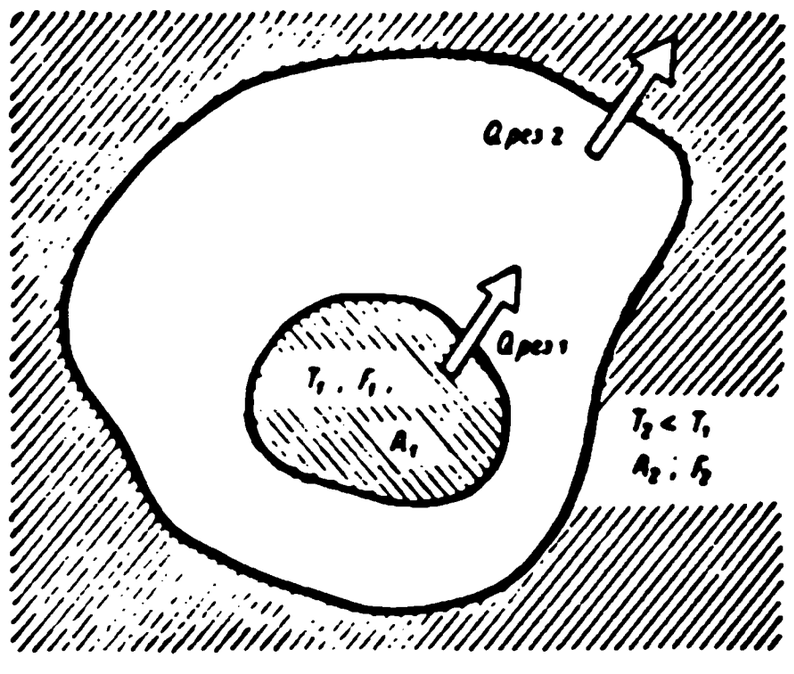

Fig. 105. Illustrating the calculation of radiant heat exchange in a closed system of two bodies.

In the latter case $F_1 = F_2$ and we have in lieu of (25.19)

$$Q_{рез2} = \frac{F \cdot C_0}{\frac{1}{A_1} + \frac{1}{A_2} - 1}\left[\left(\frac{T_1}{100}\right)^4 - \left(\frac{T_2}{100}\right)^4\right]. \qquad (25.21)$$

Relation (25.21) is valid also for a system of two equidistant curvilinear surfaces (Fig. 104), the distance between which is quite small.

If the surface of the external sphere exceeds appreciably the surface of the internal one, i.e., $F_2 \gg F_1$, and A_2 is not too small, then the quantity $F_1/F_2(1/A_2 - 1)$ can be neglected compared with $1/A_1$

$$\frac{F_1}{F_2}\left(\frac{1}{A_2} - 1\right) \ll \frac{1}{A_1}.$$

In this case $A_{1\to 2} \approx A_1$ and Relation (25.19) assumes the form

$$Q_{рез2} \approx F_1 \cdot A_1 \cdot C_0 \cdot \left[\left(\frac{T_1}{100}\right)^4 - \left(\frac{T_2}{100}\right)^4\right]. \qquad (25.22)$$

The last equation can be used to estimate Q_{rez2} in a system of two bodies of arbitrary form (Fig. 105), if the internal body is not concave and its surface does not exceed 25% of the surface of the internal body ($F_1 \leq 0.25F_2$), and each of the absorption coefficients is not smaller than 0.7 (A_1; $A_2 \geq 0.7$).

When the absorption coefficients A_1 and A_2 of the two spheres depend on the temperature of the incident-radiation source (for example in the case of graphite), the degree of blackness and the absorption coefficient are different in value and in lieu of Eq. (25.19) we can use the following relation [57]:

$$Q_{рез2} = \frac{F_1 \cdot C_0}{\frac{1}{A_1} + \frac{F_1}{F_2} \cdot \left(\frac{1}{A_2} - 1\right)} \times$$
$$\times \left[\frac{\varepsilon_1}{A_1} \cdot \left(\frac{T_1}{100}\right)^4 - \frac{\varepsilon_2}{A_2} \cdot \left(\frac{T_2}{100}\right)^4\right]. \qquad (25.23)$$

Equation (25.22) has been obtained in the same manner as (25.19), but the intensity of the intrinsic radiation was determined not by Kirchhoff's law $E_{sob} = A \cdot E_{sob0}$ but by Relation (25.4)

$$E_{\quad} = \varepsilon \cdot E_{\quad 0}.$$

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[List of Transliterated Symbols]

609 соб = sob = sobstvennyy = intrinsic

609 пад = pad = padayushchiy = incident

609 отр = otr = otrazhennyy = reflected

609 погл = pogl = pogloshchennyy = absorbed

609 эфф = eff = effektivnyy = effected

609 рез = rez = rezul'tiruyushchiy = resultant

610 прох = prokh = prokhodyashchiy = passing, transmitted

Chapter 13

HEAT-EXCHANGE APPARATUS

§26. PRINCIPLES OF THERMAL DESIGN OF HEAT-EXCHANGE APPARATUS OF THE RECUPERATIVE TYPE

Classification of Principal Types of Heat-Exchange Apparatus. Principal Assumptions Made in the Thermal Design

Heat-exchange apparatus is defined as any device intended for the transmission of heat from a hotter heat carrier (one or several) to a cooler one. Heat-exchange apparatus is subdivided into contact and surface types (Fig. 106). In apparatus of the contact type the heat is transferred as a result of direct contact between the hotter and the cooler heat carriers.

Contact apparatus are subdivided into mixing and bubbling types. In apparatus of the mixing type the hotter and the cooler heat carriers are mixed together and form solutions or mixtures.

In apparatus of the bubbling type the hotter heat carrier is driven through the bulk of the cooler one (or vice versa), without mixing the two.

Heat exchangers of the surface type have a solid heating surface, which participates in the transmission of the heat.

Apparatus of the surface type is subdivided into regenerators and recuperators.

In regenerators the solid heating surface (frequently called the checker) first receives heat from the hotter heat carrier. After the checker accumulates a sufficient amount of heat, the hotter flow is turned off, and is replaced by flow of the cooler liquid. The latter

is heated by absorbing the heat accumulated in the checker.

In recuperators, the solid heating surface separates the flows of the hotter and cooler heat carriers, and the heat is transferred through this surface.

At the present time the heat exchangers used in nuclear power are predominantly of the recuperative type. We shall therefore consider below the principles of the thermal design of recuperators. We employ here the following limitations:

1) the heat-transmission process is stationary;

2) in the recuperative heat exchanger two heat carriers, a hotter one and a cooler one, flow parallel to each other;

3) the weight flow of the heat carriers along the heating surface, G_1 and G_2, as well as their specific heat C_{p1} and C_{p2} and the heat-transmission coefficient $\underline{k}$ remain constant in value:

$$k \equiv \frac{1}{\frac{1}{\alpha_1} + \frac{\delta}{\lambda} + \frac{1}{\alpha_2}}.$$

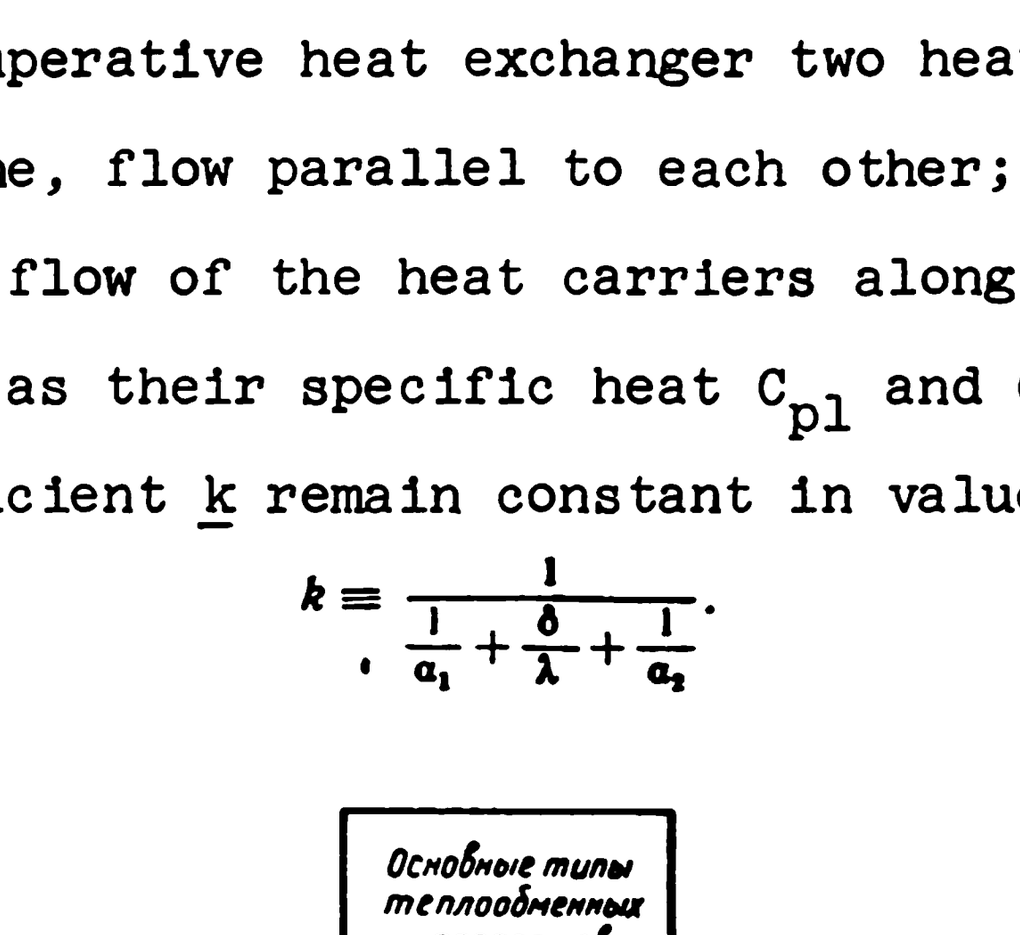

Fig. 106. Classification of the basic types of heat-exchange apparatus. 1) Basic types of heat-exchange apparatus; 2) contact; 3) surface; 4) mixing; 5) bubbling; 6) regenerators; 7) recuperators.

Fundamental Equations for Thermal Design of Recuperative Heat Exchangers

Let us consider a recuperator in which the streams of the hotter and cooler heat carriers are directed parallel to each other in the same direction (Fig. 107). This is called a direct-flow convection. If

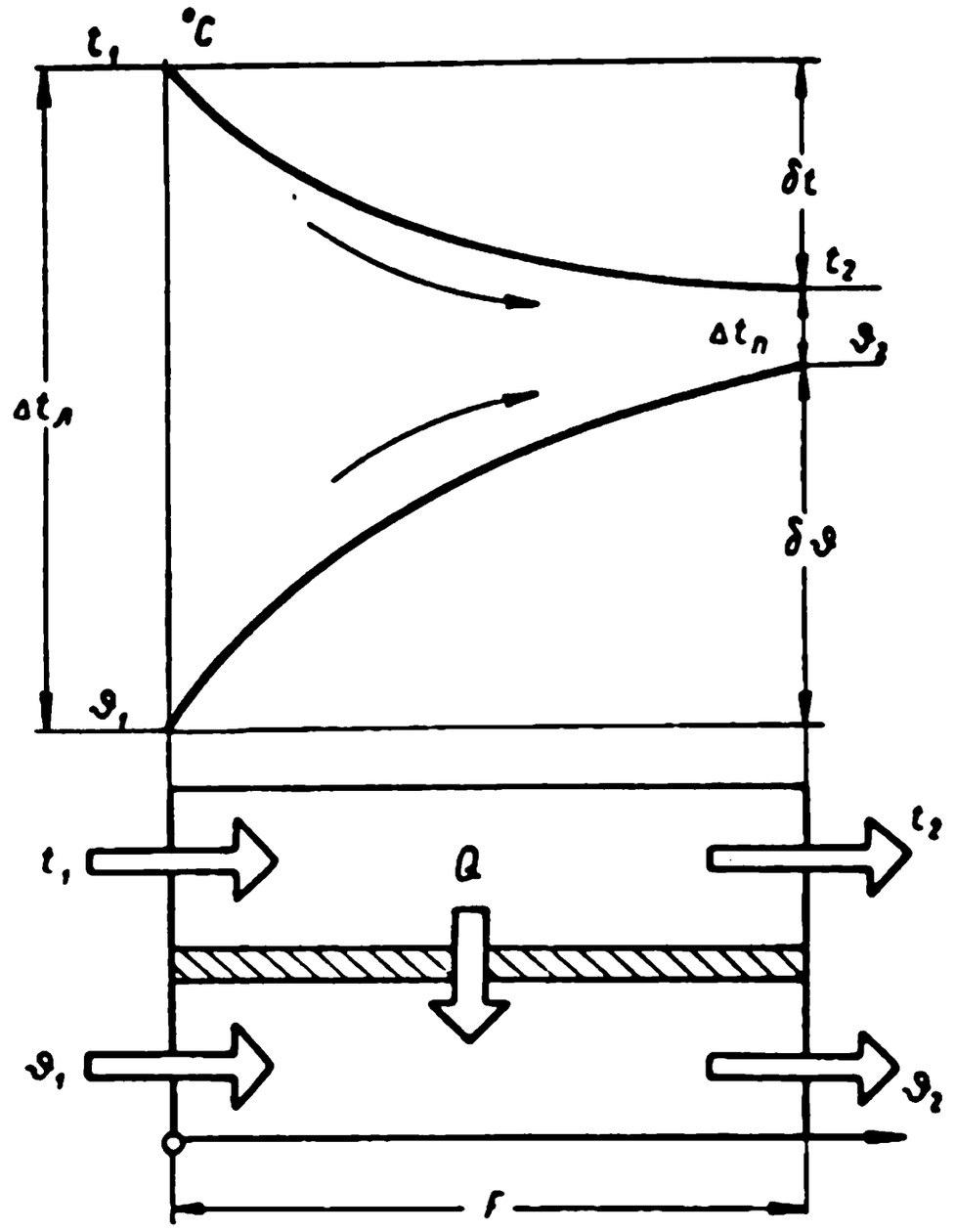

Fig. 107. Single-pass scheme of the flow of heat carriers in a recuperator.

the parallel streams are directed in opposite directions (Fig. 108), then this is called a counterflow scheme, and if they flow transversely to each other this is called a crossing scheme (Fig. 109).

To carry out the thermal calculations for the recuperator it is necessary to find an equation relating the heat flux Q in kcal/hr through the heating surface and the size of this surface F in m^2, the heat-transfer coefficient k, and also the initial and final temperatures of both carriers t_1, t_2, ϑ_1, and ϑ_2.

Let us formulate the uniqueness conditions for this problem.

The geometrical conditions specify a direct-flow motion of the

two carriers and a constant thickness δ of the heating surface.

The physical conditions specify constant specific heats C_{p1} and C_{p2} for the carriers, a heat conductivity λ for the heating surface, and also weight flows G_1 and G_2 for the carriers, in kg/hr.

The time conditions indicate that the process of heat transmission is stationary.

The boundary conditions specify constant values of the heat-transmission coefficients α_1 and α_2 along the heating surface, and also constant temperatures of the two heat carriers on entering and leaving the heat-exchange apparatus:

$$\text{for } f = 0,\ t = t_1 \text{ and } \vartheta = \vartheta_1 < t_1;$$

$$\text{for } f = F,\ t = t_2 < t_1 \text{ and } \vartheta = \vartheta_2 < t_2.$$

The heat flux dQ through an elementary portion of the heating surface df is

$$dQ = k \cdot (t - \vartheta) \cdot df, \tag{26.1}$$

where k is the heat-transmission coefficient.

Integrating this relation, we get

$$Q = k \cdot \int_0^F (t - \vartheta) \cdot df. \tag{26.2}$$

To determine the value of the local temperature difference $t - \vartheta$ as a function of the running value of the heating surface f, let us express the heat flux dQ in terms of the specific heats, the weight flows, and the elementary changes in the heat-carrier temperatures:

$$dQ = -W_1 \cdot dt = W_2 \cdot d\vartheta, \tag{26.3}$$

where $W_1 \equiv G_1 \cdot C_{p1}$ and $W_2 \equiv G_2 \cdot C_{p2}$ are the water equivalents for each of the streams.

From Eq. (26.3) we get

$$dt = -dQ/W_1 \text{ and } d\vartheta = dQ/W_2.$$

Consequently,

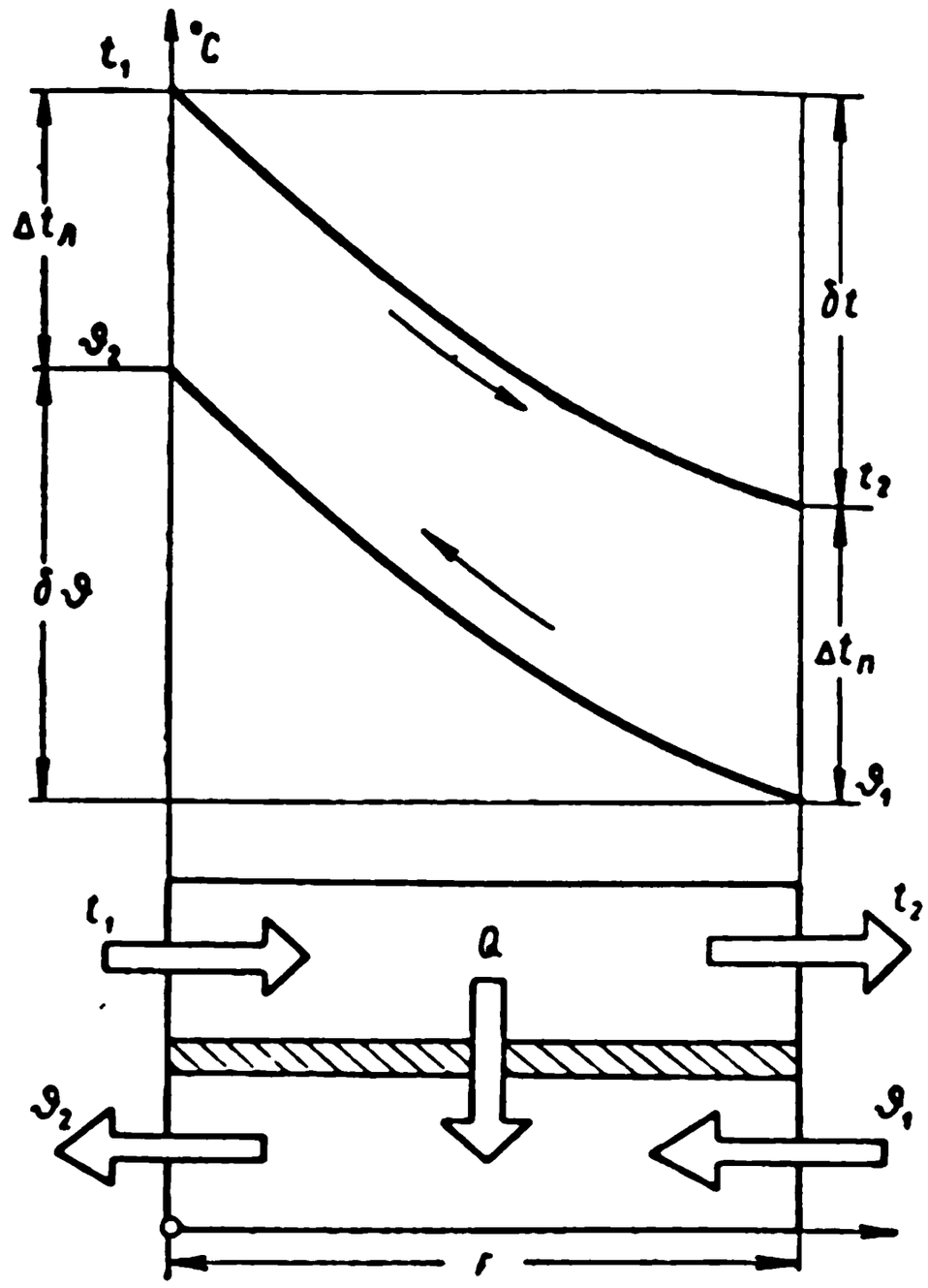

Fig. 108. Counterflow scheme of heat carriers in a recuperator.

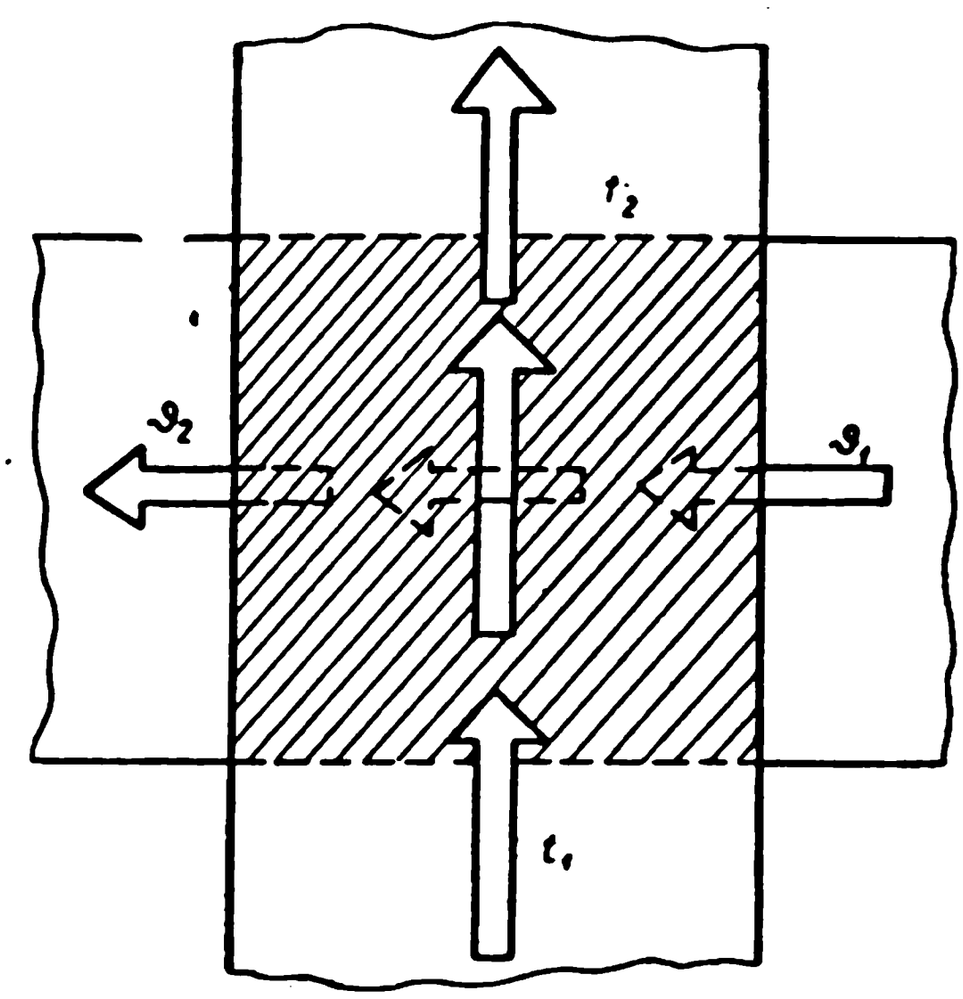

Fig. 109. Crossing flow of heat carriers in a recuperator.

$$d(t-\vartheta) = -\left(\frac{1}{W_1} + \frac{1}{W_2}\right) \cdot dQ. \qquad (26.4)$$

$$\frac{1}{W_1}+\frac{1}{W_2}\equiv m$$

and substituting (26.1) in (26.4), we obtain a first-order ordinary differential equation for the local temperature difference t − ϑ

$$\frac{d(t-\vartheta)}{df}=-m\cdot k\cdot(t-\vartheta).$$

A particular solution of this equation, satisfying the boundary condition

$$t-\vartheta\Big|_{f=0}=t_1-\vartheta_1\equiv\Delta t_n,$$

has the form

$$t-\vartheta=\Delta t_n\cdot\exp(-m\cdot k\cdot f). \qquad (26.5)$$

Here Δt_l is the local value of the temperature difference on the left side of the heat-exchange apparatus.

Let us rewrite (26.5) for the right half of the recuperator:

$$\Delta t_n=\Delta t_n\cdot\exp(-m\cdot k\cdot f). \qquad (26.6)$$

Eliminating mk from (26.5) and (26.6), we get

$$t-\vartheta=\Delta t_n\cdot\exp\left\{\frac{f}{F}\cdot\ln\left(\frac{\Delta t_n}{\Delta t_n}\right)\right\}=\Delta t_n\cdot\left(\frac{\Delta t_n}{\Delta t_n}\right)^{f/F}. \qquad (26.7)$$

Substituting Relation (26.7) in (26.2), we get

$$Q=k\Delta t_n\int_0^F\left(\frac{\Delta t_n}{\Delta t_n}\right)^{f/F}df.$$

Integrating and transforming, we obtain

$$Q=kF\Delta t_{cp}. \qquad (26.8)$$

Here

$$\Delta t_{cp}\equiv\frac{\Delta t_n-\Delta t_n}{\ln\left(\frac{\Delta t_n}{\Delta t_n}\right)}. \qquad (26.9)$$

Relation (26.7) is the principal equation of heat transmission in recuperators. The quantity Δt_{sr} is called the average logarithmic temperature difference. Expression (26.8) for Δt_{sr} is valid for both direct flow and counterflow.

For direct flow (see Fig. 107)

$$\Delta t_{cp} = \frac{(t_1 - \vartheta_1) - (t_2 - \vartheta_2)}{\ln\left(\frac{t_1 - \vartheta_1}{t_2 - \vartheta_2}\right)}. \qquad (26.10)$$

For counterflow (see Fig. 108)

$$\Delta t_{cp} = \frac{(t_1 - \vartheta_2) - (t_2 - \vartheta_1)}{\ln\left(\frac{t_1 - \vartheta_2}{t_2 - \vartheta_1}\right)}. \qquad (26.11)$$

Equation (26.8) is valid also for more complicated connections of the heat-carrier streams in the recuperator. In these cases, however, the value of the average temperature difference does not satisfy Relation (26.9).

In addition to Eq. (26.8), it is easy to obtain two other equations by integrating (26.3)

$$\left.\begin{aligned} Q &= -W_1\int_{t_1}^{t_2} dt = W_1(t_1 - t_2) = W_1\delta t; \\ Q &= W_2\int_{\vartheta_1}^{\vartheta_2} d\vartheta = W_2(\vartheta_2 - \vartheta_1) = W_2\delta\vartheta, \end{aligned}\right\} \qquad (26.12)$$

where

$$\delta t \equiv t_1 - t_2 \quad \text{and} \quad \delta\vartheta \equiv \vartheta_2 - \vartheta_1.$$

Average Temperature Difference

Let us rewrite in dimensionless form Eq. (26.9)

$$\frac{\Delta t_{cp}}{\Delta t_л} = \frac{\left(\frac{\Delta t_б}{\Delta t_м}\right) - 1}{\ln\left(\frac{\Delta t_б}{\Delta t_м}\right)} = \varphi\left(\frac{\Delta t_б}{\Delta t_м}\right).$$

The dependence $\Delta t_{sr}/\Delta t_l = \varphi(\Delta t_p/\Delta t_l)$ is shown in Fig. 110. The same figure shows by means of dashed lines three approximate expressions for the function φ. In the interval $0.5 \leq \Delta t_p/\Delta t_l \leq 1$ the value of the dimensionless temperature difference $\Delta t_{sr}/\Delta t_l$ can be approximately calculated from the following linear equation:

$$\frac{\Delta t_{cp}}{\Delta t_л} \approx 0{,}5 + 0{,}5\left(\frac{\Delta t_б}{\Delta t_м}\right)$$

or

$$\Delta t_{cp} \approx \frac{\Delta t_n + \Delta t_n}{2}. \qquad (26.13)$$

In the broader interval $0.2 \leq \Delta t_p/\Delta t_l \leq 1$ another approximate linear equation holds true [9c]

$$\frac{\Delta t_{cp}}{\Delta t_n} \approx 0{,}4 + 0{,}6 \cdot \left(\frac{\Delta t_n}{\Delta t_n}\right)$$

or

$$\Delta t_{cp} \approx \frac{\Delta t_n + \Delta t_n}{2} - 0{,}1\,(\Delta t_n - \Delta t_n). \qquad (26.14)$$

Finally, to determine the temperature difference in the broadest interval $0.1 \leq \Delta t_p/\Delta t_l \leq 1$ we can use the following approximate nonlinear relation [58a]:

$$\frac{\Delta t_{cp}}{\Delta t_n} \approx \frac{1}{3}\left[\frac{1}{2}\left(1 + \frac{\Delta t_n}{\Delta t_n}\right) + 2\sqrt{\frac{\Delta t_n}{\Delta t_n}}\right]$$

or

$$\Delta t_{cp} \approx \frac{1}{3}\left(\frac{\Delta t_n + \Delta t_n}{2} + 2\sqrt{\Delta t_n \cdot \Delta t_n}\right). \qquad (26.15)$$

Relations (26.13)-(26.15) are valid for both direct flow and counterflow.

The average temperature difference for crossing motion of the heat carriers, and also for other more complicated connections of the streams, must be determined from the following equation

$$\Delta t_{cp} = \Delta t_{cp.прот} \cdot \psi(P, R). \qquad (26.16)$$

Here $\Delta t_{sr.prot}$ is the average logarithmic value of the temperature difference for counterflow connection of the same heat carriers, which flow in the more complicated scheme.

The correction coefficient $\psi = \psi(P, R)$ is determined from auxiliary plots as a function of the values of P and R:

$$P \equiv \frac{\vartheta_2 - \vartheta_1}{t_1 - \vartheta_1};$$

$$R \equiv \frac{t_1 - t_2}{\vartheta_2 - \vartheta_1}.$$

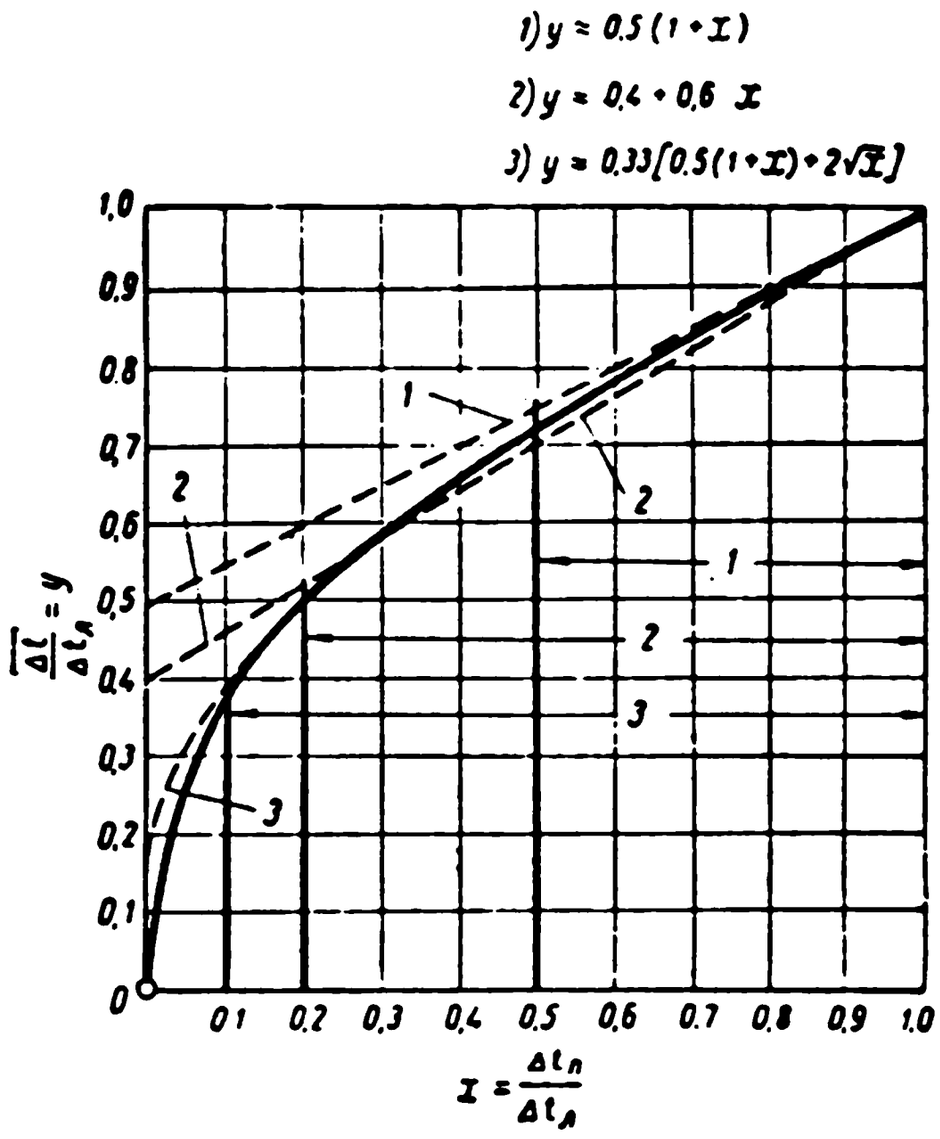

Fig. 110. Regions of applicability of the approximate formulas for the average temperature difference for "direct flow" and "counterflow."

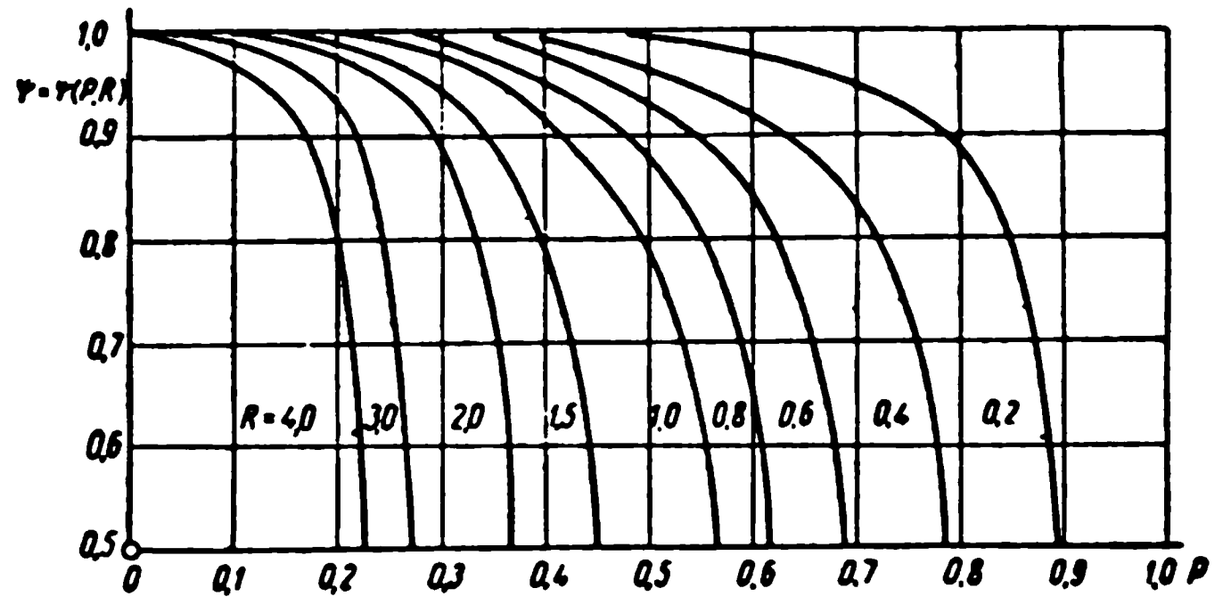

Fig. 111. Plot for the calculation of the average temperature difference in crossing flow of the heat carriers.

By way of an example Fig. 111 shows a plot of $\psi(P, R)$ for the simplest crossing flow. Analogous plots have been prepared also for more complicated stream connections [31, 35b].

Comparison of Direct Flow and Counterflow Schemes of Heat-Carrier Motion

Heat-transmission processes for the direct-flow and counterflow schemes of motion of heat carriers in recuperators are equivalent to each other in two cases:

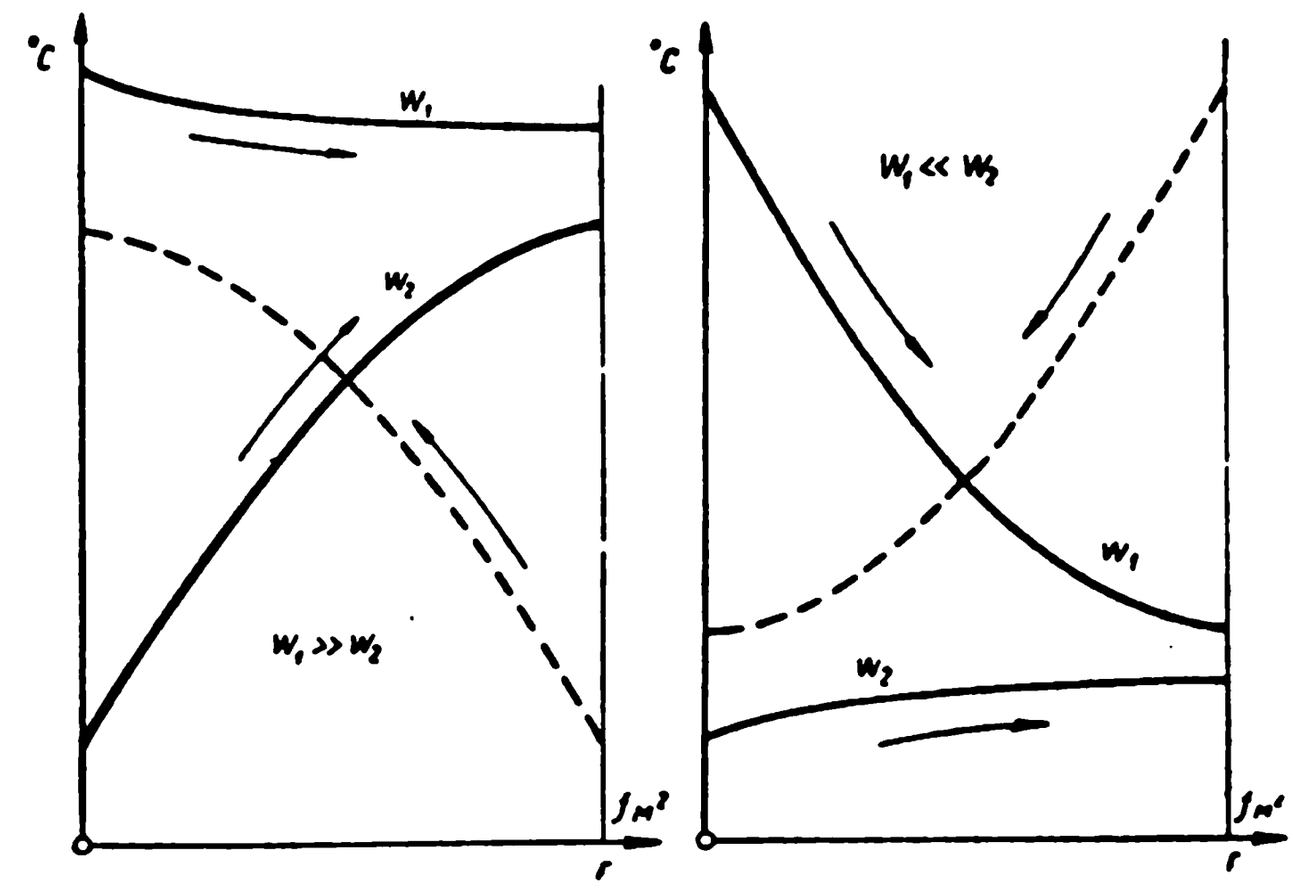

Fig. 112. Illustrating the comparison of the direct flow and counterflow heat-carrier schemes.

If the water equivalents of the hotter and colder heat carriers, W_1 and W_2, respectively, differ greatly from each other, i.e., if $W_1 \gg W_2$ or $W_1 \ll W_2$. When $W_1 \gg W_2$ we have $\delta t \ll \delta \vartheta$, and when $W_1 \ll W_2$ we have $\delta t \gg \delta \vartheta$ (Fig. 112).

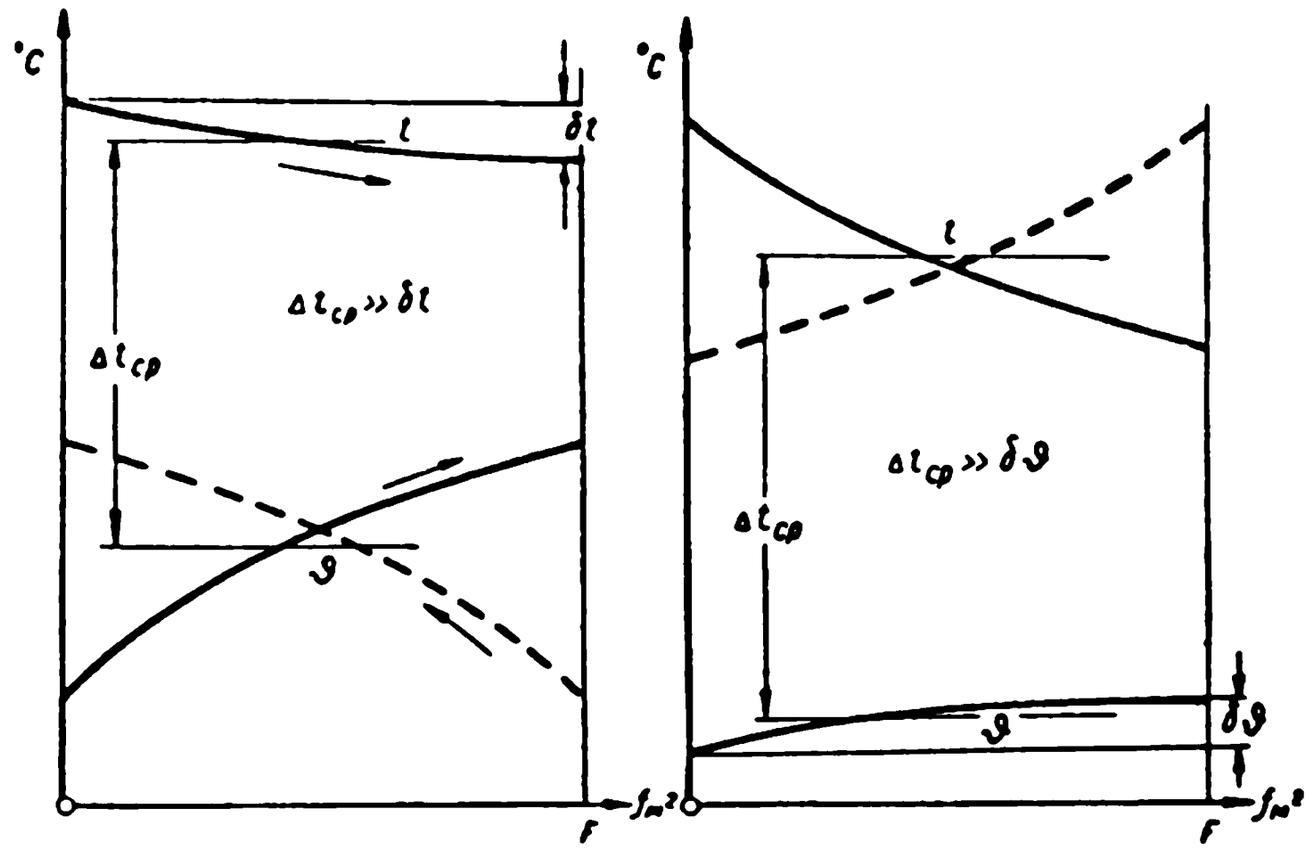

Fig. 113. Illustrating the comparison of the direct flow and counterflow heat-carrier schemes.

If the average temperature difference exceeds greatly the total change in temperature δt or $\delta \vartheta$ along the heating surface of the heat-

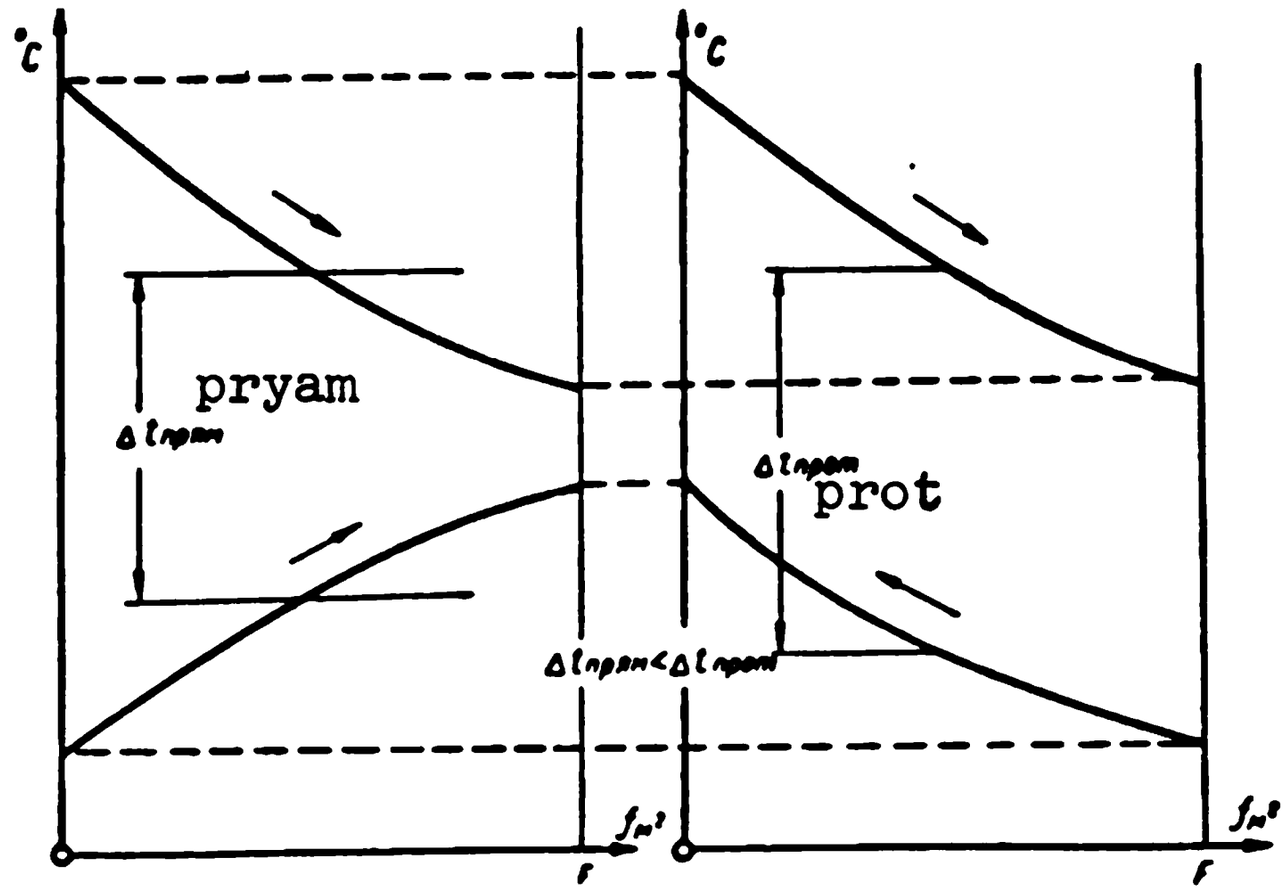

Fig. 114. Illustrating the comparison of the direct flow and counterflow heat-carrier schemes.

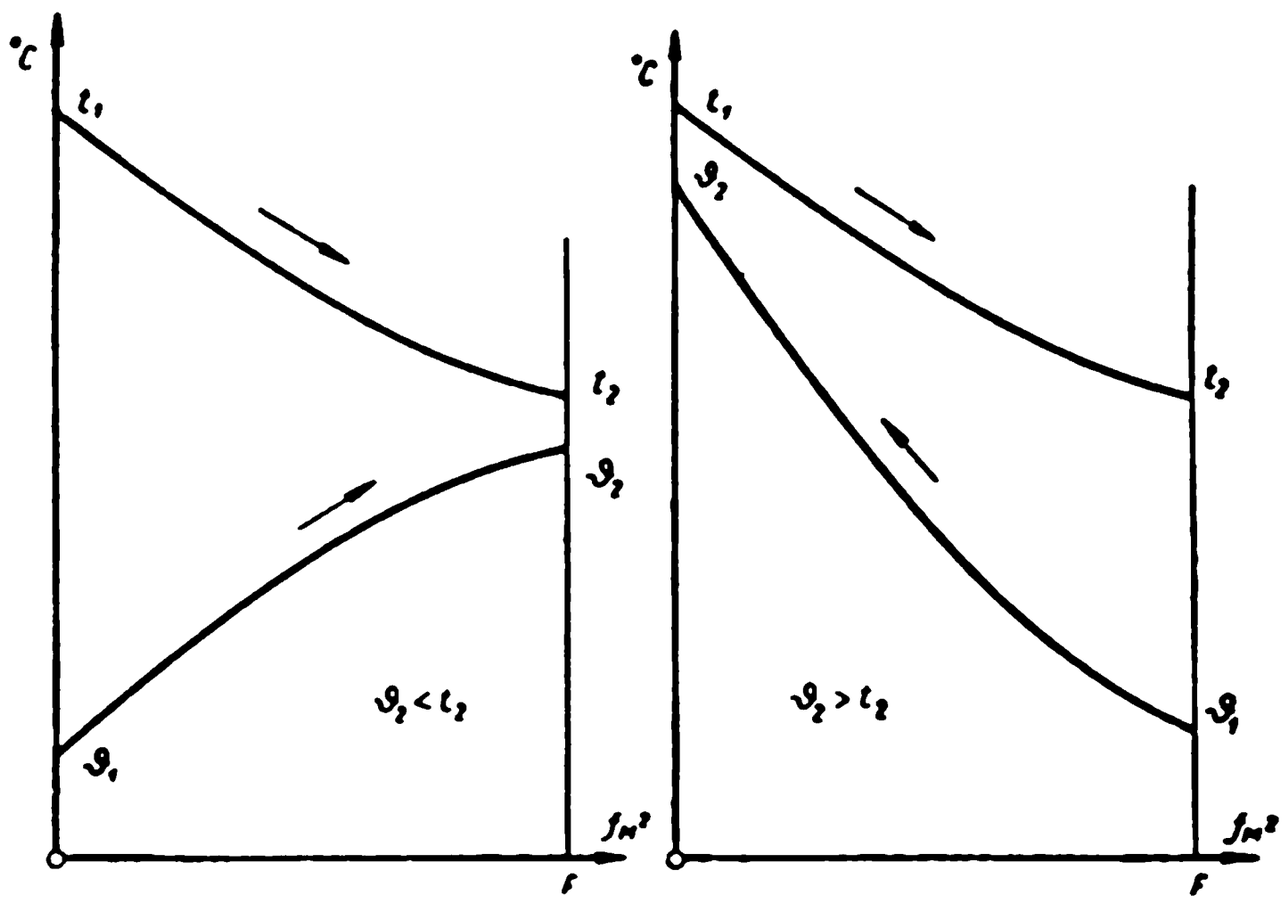

Fig. 115. Illustrating the comparison of the direct flow and counterflow heat-carrier schemes.

all other cases the counterflow scheme is more convenient than the direct flow scheme for the following reasons.

If the initial and final temperatures of the liquids are the same for both schemes, the average temperature drop for the counterflow scheme is larger than for the direct flow scheme (Fig. 114),

$$\Delta t_{sr.prot} > \Delta t_{sr.pryam}.$$

When a counterflow scheme is used, it is possible to heat the

colder liquid to a higher temperature than the final temperature of the hot liquid (Fig. 115). If the direct flow scheme is used, this is impossible, since $\vartheta_2 < t_2$.

Classification of Principal Problems in Thermal Design of Recuperators

The thermal design of heat exchangers of the recuperative type is usually subdivided into two stages. The first stage consists of determining the seven quantities t_1, t_2, ϑ_1, ϑ_2, W_1, W_2, and $k \cdot F$, which we call fundamental, and the second stage consists in determining the derivative quantities Q, $\underline{k}$, F, Δt_{sr}, C_{p1}, C_{p2}, G_1, and G_2 [58b].

To determine the seven principal quantities we have the two equations

$$W_1 \cdot (t_1 - t_2) = W_2 \cdot (\vartheta_2 - \vartheta_1) = (k \cdot F) \cdot \Delta t_{cp}(t_1, t_2, \vartheta_1, \vartheta_2), \qquad (26.17)$$

which follow from Eqs. (26.8) and (26.9). Therefore the five fundamental quantities out of the seven should be specified beforehand, and two of them should be determined from Eq. (26.17). This gives rise to 21 problems of the first stage of thermal design of recuperators. These problems are conveniently subdivided into six groups.

The first group consists of four problems and reduces to a determination of the quantity $(k \cdot F)$ and one of the four temperatures (t_1, t_2, ϑ_1, or ϑ_2), if the values of the water equivalents W_1 and W_2 and the values of three temperatures are specified.

The second group consists of two problems and reduces to a determination of $(k_1 \cdot F)$ and the water equivalent W of one of the two heat carriers (W_1 or W_2), provided the temperatures t_1, t_2, ϑ_1, ϑ_2 and the water equivalent of the second liquid are specified: $(k \cdot F) = ?$, $W_1 = ?$ (or $W_2 = ?$).

The third group consists of one problem. It is necessary here to separate the two water equivalents W_1 and W_2 for specified t_1, t_2, ϑ_1, ϑ_2, and $(k \cdot F)$.

The fourth group consists of six problems and consists of determining two arbitrary final temperatures of the heat carriers (t_1 and t_2, ϑ_1 and ϑ_2, t_1 and ϑ_1, t_1 and ϑ_2, t_2 and ϑ_1, t_2 and ϑ_2) if the two other final temperatures, the water equivalents W_1 and W_2, and the quantity (k·F) are specified.

The fifth group consists of four problems. The unknowns are the water equivalent and the final temperature of one and the same heat carrier (W_1 and ϑ_1, W_1 and ϑ_2, W_2 and t_1, W_2 and t_2) provided the second water equivalent, three values of the final temperatures of the heat carriers, and the quantity (k·F) are specified.

The sixth group also consists of four problems, each of which reduces to a determination of the water equivalent for one heat carrier and one of the final temperatures of the other heat carrier (W_1 and ϑ_1, W_1 and ϑ_2, W_2 and t_1, W_2 and t_2), provided the second water equivalent, three values of the final temperatures of the heat carriers, and the quantity (k·F) are specified.

Methods of Solving the Fundamental Problems of Thermal Design of Recuperators

First group of problems ($k \cdot F = ?$; $t_{1 \text{ or } 2} = ?$ or $\vartheta_{1 \text{ or } 2} = ?$).

From the equation

$$W_1 \cdot (t_1 - t_2) = W_2 \cdot (\vartheta_2 - \vartheta_1)$$

we obtain one of the final temperatures t_1, t_2, ϑ_1, or ϑ_2. We then calculate the average logarithmic temperature difference

$$\Delta t_{cp} = \frac{\Delta t_a - \Delta t_n}{\ln\left(\frac{\Delta t_a}{\Delta t_n}\right)}$$

and determine the second unknown quantity

$$k \cdot F = \frac{W_1 \cdot (t_1 - t_2)}{\Delta t_{cp}}.$$

Second group of problems ($k \cdot F = ?$; $W_{1 \text{ or } 2} = ?$). All four final values of the temperatures are specified. We can therefore calculate

Δt_{sr} and then determine the value of $k \cdot F$ from the formula given above. The value of the water equivalent is determined from the equation

$$W_2 = W_1 \cdot \frac{t_1 - t_2}{\vartheta_2 - \vartheta_1}. \qquad (26.18)$$

Third group of problems (W_1 = ?; W_2 = ?). We calculate the average logarithmic temperature difference Δt_{sr}. It is then easy to determine both water equivalents

$$W_1 = \frac{(k \cdot F) \cdot \Delta t_{cp}}{t_1 - t_2}; \quad W_2 = \frac{(k \cdot F) \cdot \Delta t_{cp}}{\vartheta_2 - \vartheta_1}.$$

Fourth group of problems ($t_{1 \text{ or } 2}$ = ?; $\vartheta_{1 \text{ or } 2}$ = ?). We consider first the solutions of this group of problems for the direct flow of the heat carriers. In this case (see Fig. 107) the following relation holds true:

$$\Delta t_б = \delta t + \Delta t_м + \delta\vartheta,$$

therefore

$$\Delta t_б - \Delta t_м = \delta t + \delta\vartheta$$

In addition,

$$\delta t = \frac{W_2}{W_1} \cdot \delta\vartheta.$$

Hence

$$\Delta t_б - \Delta t_м = \frac{W_2}{W_1} \cdot \delta\vartheta + \delta\vartheta = \left(\frac{W_2}{W_1} + 1\right) \cdot \delta\vartheta.$$

To solve the problem we use Eq. (26.17), from which we get

$$W_2 \cdot \delta\vartheta = (k \cdot F) \cdot \frac{\Delta t_б - \Delta t_м}{\ln\left(\frac{\Delta t_б}{\Delta t_м}\right)}. \qquad (26.19)$$

Substituting the preceding equation in (26.19), we get

$$W_2 \cdot \delta\vartheta = (k \cdot F) \cdot \frac{\left(\frac{W_2}{W_1} + 1\right) \cdot \delta\vartheta}{\ln\left(\frac{\Delta t_б}{\Delta t_м}\right)}$$

or

$$\ln\left(\frac{\Delta t_б}{\Delta t_м}\right) = m \cdot (k \cdot F),$$

where

$$m = \frac{1}{W_1} + \frac{1}{W_2}.$$

Consequently,

$$\frac{t_1 - \vartheta_1}{t_2 - \vartheta_2} = \exp[m \cdot (k \cdot F)]. \qquad (26.20)$$

The two linear algebraic equations (26.18) and (26.20) form a closed system which contains as unknowns any two final temperatures of the heat carrier.

Let us consider the <u>counterflow</u> scheme of connecting the heat exchanger on the condition that the water numbers of both streams are not the same, i.e., $W_1 \neq W_2$. From Fig. 108 we see that

$$\Delta t_л + \delta\vartheta = \delta t + \Delta t_п.$$

Therefore

$$\Delta t_л - \Delta t_п = \delta t - \delta\vartheta = \left(\frac{W_2}{W_1} - 1\right) \cdot \delta\vartheta. \qquad (26.21)$$

Substituting (26.21) in (26.19), we obtain after transformation

$$\frac{t_1 - \vartheta_2}{t_2 - \vartheta_1} = \exp \cdot \left[(k \cdot F) \cdot \left(\frac{1}{W_1} - \frac{1}{W_2}\right)\right]. \qquad (26.22)$$

Relations (26.18) and (26.22) form a closed system of two equations with two unknown final temperatures.

Finally, let us consider a counterflow scheme under the condition that both water equivalents have the same value:

$$W_1 = W_2 = W.$$

In this case Eqs. (26.10) and (26.17) simplify noticeably and assume the form

$$t_1 - t_2 = \vartheta_2 - \vartheta_1 = \frac{k \cdot F}{W} \cdot (t_1 - \vartheta_2). \qquad (26.23)$$

The system (26.23) consisting of two equations must be solved relative to any of the two final temperatures.

<u>Fifth group of problems</u> (W_1 = ?; $t_{1 \text{ or } 2}$ = ? or W_2 = ?; $\vartheta_{1 \text{ or } 2}$ =

= ?). For the <u>direct flow</u> scheme, the problems of the fifth group reduce to a solution of the transcendental equation

$$\frac{\vartheta_2-\vartheta_1}{t_2-\vartheta_2}=\frac{(kF)}{W_2}\frac{\frac{t_1-\vartheta_1}{t_2-\vartheta_2}-1}{\ln\left(\frac{t_1-\vartheta_1}{t_2-\vartheta_2}\right)}. \qquad (26.24)$$

Solving this equation by trial with respect to t_1 or t_2, we obtain the water equivalent W_1 from Eq. (26.18).

For the <u>counterflow</u> scheme with $W_1 \neq W_2$, the problems of the fifth group reduce to a solution of the transcendental equation

$$\frac{\vartheta_2-\vartheta_1}{t_2-\vartheta_1}=\frac{k\cdot F}{W_2}\cdot\frac{\frac{t_1-\vartheta_2}{t_2-\vartheta_1}-1}{\ln\left(\frac{t_1-\vartheta_2}{t_2-\vartheta_1}\right)}. \qquad (26.25)$$

After finding t_1 or t_2, we determine W_1 from Relation (26.18).

For the <u>counterflow</u> scheme with $W_1 = W_2 = W$ it is necessary to solve Eq. (26.23) with respect to W and one of the final temperatures.

<u>Sixth group of problems</u> (W_1 = ?; $\vartheta_{1 \text{ or } 2}$ = ? or W_2 = ?; $t_{1 \text{ or } 2}$ = = ?). For the <u>direct flow</u> scheme the following equation holds true:

$$\frac{t_1-\vartheta_1}{t_2-\vartheta_2}+\frac{\vartheta_2-t_1}{t_2-\vartheta_2}=\frac{k\cdot F}{W_2}\cdot\frac{\left(\frac{t_1-\vartheta_1}{t_2-\vartheta_2}\right)-1}{\ln\left(\frac{t_1-\vartheta_1}{t_2-\vartheta_2}\right)}. \qquad (26.26)$$

Solving this equation with respect to ϑ_1 or ϑ_2, we obtain from (26.18) the water equivalent W_1.

For the <u>counterflow</u> scheme with $W_1 \neq W_2$ the corresponding transcendental equation has the form

$$\frac{t_2-\vartheta_1}{t_1-\vartheta_2}+\frac{\vartheta_2-t_2}{t_1-\vartheta_2}=\frac{k\cdot F}{W_2}\cdot\frac{\left(\frac{t_2-\vartheta_1}{t_1-\vartheta_2}\right)-1}{\ln\left(\frac{t_2-\vartheta_1}{t_1-\vartheta_2}\right)}. \qquad (26.27)$$

Equations (26.27) and (26.18) enable us to determine ϑ_1 or ϑ_2 and W_1.

An effective method of solving transcendental equations, to which the problems of the fifth and sixth group lead, is proposed in [58b].

As a result of performing the first stage in the thermal design

of the recuperators, all eight primary quantities (W_1, W_2, t_1, t_2, ϑ_1, ϑ_2, k·F) are determined. The second stage of the thermal design consists of determining the secondary quantities Q, C_{p1}, C_{p2}, G_1, G_2, and k·F. The values of C_{p1} and C_{p2} are dependent on the choice of the various heat carriers, and the value of the heat-transmission coefficient $\underline{k}$ is determined by the structural scheme of the apparatus, the form of the live cross sections for the streams, and the velocities and physical properties of the heat carriers [31, 35b].

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[List of Transliterated Symbols]

626 л = l = levyy = left

626 п = p = pravyy = right

629 ср = sr = sredniy = average

631 прот = prot = protivotochnyy = counterflow

634 прям = pryam = direct

SYMBOLS

§1

$\tilde{Q}$ – amount of heat, kcal;

Q – flow of heat, kcal/hr;

$\vec{q}$ – vector of heat flux density, kcal/m^2;

q_V – strength of internal heat sources, kcal/m^3-hr;

τ – time, hours or seconds;

F – surface, m^2;

x, y, z or x_i ($i = 1, 2, 3$) – coordinates, meters;

n – normal to the surface, meters;

U – internal energy, kcal;

u – internal energy per unit volume, kcal/m^3;

p – pressure, kg/m^2;

v – specific volume, m^3/kg;

T – temperatures, ^oK;

t – temperatures, ^oC;

C_v – isochoric specific heat per unit volume, kcal/m^3-deg;

V – volume, m^3;

λ – coefficient of heat conductivity, kcal/m-hr-deg;

a_v – isochoric coefficient of temperature conductivity, m^2/hr;

q_F – heat rate on the heating surface, kcal/m^2-hr.

§2

$\vec{l}_F$ – radius vector of any point on the heating surface, meters;

t_{zh} – temperature of liquid flowing around a solid body, degrees;

α – coefficient of heat transmission, kcal/m^2-hr-deg;

t_F – temperature on the heating surface, degrees.

§3

δ – thickness of plate, meters;

ϑ – Kirchhoff variable, kcal/m-hr;

t_{F_1} and t_{F_2} – constant temperatures on the external surfaces of the plate, ^{o}C;

Θ – dimensionless temperature;

X – dimensionless abscissa;

$\Delta t_F \equiv t_{F_1} - t_{F_2}$ – temperature difference in the plate, ^{o}C;

Θ_{F_1} – dimensionless temperature of the surface of the plate;

α_1 and α_2 – coefficients of heat transmission on the surfaces of the plate, kcal/m^2-hr-deg;

t_{zh1} and t_{zh2} – temperatures of hot and cold liquids, which are separated by the plate, ^{o}C;

$\Delta t_{zh} \equiv t_{zh1} - t_{zh2}$ – temperature difference between the hot and cold liquids, ^{o}C;

k – coefficient of heat transmission through the plate, kcal/m^2-hr-deg;

F_p – value of finned surface, m^2;

Θ_{zh1} – dimensionless temperature of hot liquid;

$B_i \equiv \alpha\delta/\lambda_F$ – dimensionless Biot number.

§4

r – variable radius, meters;

r_1, r_2, d_1, d_2 – internal and external radii and diameter of the tube, meters;

$R \equiv r/r_2$ – dimensionless variable radius;

$R_1 \equiv r_1/r_2$ – dimensionless inside radius of the tube;

d_{kr} – critical diameter of tube wall, meters.

§5

f – transverse cross section of cylindrical rod, m^2;

Π – perimeter of transverse cross section $\underline{f}$, meters;

l – length of rod, meters;

$^{f}\overline{t}$ – rod temperature averaged over the cross section $\underline{f}$, ^{o}C;

$^{f}\overline{q}_{V}$ – strength of internal heat sources acting in the rod, averaged over the cross section $\underline{f}$, kcal/m^3-hr;

$\vartheta \equiv {}^{f}\overline{t} - t_{zh}$ – excess temperature in any section of the rod, ^{o}C.

§6

δ – half thickness of the plate, meters;

τ – time, hours;

t_0 – initial temperature of the plate, ^{o}C;

$\Delta t \equiv t_{zh} - t_0$ – excess temperature of plate at the initial instant of time, ^{o}C;

$T(\tau)$ – auxiliary function, which depends only on the time;

$X(x)$ – auxiliary function, which depends only on the abscissa;

$\vartheta \equiv t_{zh} - t$ – excess temperature, ^{o}C;

C_1 and C_2 – constants;

k – serial number of the eigenvalue;

n_k – eigenvalue with serial number $\underline{k}$;

$\theta \equiv \frac{t_{ж} - t}{t_{ж} - t_0}$ – dimensionless temperature at any point of the plate at any instant of time;

$F_0 \equiv a_v \cdot \tau/\delta^2$ – Fourier number for the plate;

Θ_0 and Θ_1 – dimensionless temperature at the center of the plate and on its surface;

A_k – thermal amplitude;

$\theta = \int_{t_0}^{t} \lambda(t)\,dt$ – Kirchhoff variable.

§7

F and V – surface and volume of body of arbitrary shape;

m – tempo of regular mode, reciprocal hours;

$\vec{l}$ – radius vector of any point inside the body, meters;

$\bar{\alpha}$ – coefficient of heat transfer averaged over the surface of the body, kcal/m^2-hr-$^\circ$C;

$\bar{t}$ – temperature averaged over the volume of the body, $^\circ$C.

§8

τ_0 – time scale, a finite and positive quantity, hours;

T – dimensionless time;

X – dimensionless abscissa;

A_l – final positive coefficient of proportionality between two different units used to measure distances;

A_ϑ, A_τ – analogous coefficients for units used to measure the temperature and the time;

n – number of independent dimensional quantities contained in the equation;

k – number of groups of homogeneous quantities with independent measurement units;

π – dimensionless power function of the dimensional quantities of which the equation is made up.

§9

C_φ – similarity constant for the quantity φ;

π – similarity criterion.

§10

μ – dynamic viscosity, kg-sec/m^2;

ρ_{zh} – density of liquid, kg-sec^2/m^4;

C_{pzh} – isobaric bulk specific heat of the liquid, kcal/m^3-deg;

λ_{zh} – coefficient of heat conductivity of the liquid, kcal/m-hr-deg;

t_{zh} – temperature of the liquid either far from the heating surface (external problem) or averaged over the cross section of the channel (internal problem);

n – normal to the solid surface about which the liquid flows, meters;

k – thickness of thermal boundary layer, meters;

p – pressure in the liquid kg/m^2;

v – specific volume of the liquid, m^3/kg;

I – enthalpy of the entire volume of the liquid, kcal;

i – enthalpy per unit volume, $kcal/m^3$;

$\vec{u}$ – vector of velocity of the liquid, m/sec;

a_p – isobaric thermal diffusivity, m^2/hr;

$\vec{m}$ – vector of mass flow density of the liquid, $kg\text{-}sec/m^3$;

P_{xx}, P_{yx} – stresses on the faces of the elementary parallelepiped of liquid, kg/m^2;

τ_{xz} – tangential stress or viscous-friction stress in the liquid, kg/m^2;

$\nu \equiv \mu/\rho$ – kinematic viscosity, m^2/sec;

T – temperature, ^oK;

n – exponent in the exponential dependence of the physical parameters on the temperature;

τ – time, hours.

§12

$\vec{l}_F$ – radius vector of point on the surface of the body, meters;

l_1, l_2, ..., l_m – characteristic dimensions of body in the stream, m;

ψ_x, ψ_y, ψ_z – angles of attack or direction angles of the velocity vector of the liquid away from the body;

$\vec{u}_0$ – velocity of the liquid far from the body, m/sec;

$\vartheta \equiv t - t_{zh}$ – excess temperature, ^oC;

$\Theta = \frac{t - t_ж}{t_F - t_ж}$ – dimensionless temperature of the liquid;

$Pe = \frac{u \cdot l_1}{a_{pж}}$ – Peclet number;

$Re = \frac{u \cdot l_1}{\nu_ж}$ – Reynolds number;

$\vec{L}_F \equiv l_F/l_1$ – dimensionless radius vector of any point on the surface

$Nu = \frac{q_F \cdot l_1}{\lambda_{\text{ж}} \Delta t}$ – Nusselt number;

$Pr \equiv \nu/a_{pzh}$ – Prandtl number (isobaric);

$\Delta t \equiv t_F - t_{zh}$ – temperature difference, ^{o}C.

§13

d – diameter of cylinder, meters;

r and φ – polar coordinates;

$Re_f \equiv u_0 d/\nu_f$ – Reynolds number with viscosity referred to the temperature of the incoming stream;

$\bar{Nu_l} = \frac{q_F \cdot d}{\lambda_{\text{ж}l} \cdot \Delta t}$ – Nusselt number with heat conduction of the liquid referred to the temperature of the incoming stream;

Pr_f – Prandtl number of the liquid at the temperature of the incoming stream;

Pr_w – Prandtl number of the liquid at the temperature of heating surface;

$\varepsilon(\psi_x)$ – correction taking into account the influence of the angle of attack ψ_x on the heat transmission of a cylinder;

$\bar{\varphi}_\alpha$ – coefficient of heat transfer averaged over the perimeter of the cylinder, $kcal/m^2$-hr-deg.

§14

β_ρ – coefficient of volume expansion, deg^{-1};

$\vec{g}$ – vector of gravitational acceleration, m/sec^2;

p_∞ – pressure of liquid far from the body, kg/m^2;

Gr – Grashof number.

§16

Ry – Rayleigh number;

δ – thickness of hydrodynamic boundary layer, meters;

u_x and u_y – projections of the liquid velocity vector on the coordinate axes $\underline{x}$ and $\underline{y}$;

u_0 – velocity of liquid far from the plate, m/sec;

p_∞ – pressure of liquid far from the plate, kg/m^2.

$\vec{\Pi}(\varphi)$ – symbol denoting the order of the quantity φ;

ε – positive quantity, small compared with unity;

$Y \equiv y/\delta$ – dimensionless ordinate;

a_0, a_1, a_2 – constant coefficients in the power polynomial for the velocity;

τ_F – tangential stress or stress of friction against the plate, kg/m^2;

$Re_x = \frac{u_0 \cdot x}{\nu}$ – "longitudinal" Reynolds number;

Re_{xkr} – critical value of the "longitudinal" Reynolds number at which laminar flow in the boundary layer is replaced by turbulent flow;

Re_{xmin} – the smallest value of the "longitudinal" Reynolds number, starting with which the approximate theory of the boundary layer is valid.

§17

k – thickness of thermal boundary layer, meters;

$\vartheta_0 \equiv t_F - t_{zh}$ – temperature difference, ^oC;

b_0, b_1, b_2, ... – constant coefficients in the power polynomial for the excess temperature;

$Nu_x = \frac{q_F \cdot x}{\vartheta_0 \cdot \lambda_{zh}}$ – "longitudinal" Nusselt number.

§18

d – internal diameter of tube, meters;

l – length of tube, meters;

t_{zh1} – constant temperature of the liquid on entering the tube, ^oC;

q_F – heat rate on the inner surface of the tube, $kcal/m^2$-hr;

$\vartheta \equiv t - t_{zh1}$ – excess temperature, ^oC;

u_1 – constant velocity of liquid on entering the tube, m/sec;

$\Delta t_0 \equiv t_F - t_{zh1}$ – "initial" temperature difference, ^oC;

$\vec{U} \equiv \vec{u}/u_1$, $R \equiv r/r_0$, $X \equiv x/r_0$ – dimensional velocity, radius, and abscissa;

f – "dim" [illegible] ^{2}

$^{f}\overline{t}_{zh}(x)$ – liquid temperature averaged over the cross section $\underline{f}$, ^{o}C;

$\Delta t = t_F - {}^{f}\overline{t}_{zh}(x)$ – local temperature difference at any cross section of the tube, ^{o}C;

$^{l}\overline{q}_F$ – heat rate on the internal surface of a tube, averaged over its length, $kcal/m^2$-hr;

$^{l}\overline{\Delta t}$ – temperature difference averaged over the length of the tube, ^{o}C;

$^{l}\overline{\alpha}$ – coefficient of heat transfer averaged over the length of the tube, $kcal/m^2$-hr-deg;

$Re = \frac{u_1 \cdot d}{\nu}$ – Reynolds number;

$Pe = \frac{u_1 \cdot d}{a_{px}}$ – Peclet number;

$\theta_F = \frac{\vartheta_F \cdot \lambda_{ж}}{q_F \cdot d}$ – dimensionless temperature of tube wall;

$L = l/d$ – dimensionless length of tube;

$\Lambda = \frac{\lambda_{ж} - \lambda_{ж1}}{\lambda_{ж2} - \lambda_{ж1}}$ – dimensionless coefficient of heat conductivity of the liquid;

$C_p = \frac{C_p - C_{p1}}{C_{p2} - C_{p1}}$ – dimensionless specific heat of the liquid;

$M = \frac{\mu - \mu_1}{\mu_2 - \mu_1}$ – dimensionless viscosity of the liquid;

$\varrho = \frac{\varrho - \varrho_1}{\varrho_2 - \varrho_1}$ – dimensionless mass density of the liquid;

T_{zh1} – constant temperature of the liquid on entering the tube, ^{o}K;

T_{zh2} – temperature of the liquid leaving the tube averaged over the section of the tube, ^{o}K;

$Re_1 = \frac{u \cdot d}{\nu_1}$ – Reynolds number for the liquid stream on entering the tube;

$Pe_1 = \frac{u_1 \cdot d}{a_{px1}}$ – Peclet number for the liquid stream on entering the tube;

$\theta = \frac{t - t_{ж1}}{t_{ж2} - t_{ж1}}$ – dimensionless temperature of the liquid.

t' – turbulent pulsation of the temperature, ^{o}C;

$\vec{u}'$ – turbulent pulsation of the velocity, m/sec;

a_{rt} – turbulent heat conductivity in the radial direction, m^2/sec;

a_{xt} – turbulent temperature conductivity in the axial direction, m^2/sec;

$\tau\vec{\vec{u}}$ – velocity in the turbulent stream averaged over the time, m/sec;

τt – temperature in the turbulent stream averaged over the time, ^oC;

t, $\vec{u}$ and $\vec{q}$ – instantaneous values of the temperature, velocity, and density of the heat flow in the turbulent mode;

$\tau\vec{q}$ – time-averaged value of the density of the heat flow in the turbulent mode, kcal/m^2-hr;

l_r – path of turbulent mixing in radial direction, meters;

τ_t – tangential stress of turbulent friction, kg/m^2;

μ_{rt} – turbulent viscosity due to radial turbulent transport of momentum;

$P_{rt} \equiv \nu_{rt}/a_{rt}$ – turbulent Prandtl number, due to radial turbulent transport of heat and momentum;

Δp – pressure drop in stabilized liquid stream, kg/m^2;

τ_F – friction stress on the wall of the tube, kg m^2;

τ_r – friction stress on the cylindrical surface with radius $r < r_0$;

$\varphi = \tau\bar{u}(x) \cdot \sqrt{\frac{\varrho}{\tau_F}}$ – dimensionless velocity;

$\eta = \frac{y}{\nu} \cdot \sqrt{\frac{\tau_F}{\varrho}}$ – dimensionless distance from tube wall;

$R \equiv r/r_0$ – dimensionless radius;

$\vartheta_x(x)$ and $\vartheta_r(r)$ – functions, each of which depends only on one argument;

m – constant quantity, the tempo of the regular variation of the temperature field;

$\alpha_{0\ stab}$ – stabilized heat-transmission coefficient, referred to the initial temperature difference, kcal/m^2-hr-deg;

α_{stab} – stabilized heat-transmission coefficient, referred to the local temperature difference, kcal/m^2-hr-deg;

$Q_r \equiv q_r/q_F$ – dimensionless density of heat flow on the surface of a

cylinder with radius $\underline{r}$;

Nu_0 – stabilized Nusselt number for turbulent flow of liquids with vanishingly small Prandtl numbers in tubes;

$\Lambda = \frac{\lambda_m + \lambda_T}{\lambda_m}$ – ratio of overall heat conductivity in turbulent stream to the "molecular" heat conductivity;

$d_e \equiv 4f/\Pi$ – equivalent diameter, meters;

Π – "wetted" perimeter of cross section of the channel, meters;

f – live cross section of the channel, m^2;

$\zeta(Re_e)$ – coefficient of hydraulic friction resistance in the channel;

$f_{\overline{U}^2}$ – dimensionless velocity averaged over the cross section $\underline{f}$ of the channel;

Δn – length of the normal to any point of the channel perimeter, i.e., distance along the normal from the point on the perimeter to the second point of crossing of the normal with the perimeter, meters;

R – in Eq. (19.58) – radius of curvature at any point of the perimeter of the channel cross section, meters;

u_0 – liquid velocity averaged over the cross section of the channel, m/sec. The index "e" refers the quantity which this index designates to the liquid stream in the equivalent round tube.

§20

t_n – temperature of liquid in the saturation state, ^oC;

$^l\overline{t}_{zh}$ – temperature of the liquid averaged over the length of the tube, ^oC;

$^l\overline{t}_f$ – temperature of the wall averaged over the length of the tube, ^oC;

μ_f and λ_{zhf} – viscosity and heat conductivity of the liquid at its average temperature;

μ_w and λ_{zhw} – viscosity and heat conductivity of the liquid at the average temperature of the internal surface of the tube;

$^l\overline{\Delta t} \equiv {}^l\overline{t}_F - {}^l\overline{t}_{zh}$ – temperature difference averaged over the length of

the tube, ^{o}C;

$\bar{\alpha}_l$ – coefficient of heat transfer, averaged over the length of the tube, kcal/m^2-hr-deg;

ε_l – correction coefficient which takes into account the influence of the length of the tube on the average heat-transmission coefficient;

R – radius of turn of bent tube;

φ – angle of turn of bent tube;

P – exponent which depends on the dimensionless distance from the entrance to the tube;

d_1 and d_2 – inside and outside diameters of channel with annular form of transverse cross section;

$Pe_e = u_0 d_e/\nu$ – Peclet number for equivalent round tube.

p_n – saturation pressure, kg/m^2;

t_n – saturation temperature, ^{o}C;

q_F – heat rate on a heating surface cooled by a boiling liquid, kcal/m^2-hr;

G_F – gravimetric rate of formation of the vapor phase in bubble boiling on the heating surface, kg/m^2-hr;

G_* – maximum permissible rate of removal of vapor from the heating surface under given boiling conditions, kg/m^2-hr;

ρ – radius of nucleus of vapor phase, meters;

ρ_0 – radius of neutral nucleus of vapor phase, meters;

p_ρ – vapor pressure inside the nucleus, kg/m^2;

σ – coefficient of surface tension, kg-m/m^2;

γ_1 and γ_2 – specific gravities, liquid and vapor, kg/m^3;

p_{n_F} – saturation pressure at the temperature of the heating surface,

r_n – latent heat of formation of the vapor phase, kcal/kg;

d_0 – breakaway diameter of vapor bubble, meters;

τ_0 – average value of the period of formation and detachment of a vapor bubble from the heating surface, kg/m^2;

Θ – wetting angle.

§22

H – height of boiling layer, meters;

$N_{\rho,\rho+\Delta\rho}$ – number of nuclei of the vapor phase, which have a radius in the interval $\Delta\rho$, i.e., for $\rho - \Delta\rho \leq \rho \leq \rho + \Delta\rho$;

$\vec{u}_0$ – constant rate of rise of the vapor bubble, m/sec;

d – diameter of vapor bubble, meters;

ζ – coefficient of hydraulic resistance of vapor bubble as it rises;

τ – time, hours;

d_0 – breakaway diameter of the bubble, meters;

$^{\tau}\bar{t}_F$ – temperature of a surface, averaged over the time and over the area, $^{\circ}C$;

a_l and ν_l – thermal diffusivity and kinematic viscosity of the liquid phase;

$\vartheta_l \equiv t_l - t_n$ – excess temperature of the liquid, $^{\circ}C$;

$\overline{\Delta t} \equiv \bar{t}_F - \bar{t}_n$ – average temperature difference, $^{\circ}C$;

$\overline{Nu} = \frac{q_F}{\lambda_2 \cdot \Delta t} \times$

$\times \left(\frac{\sigma}{\gamma_1 - \gamma_2}\right)^{1/2}$ – Nusselt number or reciprocal of the dimensionless temperature difference;

$Ar = \frac{g}{\nu_1^2}\left(\frac{\sigma}{\gamma_1 - \gamma_2}\right)^{3/2} \times$

$\times \left(\frac{\gamma_1 - \gamma_2}{\gamma_1}\right)$ – Archimedes number (a measure of the ratio of the buoyant forces to the viscosity forces).

§23

$\delta(\varphi)$ – thickness of condensate film, meters;

μ_{1n} and λ_{1n} – viscosity and heat conductivity of condensate at saturation temperature;

μ_{1F} and λ_{1F} – viscosity and heat conductivity of the condensate at the heating-surface temperature.

§24

$d \equiv 2r_0$ – diameter of cylinder, meters;

$\vec{u}_{02}$ – velocity of the vapor far from cylinder, m/sec;

r and φ – polar coordinates;

$\varphi\bar{q}_F$ – heat rate of the cylinder averaged over the perimeter, kcal/m^2-hr;

$Re_2 \equiv u_{02}d/\nu_2$ – Reynolds number for the vapor flow;

$\overline{Nu}_0$ – Nusselt number for condensation of stationary vapor;

$\overline{Nu} \equiv \bar{\alpha}d/\lambda_1$ – Nusselt number for condensation of moving vapor.

§25

λ – wavelength of thermal radiation, μ;

E – integral density of hemispherical radiation, kcal/m^2-hr;

E_λ – spectral density of hemispherical radiation, kcal/m^2-hr-μ;

B – radiation brightness, kcal/m^2-hr-steradian;

A, R, D – coefficients of absorption, reflection, and transmission, respectively;

ε – average emissivity;

ε_λ – spectral emissivity;

φ_{1-2} – angle coefficient of a system of bodies 1 and 2;

C – radiation coefficient, kcal/m^2-hr-$^oK^4$;

$C_0 = 4.9$ – black-body radiation coefficient, kcal/m^2-hr-$^oK^4$;

F_1 and F_2 – surfaces of bodies 1 and 2, m^2;

T_1 and T_2 – temperatures of the bodies, ^oK;

A_{1-2} – reduced coefficient of absorption of the system of bodies 1 and 2.

§26

t_1 and t_2 – temperature of the hotter liquid on entering and leaving

the heat-exchange apparatus, ^{o}C;

ϑ_1 and ϑ_2 – temperature of the cooler liquid entering and leaving the heat-exchange apparatus, ^{o}C;

G_1 and G_2 – weight flows of the hotter and cooler liquids, kg/hr;

F – total heating surface of the heat-exchange apparatus, m^2;

f – fraction of the heating surface of the heat-exchange apparatus, m^2;

W_1 and W_2 – water equivalents of the hotter and cooler liquids, kcal/hr-deg;

Δt_l – temperature difference on the left side of the heat-exchange apparatus, ^{o}C;

Δt_p – temperature difference on the right side of the heat-exchange apparatus, ^{o}C;

Δt_{sr} – average temperature difference, ^{o}C;

P and R – auxiliary quantities;

δt and $\delta\vartheta$ – longitudinal temperature drop of the hotter and cooler liquids, ^{o}C.

REFERENCES

1. Alad'yev, I.T., Izv. AN SSSR, OTN [Bulletin of the Acad. Sci. USSR, Division of Technical Sciences], No. 11 (1951).
2. Aref'yeva, Ye.I. and Alad'yev, I.T., O vliyanii smachivayemosti na teploobmen pri kipenii [The Effect of Wettability on Heat Transfer in Boiling], Inzhenerno-fizicheskiy zhurnal [Physical Engineering Journal], 1, No. 7 (1958).
3. Berman, L.D., Izv. Vsesoyuz. teplotekhn. in-ta [Bulletin of the All-Union Thermal Engineering Institute], No. 3 (1953).
4. Bridzhmen, P.V., Analiz razmernostey [Dimensional Analysis], Moscow, Gostekhizdat [State Scientific and Technical Publishing House], 1934.
5. Bronshteyn, I.N. and Semendyayev, K.A., Spravochnik po matematike [Handbook on Mathematics], Moscow, Gostekhizdat, 1953.
6. Banichev, A.P., a) Zh. tekhn. fiz. [Journal of Technical Physics], No. 2 (1938); b) Izv. AN SSSR, OTN, No. 12 (1946); Tr. NII-1 [Trans. of the Scientific Research Institute No. 1], No. 2 (1947).
7. Vargaftik, N.B. (Editor), Teplofizicheskiye svoystva veshchestv (spravochnik) [Heat-Physical Properties of Materials (Handbook), Moscow, Gosenergoizdat [State Power Engineering Publishing House], 1956.
8. Venikov, V.A., Primeneniye teoriya podobiya i fizicheskogo modelirovaniya v elektrotekhnike [The Use of Similarity Theory and Physical Modeling in Electrical Engineering], Moscow, Gosener-

9. Voskresenskiy, K.D., a) Sbornik raschetov i zadach po teploperedache [Collection of Calculations and Problems in Heat Transfer], Moscow, Gosenergoizdat, 1959; b) Izv. AN SSSR, OTN, No. 7 (1948); c) Prom. energetika [Power Engineering Industry], No. 2 (1946).

10. Voskresenskiy, K.D. and Turilina, Ye.S., a) Priblizhennyy raschet teplootdachi rasplavlennykh metallov. Sb.: Teploperedacha i teplovoye modelirovaniye [Approximate Calculation of Heat Transfer of Fused Metals. Collection entitled Heat Transfer and Heat Modeling], Moscow, Izd-vo AN SSSR, 1959; b) Usloviya podobiya dlya protsessov teplootdachi pri techenii v trubakh zhidkosti s peremennymi fizicheskimi svoystvami. Otchet ENIN AN SSSR [Similarity Conditions for Heat-Transfer Processes With Flow of Liquid With Changing Physical Properties in Tubes. Report of the Institute of Power Engineering (im. G.M. Krizhanovskiy) of the Acad. Sci. USSR], No. 4 (1959).

11. Bukalovich, M.P. and Novikov, I.I., Tekhnicheskaya termodinamika [Technical Thermodynamics], Moscow, Gosenergoizdat, 1955.

12. Gol'dshteyn, S. (Editor), Sovremennoye sostoyaniye problem gidrodinamiki vyazkoy zhidkosti [Contemporary State of Problems of Hydrodynamics of the Viscous Fluids], Vol. II, Moscow, Izd-vo inostr. lit., 1948.

13. Gutenmakher, L.I., Elektricheskiye modeli [Electric Models], Moscow, AN SSSR, 1949.

14. Gukhman, A.A., Fizicheskiye osnovy teploperedachi [Physical Bases of Heat Transfer], Moscow, Gosenergoizdat, 1934.

15. Dolidze, D.Ye., Prikl. matem. i mekh. [Applied Mathematics and Mechanics], XI, No. 2 (1947); XII, No. 2, (1948); Dokl. AN SSSR, XCVI. No. 3 (1954).

16. Dyadyatkin, B.V. and Lel'chuk, V.L., Teploenergetika [Thermal Power Engineering], No. 9 (1958).

17. Zhukauskas, A.A., Teploenergetika, No. 4 (1955).

18. Zozulya, N.V., Kandidatskaya dissertatsiya [Candidate's Dissertation], Kiev, 1955.

19. Ivantsov, G.M., a) Tr. Ural'skogo politekhn. in-ta im. S.M. Kirova [a) Trans. of the Urals Polytechnic Inst. im S.M. Kirov], Collection 20, 1945; b) Nagrev metalla [Heating of Metal], Moscow, Metallurgizdat, 1949.

20. Kalafati, D.D. and Petukhov, B.S., Razvitiye teoreticheskikh osnov teplotekhniki [Development of the Theoretical Bases of Thermal Engineering], Moscow, Gosenergoizdat, 1954.

21. Kirpichev, V.L., Besedy o mekhanike [Conversations on Mechanics], Moscow, Gostekhizdat, 1951.

22. Kirpichev, B.L. and Gukhman, A.A., Prilozheniye teorii podobiya k opytu [Application of Similitude Theory to Experiment], Tr. Leningr. otd. Teplotekhn. in-ta [Trans. of the Leningrad Division of the Thermal Engineering Inst.], No. 1 (1931).

23. Kirpichev, M.V. and Mikheyev, M.A., Modelirovaniye teplovykh ustroystv [Modeling of Thermal Devices], Moscow, Izd-vo AN SSSR, 1936.

24. Kirpichev, M.V., Mekheyev, M.A. and Eygenson, L.S., Teploperedacha [Heat Transfer], Moscow, Gosenergoizdat, 1940.

25. Kondrat'yev, G.M., a) Zh. prikl. fiz. [Journal of Applied Physics], V, Nos. 3-4 (1928); b) Regulyarnyy teplovoy rezhim [Regular Heat Regime], Moscow, Gostekhizdat, 1954.

26. Kruzhilin, G.N., a) Issledovaniye teplovogo pogranichnogo sloya. Zh. tekhn. fiz. [a) Investigation of Thermal Boundary Layer. Journal of Technical Physics], VI, No. 3 (1936); b) Izv. AN SSSR,

OTN, No. 7 (1948), No. 5 (1949).

27. Kruzhilin, G.N. and Subbotin, V.I., Okhlazhdeniye vodo-vodyanykh reaktorov. Doklad No. 2054, predstavlennyy na Vtoruyu mezhdunarodnuyu konferentsiyu po mirnomu ispol'zovaniyu atomnoy energii (Zheneva, 1958) [Cooling of Water-Cooled Reactors. Report No. 2054 Presented at the Second International Conference on the Peaceful Uses of Atomic Energy (Geneva, 1958).

28. Kruzhilin, G.N. and Shvab, V.A., Zh. tekhn. fiz. [Journal of Technical Physics], V, No. 3 (1935); V, No. 4 (1935).

29. Kutateladze, S.S., Borishanskiy, V.M., Novikov, I.I. and Fedynskiy, O.S., Zhidkometallicheskiye teplonositeli [Liquid-Metal Heat Carriers], Moscow, Atomizdat, 1958.

30. Kutateladze, S.S., a) Gidromekhanicheskaya model' krizisa teploobmena v kipyashchey zhidkosti pri svobodnoy konvektsii [Hydromechanical Model of Heat-Transfer Crisis in Boiling Liquid with Free Convection], Zh. tekhn. fiz., XX, No. 11 (1950); b) Teploperedacha pri kondensatsii i kipenii [Heat Transfer in Condensation and Boiling], Moscow, Mashgiz, 1949 and 1957; c) Osnovy teorii teploobmena [Fundamentals of Heat Transfer Theory], Moscow-Leningrad, Mashgiz, 1957.

31. Kutateladze, S.S. and Borishanskiy, V.M., Spravochnik po teploperedache [Handbook on Heat Transfer], Moscow, Gosenergoizdat, 1959.

32. Labuntsov, D.A., a) Teplootdacha pri plenochnoy konedensatsii chistykh parov. Kandidatskaya dissertatsiya, MEI [a) Heat Transfer in Film Condensation of Pure Vapors. Candidate's Dissertation, Moscow Institute of Power Engineering (im. Molotov)], 1956; b) Teploenergetika [Thermal Power Engineering], No. 12 (1959).

33. Loytsyanskiy, L.G., Mekhanika zhidkosti i gaza [Mechanics of

Liquids and Gases], Moscow, Fizmatgiz [State Press for Physico-mathematical Literature], 1959.

34. Lykov, A.V., Teoriya teploprovodnosti [Theory of Thermal Conductivity], Moscow, Gostekhizdat, 1952.

35. Mikheyev, M.A., a) Izv. AN SSSR OTN, No. 10 (1952); b) Osnovy teploperedachi [Fundamentals of Heat Transfer], Moscow, Gosenergoizdat, 1956.

36. Mikheyev, M.A., Fedynskiy, O.S. and Voskresenskiy, K.D., Teplootdacha rasplavlennykh metallov. Sb. Reaktorostroyeniye i teoriya reaktorov [Heat Transfer of Molten Metals. In Collection Entitled Reactor Building and Reactor Theory], Moscow, Izd-vo AN SSSR, 1955.

37. Mijheyeva, I.M., Teploenergetika, No. 4 (1956).

38. Nikuradze, I.I., Sb.: Problemy turbulentnosti [Collection entitled Problems of Turbulence], ONTI, 1936.

39. Ostrogradskiy, M.V., Ob uravenii rasprostraneniya tepla v zhidkosti [Concerning the Equation of Heat Propagation in Liquids], Mem. l'Acad. [Memoirs of the Academy], 1, No. 4, 25-26 (1836).

40. Petrovskiy, I.G., Lektsiya ob uravneniyakh s chastnymi proizvodnymi [Lecture on Partial Differential Equations], Moscow, Gostekhizdat, 1950.

41. Petukhov, B.S., a) Opytnoye izucheniye protsessov teploperedachi [Experimental Study of Heat Transfer Processes], Moscow, Gosenergoizdat, 1952; b) Teploenergetika, No. 7 (1954).

42. Petukhov, B.S., Krasnoshchekov, Ye.A. and Nol'de, L.D., Tr. Mosk. energ. in-ta [Trans. of the Moscow Power Engineering Institute], No. 25, Moscow, Gosenergoizdat, 1955.

43. Plank, M., Vvedeniye v teoreticheskuyu fiziku [Introduction to Theoretical Physics], Ch. V-Teoriya teploty [Part V-Heat Theory],

ONTI NKTP SSSR [United Scientific and Technical Publishing House, People's Commissariat of the Post and Telegraph USSR], 1935.

44. Polyak, G.L., Teoriya luchistogo teploobmena. [Theory of Radiant Heat Transfer], Doctorate Dissertation, ENIN AN SSSR [Institute of Power Engineering (im. G.M. Krzhizhanovskiy) of the Acad. Sci. USSR], 1938.

45. Sedov, L.I., Metody Podobiya i razmernostey v mekhanike [Similitude and Dimensional Methods in Mechanics], Moscow, Gostekhizdat, 1957.

46. Sinel'nikov, A.S. and Chashchikhin, A.S., Zh. tekhn. fiz., 2, No. 9-10 (1932).

47. Spivak, G.V., (Editor), Spetsial'nyy fizicheskiy praktikum [Special Practical Course in Physics], Moscow, Gostekhizdat, 1945.

48. Surinov, Yu.A., Analiticheskiye metody v teorii luchistogo teploobmena i ikh primeniye v teplotekhnike [Analytical Methods in the Theory of Radiant Heat Transfer and Their Application to Thermal Engineering], Doctorate Dissertation, ENIN AN SSSR, 1953.

49. Treshchov, G.G., Teploenergetika, No. 5 (1957).

50. Turilina, Ye.S., Candidate's Dissertation, ENIN AN SSSR, 1952.

51. Umov, N.A., Izbrannyye sochineniya [Selected Works], Edited by A.S. Predvoditelev, Moscow, Gostekhizdat, 1950.

52. Frank, F. and Mizes, R., Differentsial'nyye i integral'nyye uravneniya matematicheskoy fiziki [Differential and Integral Equations of Mathematical Physics], Moscow, ONTI, 1937.

53. Fuks, S.N., Izv. Vsesoyuz. teplotekhn. in-ta [Bulletin of the All-Union Thermal Engineering Institute], No. 3 (1953).

54. Chirkin, V.S., Teplofizicheskiye svoystva veshchestva. Spravochnoye rukovodstvo [Thermophysical Properties of Matter. Reference

Manual], Moscow, Fizmatgiz, 1959.

55. Shevel'kov, V.L., Teplofizicheskiye kharakteristiki izolyatsionnykh materialov [Thermophysical Characteristics of Insulation Materials], Moscow, Gosenergoizdat, 1958.

56. Sheffer, K., Teoriya teploty [Heat Theory], Moscow-Leningrad, Gostekhizdat, 1933.

57. Shorin, S.N., Teploperedacha [Heat Transfer], Moscow, Gosstroyizdat, 1952.

58. Shubin, Ye.P., a) Teploenergetika, No. 2 (1957); b) Teploenergetika, No. 11 (1955).

59. Eygenson, L.S., Modelirovaniye [Simulation], Moscow, Izd-vo Sovetskaya nauka [Soviet Science Publishing House], 1951.

60. Yakovlev, V., Atomnaya energiya [Atomic Energy], 8, No. 3 (1960).

61. Deissler, R., Trans. ASME, 73, 2 (1951).

62. Einstein, A., Ann. d. Phys., 35, 686 (1911).

63. Fourier, I.B.J., Theorie analytique de la chaleur [Analytical Theory of Heat], Paris, 1922.

64. Grundlagen der Physik [Foundations of Physics], Vol. 5, Theorie der Waerme [Heat Theory], 1894.

65. Lyon, R., Chem. Engng. Progr., 47, 2 (1951).

66. Lyon, R., Handbook Liquid Metals, 1955.

67. Lichtenstein, L., Grundlagen der Hydrodynamik [Fundamentals of Hydrodynamics], Berlin, 1929.

68. Miller, B., Trans. ASME, 71, 4 (1949).

69. Nusselt, W., VDI-Zeitschrift [J. Soc. Germ. Engineers], 60, 27 (1916).

70. Pepperhoff, W., Temperaturstrahlung u. Wissenschaftliche Forschungs-berichte, Naturwissenschaftliche Reihe [Thermactinic

Vol. 65, 1956.

71. Reichardt, H., Z. angew. Math. und Mech. [Journal of Applied Math. and Mech.], 20, 6 (1940).

72. Reichardt, A.Z., angew. Math. und Mech., 31, 7 (1951).

73. Schneider, P.J., Conduction Heat Transfer, 1955.

74. Stromquist, Effect of Wetting on Heat Transfer Characteristics of Liquid Metals, 1953, Univ. Tenn. Knoxville; United States Atomic Energy Comm.

75. Zuber, N., Trans. ASME, 80 (1958).

76. Elser, K., Schweiz. Bauzeitung [Construction Newsletter], V, 46, 69 (1951).

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